

The Schrödinger method for water wave kinematics

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Abstract

The Schrödinger method is developed for computation of water wave kinematics on finite depth, and is applied to laboratory experimental data for irregular waves.

Introduction

Conventional methods for computation of the velocity field under water waves typically fail near the wave crest due to uncertainty in how to treat the high-frequency portion of the spectrum. The failure of the linear Airy method is due to high-frequency contamination: The high-frequency tail of the spectrum is treated as representing linear free-wave oscillations around the mean water level, and is hence assigned too much weight.

The Wheeler stretching method was designed to compensate for the poor performance of the linear Airy method near the crest by a stretching modification that has little foundation in wave hydrodynamics. There is a large number of other similar empirical modifications of the linear Airy method to yield improved results.

Due to limited accuracy of field measurements, the high-frequency tail is the least reliable part of the measured spectrum. The energy carried by the high-frequency tail is in any case a small portion of the total energy. If only the most reliable portion of the measured spectrum is retained, the bandwidth is no longer wide. The high-frequency portion of the spectrum will only be important to the extent that it gives contributions coherent with the spectral peak.

The *nonlinear* Schrödinger method accounts for linear free waves near the spectral peak, in addition to nonlinear waves that are forced by the free waves near the peak. Hence the high-frequency tail is accounted for to the extent that it

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consists of nonlinear waves coherent with, and forced by, the linear free waves near the spectral peak. The *linear* Schrödinger method only accounts for linear free waves near the spectral peak. The philosophy behind the Schrödinger method is to take advantage of the weakly nonlinear narrow banded nature of typical ocean waves, in order to predict the velocity and acceleration fields by an asymptotic perturbation method with acceptable accuracy. In this paper we show that even a leading order *linear* Schrödinger method performs remarkably well in this respect.

Recently, the nonlinear Schrödinger method has been described for computation of the velocity field under *deep-water* waves (Trulsen & Dysthe 1997). Further analysis of synthetic irregular wave data on deep water has demonstrated that the nonlinear Schrödinger method predicts a horizontal velocity under a crest between the predictions of the linear Airy method and the Wheeler stretching method. The nonlinear Schrödinger method generally predicts a horizontal velocity 12–25% larger than the prediction of the Wheeler stretching method near the mean water level, in good agreement with the recommendation of Gudmestad & Haver (1993) and Gudmestad & Karunakaran (1994).

For many applications of practical interest, the waves under consideration are *finite-depth* waves on the continental shelf. In this paper, we extend the Schrödinger method to finite depth. The present theory uses the same scaling assumptions for broader bandwidth weakly nonlinear waves that were first introduced by Trulsen & Dysthe (1996). We believe that these scaling assumptions are representative for typical ocean waves.

Governing equations

The inviscid hydrodynamic equations for the velocity potential $\phi(x, z, t)$ and surface displacement $\zeta(x, t)$ of an incompressible fluid with uniform depth h are

$$\nabla^2 \phi = 0 \quad \text{for} \quad -h < z < \zeta, \quad (1)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = \zeta, \quad (2)$$

$$\frac{\partial \phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla \phi)^2 = 0 \quad \text{at} \quad z = \zeta, \quad (3)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h. \quad (4)$$

Here g is the acceleration of gravity, the horizontal coordinate is x , the vertical coordinate is z and $\nabla = (\partial/\partial x, \partial/\partial z)$.

In comparison with Trulsen & Dysthe (1996), the assumption on the depth is relaxed to order unity:

$$k_c a = \mathcal{O}(\epsilon), \quad |\Delta k|/k_c = \mathcal{O}(\epsilon^{1/2}), \quad (k_c h)^{-1} = \mathcal{O}(1). \quad (5)$$

Here k_c , a and Δk are characteristic scales for the wavenumber, amplitude and modulation wavenumber, respectively, and $\epsilon \ll 1$ is a small perturbation parameter. In practical terms, the steepness $k_c a$ is typically in the range 0.1–0.12, and the bandwidth $|\Delta k|/k_c$ is typically in the range 0.35–0.45.

The linearized equations (1)–(4) yield the dispersion relation for gravity waves on finite depth

$$\omega^2 = gk \tanh kh. \quad (6)$$

All of the following results have been made dimensionless by using the wave-number and frequency of the central wave, k_c and ω_c , related by the linear dispersion relation (6).

The following harmonic expansions for the velocity potential and surface displacement are employed:

$$\phi = \epsilon^{3/2} \bar{\phi} + \frac{1}{2} (\epsilon A'_1 e^{i\theta} + \epsilon^2 A'_2 e^{2i\theta} + \dots + \text{c.c.}), \quad (7)$$

$$\zeta = \epsilon^2 \bar{\zeta} + \frac{1}{2} (\epsilon B e^{i\theta} + \epsilon^2 B_2 e^{2i\theta} + \dots + \text{c.c.}). \quad (8)$$

Here c.c. denotes the complex conjugate, and the rapid phase is $\theta = x - t$. The slow drift $\bar{\phi}$ and set-down $\bar{\zeta}$ as well as the complex harmonic amplitudes $A'_1, A'_2, \dots, B, B_2, \dots$ are functions of the slow modulation variables $\epsilon^{1/2}x$ and $\epsilon^{1/2}t$. For finite depth, it is convenient to assume that the variables $\bar{\phi}, A'_1, A'_2, \dots$ depend on the basic vertical coordinate z .

The vertical dependence of the harmonic amplitudes of the velocity potential is found from the Laplace equation and the bottom boundary condition:

$$\frac{\partial^2 A'_n}{\partial z^2} - n^2 A'_n + 2ni\epsilon^{\frac{1}{2}} \frac{\partial A'_n}{\partial x} + \epsilon \frac{\partial^2 A'_n}{\partial x^2} = 0 \quad \text{for} \quad -h < z \quad (9)$$

$$\frac{\partial A'_n}{\partial z} = 0 \quad \text{at} \quad z = -h \quad (10)$$

The vertical dependence can then be found by the perturbation expansion

$$A'_n = A_{n,0} + \epsilon^{\frac{1}{2}} A_{n,1} + \dots \quad (11)$$

The leading-order solution is

$$A_{1,0} = \frac{\cosh(z+h)}{\cosh h} A \quad (12)$$

which evaluates to A at $z = 0$. A is not a function of z . At the next order, the particular solution that vanishes at $z = 0$ can be written as

$$A_{1,1} = -i \frac{(z+h) \sinh(z+h) - h \tanh h \cosh(z+h)}{\cosh h} \frac{\partial A}{\partial x}. \quad (13)$$

Substituting the expansions (7), (8) and (11) into the two surface conditions (2) and (3), we get

$$A = -\frac{i}{\sigma} B - \frac{\epsilon^{\frac{1}{2}}}{\sigma} \frac{\partial B}{\partial t} \quad (14)$$

and the evolution equation

$$\frac{\partial B}{\partial x} + \frac{1}{c_g} \frac{\partial B}{\partial t} + \frac{i\epsilon^{\frac{1}{2}}}{8c_g^3} \left(1 + 2(\sigma^2 - 1) \frac{h}{\sigma} \left(1 - (3\sigma^2 + 1) \frac{h}{\sigma} \right) \right) \frac{\partial^2 B}{\partial t^2} = 0 \quad (15)$$

where the group velocity is

$$c_g = \frac{1}{2} \left(1 - (\sigma^2 - 1) \frac{h}{\sigma} \right) \quad (16)$$

and $\sigma = \tanh h$.

The velocity and acceleration fields

The velocity field is found explicitly by taking the gradient of the velocity potential, $(u, w) = \nabla \phi$, and is given by

$$u = \text{Re} \left\{ \left(\mathcal{C}_1 B + \frac{i\epsilon^{\frac{1}{2}}}{c_g} \left[\frac{1}{2} \left(1 - (\sigma^2 + 1) \frac{h}{\sigma} \right) \mathcal{C}_1 + (z + h) \mathcal{S}_1 \right] \frac{\partial B}{\partial t} \right) e^{i\theta} \right\} \quad (17)$$

and

$$w = \text{Re} \left\{ \left(-i\mathcal{S}_1 B + \frac{\epsilon^{\frac{1}{2}}}{c_g} \left[\frac{1}{2} \left(1 - (\sigma^2 + 1) \frac{h}{\sigma} \right) \mathcal{S}_1 + (z + h) \mathcal{C}_1 \right] \frac{\partial B}{\partial t} \right) e^{i\theta} \right\}, \quad (18)$$

where we have used the notation

$$\mathcal{C}_1 = \frac{\cosh(z + h)}{\sinh h} \quad \mathcal{S}_1 = \frac{\sinh(z + h)}{\sinh h}.$$

Within the present level of approximation, the Eulerian and Lagrangian acceleration fields are identical and are given by

$$\frac{\partial u}{\partial t} = \text{Re} \left\{ \left(-i\mathcal{C}_1 B + \frac{\epsilon^{\frac{1}{2}}}{c_g} [(1 - h\sigma)\mathcal{C}_1 + (z + h)\mathcal{S}_1] \frac{\partial B}{\partial t} \right) e^{i\theta} \right\} \quad (19)$$

and

$$\frac{\partial w}{\partial t} = \text{Re} \left\{ \left(-\mathcal{S}_1 B - \frac{i\epsilon^{\frac{1}{2}}}{c_g} [(1 - h\sigma)\mathcal{S}_1 + (z + h)\mathcal{C}_1] \frac{\partial B}{\partial t} \right) e^{i\theta} \right\}. \quad (20)$$

Application to laboratory measurements

We use wave data from an experiment at the Norwegian Hydrotechnical Laboratory in Trondheim, see Skjelbreia et al. (1991). The experiment was carried out in a 33 m long and 1 m wide wave tank with a water depth of 1.302 m. Letting x and z be the horizontal and vertical coordinates in the tank, the wave generator was located at $x = -19.05$ m, while simultaneous surface elevation and velocity field measurements were made at $x = 0$. The velocity field could only

be measured at one vertical height at a time. Hence the same wave generator signal was used several times in order to obtain a vertical profile of the velocity field under the same wave train. Velocity measurements were carried out at 15 different vertical positions. There are altogether 41 time series denoted by `i24_#`, where `#` is a number between 1 and 41. Each time series has 32768 scans with sample period $\tau = 0.025$ s.

While the peak frequency is often used to characterize the frequency spectrum, it is here more relevant to characterize the power spectrum by its mean. For the presented cases, the mean frequency is approximately 25% larger than the peak frequency. The characteristic frequency ω_c is hence determined from the Fourier transform of the surface displacement, $\hat{\zeta}$, by

$$\omega_c = \frac{2\pi M_0}{T} \approx \frac{\sum_j |\omega_j| |\hat{\zeta}_j|^2}{\sum_j |\hat{\zeta}_j|^2}, \quad (21)$$

such that M_0 is an integral number of central wave oscillations in the periodic computational domain of length T . The corresponding characteristic wavenumber k_c is computed from the linear dispersion relation (6). The non-dimensional depth $k_c h$ is in all cases between 1.4 and 1.6, and hence the waves are on strictly finite depth.

We have computed the velocity field under the highest wave in `i24_15`, which occurred at scan 3903. The velocity field was measured at the vertical position $z = -0.20$ m. The first 3000 scans are skipped, and the next 2048 scans are used for a periodic domain of length $T = 2048\tau = 51.2$ s. The number of central wave oscillations is set to be $M_0 = 25$, and the first harmonic B of the harmonic expansion (8) is defined by bandpassing $M_{bp} = 33$ spectral components of $\hat{\zeta}$ around the central wave. This gives a normalized bandpass width of the approximation $M_{bp}/(2M_0) = 0.66$. The exact choice of the bandpass width should not be critical provided the power spectrum decays sufficiently fast away from the central frequency. The bandpass width should be chosen appropriately to accommodate the scaling assumption for the bandwidth of the power spectrum (5b).

The computations of the Airy and Wheeler methods have been performed with a lowpass cutoff frequency such that 99% of the total energy in the frequency power spectrum is retained. We employ linear extrapolation for the Airy velocity profile above the mean water level.

The measured surface displacement is shown in figure 1 for the periodic time domain. The frequency power spectrum is shown in figure 2, where we have indicated the bandpass filter employed for the first harmonic of the linear Schrödinger method at frequencies $(M_0 \pm M_{bp}/2)/T$ and the lowpass filter frequency employed for the Airy and Wheeler methods.

In figure 3 we show the horizontal velocity profile under the highest wave. The prediction of the Airy and Wheeler methods, as well as the Schrödinger method at relative orders 1, $\epsilon^{1/2}$ and ϵ (we intend to describe the relative order ϵ nonlinear

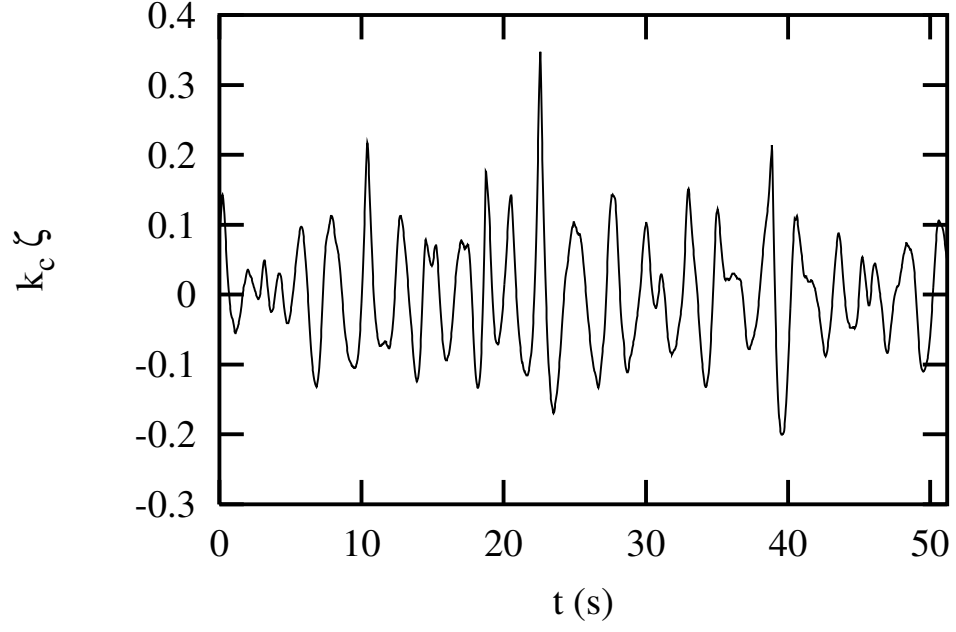


Figure 1: Measured surface displacement within the periodic time domain for computation.

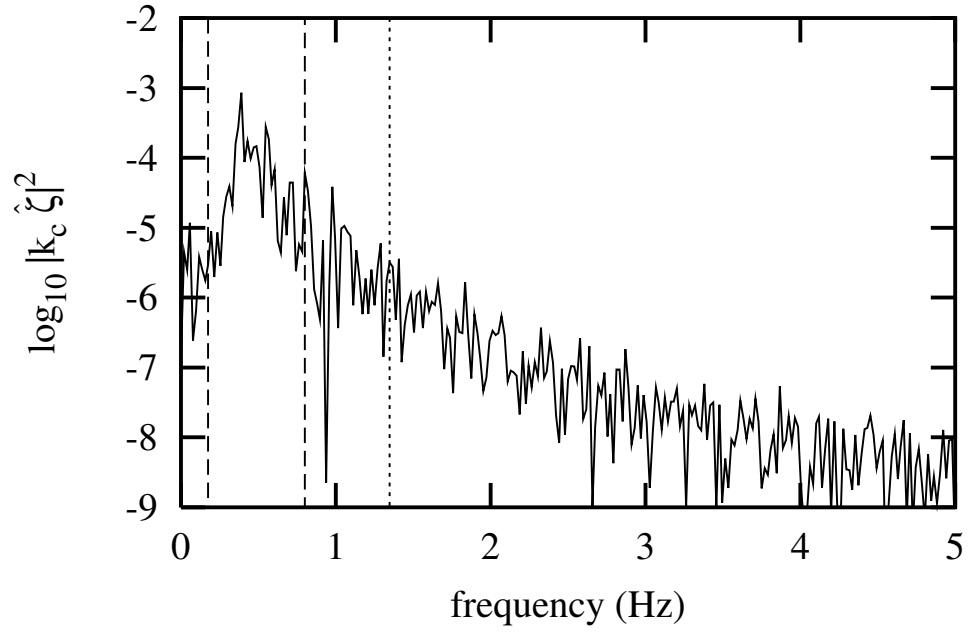


Figure 2: Frequency power spectrum (solid line) with Schrödinger bandpass filter (long dashed lines) and Airy lowpass filter (short dashed line).

Schrödinger method elsewhere). The measured velocity field is also indicated for the actual experiment i24_15 and all the other experiments.

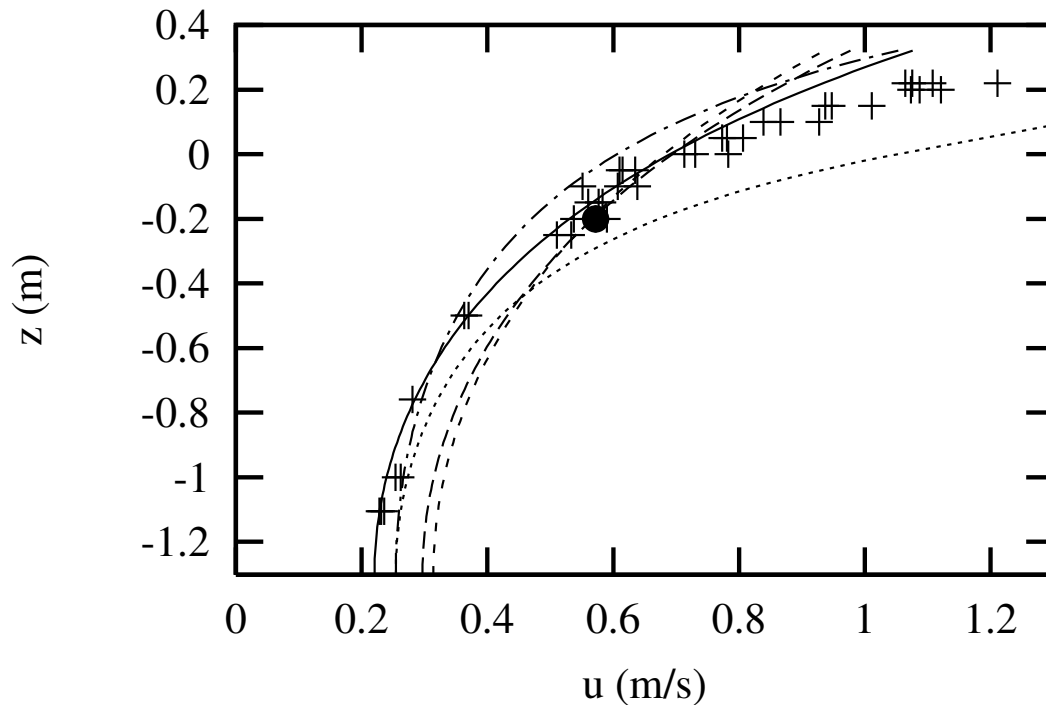


Figure 3: Horizontal velocity field under highest crest: Schrödinger order 1 (solid line), order $\epsilon^{1/2}$ (long dashed line), order ϵ (short dashed line); Airy (dotted line); Wheeler (dashed-dotted line); velocity measured in this experiment (diamond); velocities measured in the other experiments (cross).

Due to the problem of repeating the experiment exactly, while measuring the velocity field at only one location at a time, comparison of the velocity measurements from various experiments is subject to uncertainty. This is evidenced by the spread of the velocity measurements in the figure.

The correction to the Schrödinger method at relative order $\epsilon^{1/2}$ is seen not to give a significant contribution to the predicted velocity field. This is due to the fact that we are considering the velocity field near the global maximum of the time series, and hence $\partial B/\partial t$ should by definition be small. In general, the contribution from the term of order $\epsilon^{1/2}$ in (17) is larger than suggested by the figure.

The correction to the Schrödinger method at relative order ϵ , which we intend to describe elsewhere, includes the leading nonlinear contributions to the velocity field from the zeroth and second harmonics. The nonlinear contributions are most significant near the crest and the bottom. It is seen that for each consecutive order of the asymptotic theory, the approximation becomes slightly better.

Near the equilibrium water level, where the Wheeler method typically suffers from its greatest underprediction of the velocity field, even a leading order linear Schrödinger method gives remarkably good agreement with laboratory data.

Further developments

We have extended the Schrödinger method for water wave kinematics in two dimensions to finite depth, as is required for realistic applications on the continental shelf. Only the leading two linear orders are summarized here, yet we already achieve good agreement with laboratory data.

At the next order, there are nonlinear contributions to the velocity field from the zeroth and second harmonics, as well as additional linear contributions from the first harmonic. The nonlinear contributions are most significant near the crest and the bottom.

We have found that when the leading nonlinear contributions are taken into account, the velocity profile predicted by the nonlinear Schrödinger method corresponds remarkably well with the empirical correction to the Wheeler method suggested by Gudmestad & Haver (1993) (their equation 2). The nonlinear Schrödinger method for finite depth in two and three dimensions is currently being developed and will be presented in the future.

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