

Experimental evidence of transversal modulation and frequency down-shift of uniform waves in a long tank

Karsten Trulsen

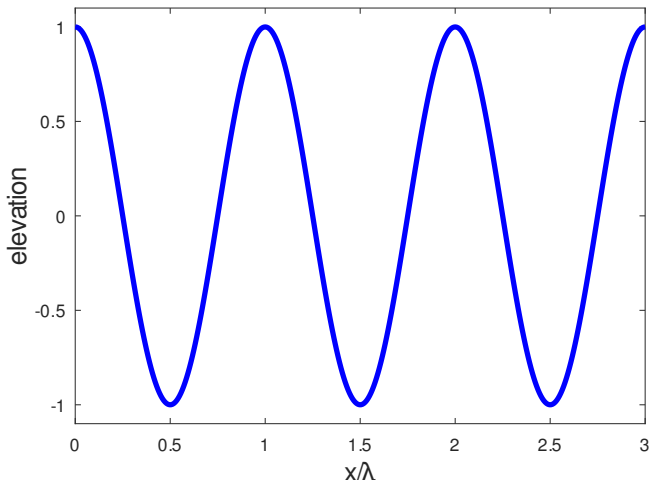


UiO : **Department of Mathematics**
University of Oslo

13 March 2025

joint work with Carl Trygve Stansberg and Kristian B. Dysthe

Uniform linear wave (Airy wave theory) — irrotational flow



Can be generalized to uniform nonlinear wave in various ways ...



Trochoidal wave or Gerstner wave — an exact solution of the Euler equations — Gerstner (1802) — with vorticity

IV.

THEORIE DER WELLEN

von

FRANZ GERSTNER,

ordentl. Prof. d. höhern Mathem. zu Prag *).

Keine mathematische Untersuchung der Natur hat bisher grössere Schwierigkeiten gezeigt, als die Erörterung der Bewegungsgesetze des Wassers, der Luft, und überhaupt aller Flüssigkeiten. Es ist der Analysis gelungen, die Mechanik des Himmels den genauesten Rechnungen zu unterwerfen; für die Bewegung des Wassers aber hat man bis jetzt nur genaue Differenzialgleichungen gefunden, zu deren endlicher Auflösung alle bekannte Näherungsmethoden nicht ausreichen. Da wir so oft vom Wasser Gebrauch zu machen, oder den schädlichen Wirkungen desselben zu begegnen haben, so wäre es sehr zu wünschen, daß man für alle Fälle der Hydraulik ähnliche Gesetze und analytische Rechnungsformeln finden möchte, als man in

*) *Abhandlungen der königl. böhmischen Gesellschaft der Wissenschaften zu Prag für das Jahr 1802.* Ich halte es



Stokes wave — an approximate solution of the Euler equations — Stokes (1847) — without vorticity

[From the *Transactions of the Cambridge Philosophical Society*,
Vol. VIII. p. 441.]

ON THE THEORY OF OSCILLATORY WAVES.

[Read March 1, 1847.]

IN the Report of the Fourteenth Meeting of the British Association for the Advancement of Science it is stated by Mr Russell, as a result of his experiments, that the velocity of propagation of a series of oscillatory waves does not depend on the height of the waves*. A series of oscillatory waves, such as that observed by Mr Russell, does not exactly agree with what it is most convenient, as regards theory, to take as the type of oscillatory waves. The extreme waves of such a series partake in some measure of the character of solitary waves, and their height decreases as they proceed. In fact it will presently appear that it is only an indefinite series of waves which possesses the property of being propagated with a uniform velocity, and without change of form: at least this is the case when the waves are



Stokes waves are unstable — modulational instability — Benjamin & Feir (1967)

J. Fluid Mech. (1967), vol. 27, part 3, pp. 417–430
Printed in Great Britain

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The disintegration of wave trains on deep water

Part 1. Theory

By T. BROOKE BENJAMIN AND J. E. FEIR†

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

(Received 21 June 1966)

The phenomenon in question arises when a periodic progressive wave train with fundamental frequency ω is formed on deep water—say by radiation from an oscillating paddle—and there are also present residual wave motions at adjacent side-band frequencies $\omega(1 \pm \delta)$, such as would be generated if the movement of the paddle suffered a slight modulation at low frequency. In consequence of coupling through the non-linear boundary conditions at the free surface, energy is then transferred from the primary motion to the side bands at a rate that, as will be shown herein, can increase exponentially as the interaction proceeds. The result is that the wave train becomes highly irregular far from its origin, even when the departures from periodicity are scarcely detectable at the start.

In this paper a theoretical investigation is made into the stability of periodic wave trains to small disturbances in the form of a pair of side-band modes, and Part 2 which will follow is an account of some experimental observations in accord with the present predictions. The main conclusion of the theory is that infinitesimal disturbances of the type considered will undergo unbounded magnification if

$$0 < \delta \leq (\sqrt{2})ka,$$

where k and a are the fundamental wave-number and amplitude of the perturbed wave train. The asymptotic rate of growth is a maximum for $\delta = ka$.



Unstable growth rate according to Benjamin & Feir (1967)

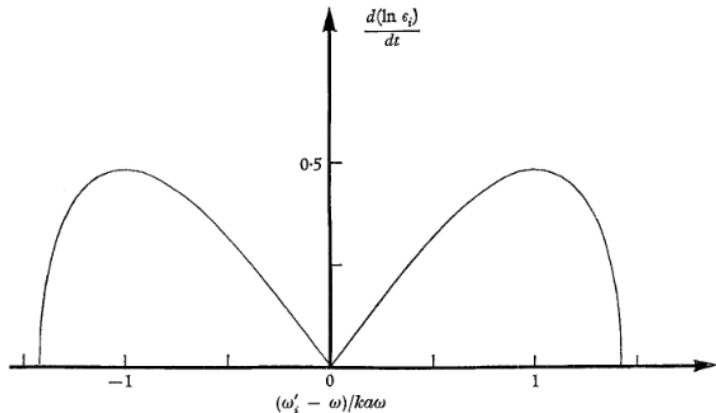


FIGURE 1. The asymptotic growth rate of the side-band amplitudes as a function of frequency.



The Fermi–Pasta–Ulam (FPU) recurrence — 1955

A long chain of masses linked by springs that obey Hooke's law (a linear interaction), but with a weak nonlinear term.

One day the computer wasn't stopped as intended, and the calculation was left running. The researchers found to their great surprise that nearly all of the energy returned to the initial mode, and the original state was almost perfectly recovered.

The energy does not drift toward the equipartition predicted by statistical physics but periodically returns to the original mode.

The first page of the report states the following:

Report written by Fermi, Pasta, and Ulam.

Work done by Fermi, Pasta, Ulam, and Tsingou.

quotes from Thierry Dauxois, Physics Today (2008)



Now called the Fermi–Pasta–Ulam–Tsingou recurrence



Figure 1. Mary Tsingou in 1955 (left) and in 2007 (right).

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January 2008 Physics Today 55



Example of Fermi–Pasta–Ulam–Tsingou recurrence (1965)

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266. - *Studies of non Linear Problems*

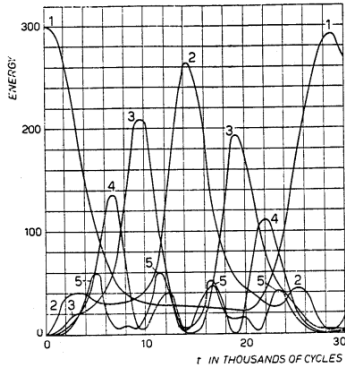


Fig. 1. - The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.



Look for Fermi–Pasta–Ulam–Tsingou recurrence in water waves — Lake, Yuen, Rungaldier & Ferguson (1977)

J. Fluid Mech. (1977), vol. 83, part 1, pp. 49–74

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Printed in Great Britain

Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train

By BRUCE M. LAKE, HENRY C. YUEN,
HARALD RUNGALDIER AND WARREN E. FERGUSON†

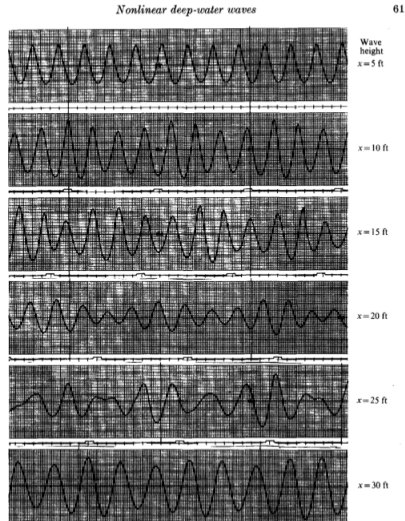
Fluid Mechanics Department, TRW/DSSG, One Space Park,
Redondo Beach, California 90278

(Received 13 December 1976 and in revised form 16 May 1977)

Results of an experimental investigation of the evolution of a nonlinear wave train on deep water are reported. The initial stage of evolution is found to be characterized by exponential growth of a modulational instability, as was first discovered by Benjamin & Feir. At later stages of evolution it is found that the instability does not lead to wave-train disintegration or loss of coherence. Instead, the modulation periodically increases and decreases, and the wave train exhibits the Fermi–Pasta–Ulam recurrence phenomenon. Results of an earlier study of nonlinear wave packets by Yuen & Lake, in which solutions of the nonlinear Schrödinger equation were shown to provide quantitatively correct descriptions of the properties of nonlinear wave packets, are applied to describe the experimentally observed wave-train phenomena. A comparison between the laboratory data and numerical solutions of the nonlinear Schrödinger equation for the long-time evolution of nonlinear wave trains is given.



Experiment in long tank shows recurrence with downshift, different from the Fermi–Pasta–Ulam–Tsingou recurrence — Lake, Yuen, Rungaldier & Ferguson (1977)



One-dimensional conservative nonlinear models predict perfect Fermi–Pasta–Ulam–Tsingou recurrence

For example:

- NLS = Nonlinear Schrödinger equation
- MNLS = Modified Nonlinear Schrödinger equation
= Dysthe equation
- etc.



With the wave modulations described by the Dysthe equation,
can dissipation, induced by wave breaking, as a consequence of
wave modulation, produce a permanent downshift?



FREQUENCY DOWN-SHIFT THROUGH SELF MODULATION AND BREAKING

KARSTEN TRULSEN AND KRISTIAN B. DYSTHE

University of Tromsø

P.O.Box 953

N-9001 TROMSØ

NORWAY

ABSTRACT. We simulate the development of a moderately steep wavetrain, using evolution equations correct to fourth order in the wave steepness (see Dysthe (1979) and Lo and Mei (1985)), with an added term simulating effects of wave breaking. It is found that the breaking damps the developing sidebands selectively, such that the most unstable lower sideband comes out of the modulation-breaking process being the dominant one. The reason for this, in our opinion, is the tendency towards spatial localization of the part of the wavetrain contributing to the upper sidebands. This “high-frequency” part of the signal seems to concentrate around the steepest portions of the wavetrain, where breaking occurs. This is found both in our simulations and in experimental records (Melville 1983).



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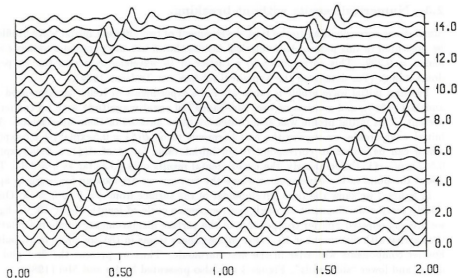


Figure 2: Time development of wave profiles, corresponding to the initial condition of Lo and Mei. $\xi/2\pi$ is taken along the horizontal axis and η is taken along the vertical axis.

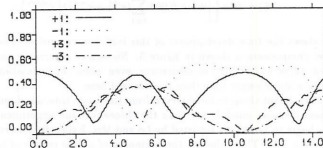


Figure 3: Time development of spectral components, corresponding to the initial condition of Lo and Mei. Component n corresponds to the n th sideband relative to the carrier $1/\epsilon\gamma$.



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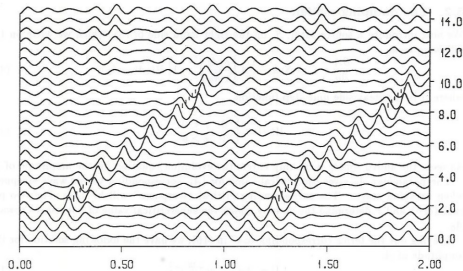


Figure 7: Wave profiles corresponding to the initial condition of Lo and Mei, with breaking. The vertically hatched areas show the localization of the breaking.

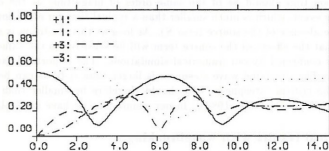


Figure 8: Spectral development corresponding to the initial condition of Lo and Mei, with breaking.



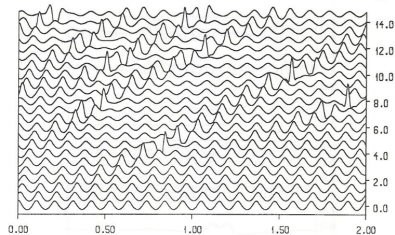


Figure 4: Wave profiles corresponding to the perturbed carrier wave.

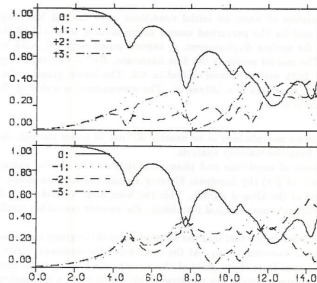


Figure 5: Spectral development for the three upper and lower sidebands, corresponding to the perturbed carrier wave.



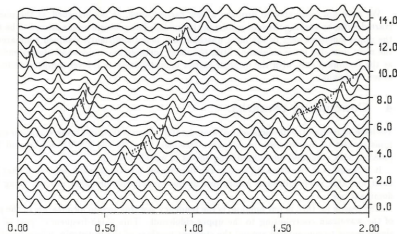


Figure 9: Wave profiles corresponding to the perturbed carrier wave, with breaking.

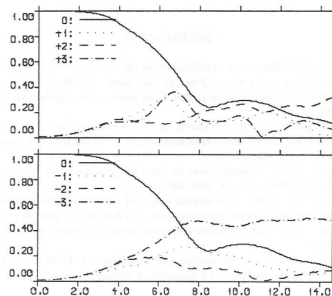


Figure 10: Spectral development for the three upper and lower sidebands, corresponding to the perturbed carrier wave, with breaking.



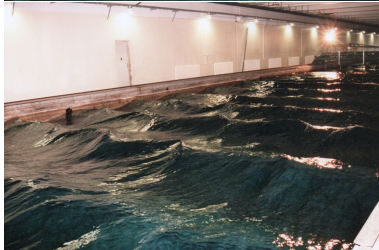
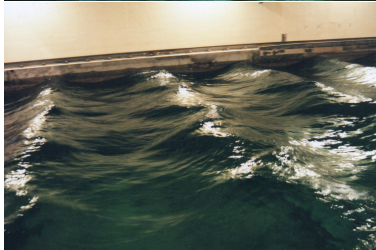
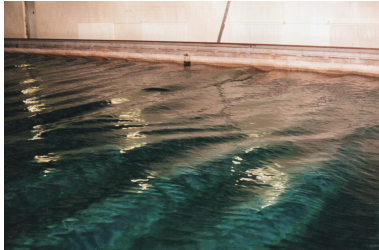
Very many publications follow this up ...

The one-dimensional downshift is not sensitive to the fine details of the heuristic damping / non-conservative mechanism.

(Here I should have included a long list of references. . .)



How to make a wave tank long :
Employ high-frequencies / short-wavelengths
but then the modulation is not long-crested



Hierarchy of modulational instabilities — collinear and transversal — type-I and type-II — McLean et al. (1981)

VOLUME 46, NUMBER 13

PHYSICAL REVIEW LETTERS

30 MARCH 1981

Three-Dimensional Instability of Finite-Amplitude Water Waves

J. W. McLean, Y. C. Ma, D. U. Martin, P. G. Saffman,^(a) and H. C. Yuen

Fluid Mechanics Department, TRW Defense and Space Systems Group, Redondo Beach, California 90278

(Received 26 September 1980)

Computations based on the full water-wave equations reveal that there are two distinct types of instabilities for gravity waves of finite amplitude on deep water. One is predominantly two dimensional and is related to all the known results for special cases. The other is predominantly three dimensional and becomes dominant when the wave steepness is sufficiently large.

PACS numbers: 47.35.+i, 03.40.Kf, 92.10.Hm

The set of equations governing surface gravity waves on irrotational, inviscid, incompressible fluid of great depth is

$$\nabla^2 \varphi = 0, \quad -\infty < z < \eta(x, y, t); \quad (1)$$

$$\varphi_t + \frac{1}{2}(\nabla \varphi)^2 + gz = 0, \quad \eta_t + \nabla \varphi \cdot \nabla \eta - \varphi_z = 0, \\ \text{on } z = \eta(x, y, t), \quad (2)$$

modes, substituting $\eta_s + \eta'$ and the corresponding $\varphi_s + \varphi'$ into (2), and satisfying the boundary conditions at $2N+1$ points. The resulting homogeneous linear system of order $4N+2$ was solved as an eigenvalue problem for σ by standard methods. The accuracy of the solutions was improved by Newton's method when necessary. For small values of h/λ (less than 0.1), $N=20$ sufficed to

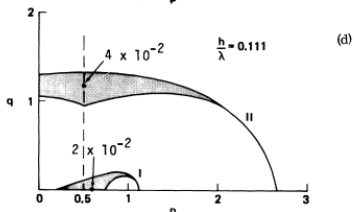
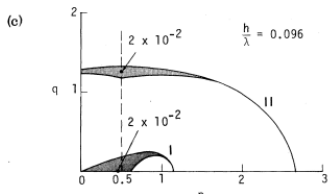
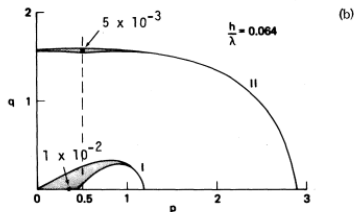
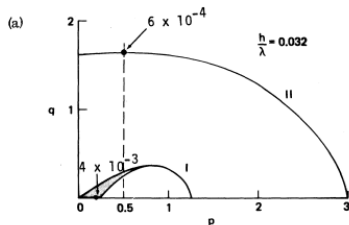


Hierarchy of modulational instabilities — collinear and transversal — type-I and type-II — McLean et al. (1981)

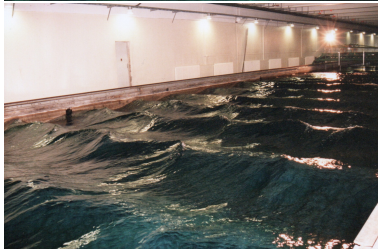
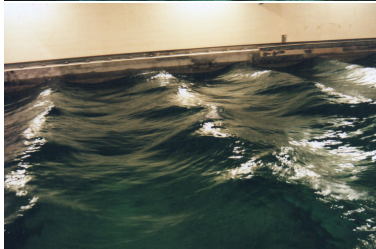
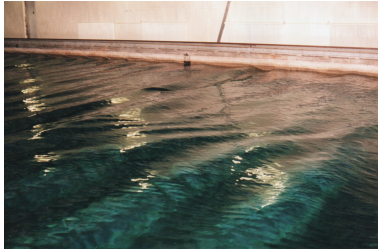
VOLUME 46, NUMBER 13

PHYSICAL REVIEW LETTERS

30 MARCH 1981



We can easily distinguish type-I and type-II transversal instability in experiments by visual inspection



Permanent downshift in wave modulations of Stokes wave without dissipation — Trulsen & Dysthe (1997)

J. Fluid Mech. (1997), vol. 352, pp. 359–373. Printed in the United Kingdom
© 1997 Cambridge University Press

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Frequency downshift in three-dimensional wave trains in a deep basin

By KARSTEN TRULSEN[†] AND KRISTIAN B. DYSTHE

Department of Mathematics, University of Bergen, Johannes Brunsgate 12,
N-5008 Bergen, Norway

(Received 5 November 1996 and in revised form 4 August 1997)

The conservative evolution of weakly nonlinear narrow-banded gravity waves in deep water is investigated numerically with a modified nonlinear Schrödinger equation, for application to wide wave tanks. When the evolution is constrained to two dimensions, no permanent shift of the peak of the spectrum is observed. In three dimensions, allowing for oblique sideband perturbations, the peak of the spectrum is permanently downshifted. Dissipation or wave breaking may therefore not be necessary to produce a permanent downshift. The emergence of a standing wave across the tank is also predicted.



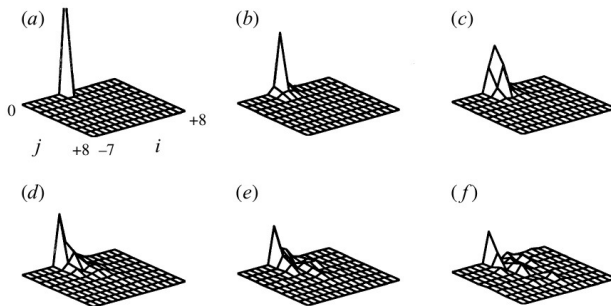


FIGURE 6. Snapshots of the spectrum $|\hat{B}(\lambda_i, \mu_j, \eta)|^2$ for $-7 \leq i \leq 8$ and $0 \leq j \leq 8$ (the spectrum is symmetric for positive and negative j): (a) initial state at $\eta = 0$; (b) $\epsilon^2 \eta = 3$; (c) $\epsilon^2 \eta = 4$ just before the downshift; (d) $\epsilon^2 \eta = 5$ just after the downshift; (e) $\epsilon^2 \eta = 9$; (f) $\epsilon^2 \eta = 16$. The vertical scale has been magnified by a factor 2 in (c-f).



Frequency downshift in three-dimensional wave trains

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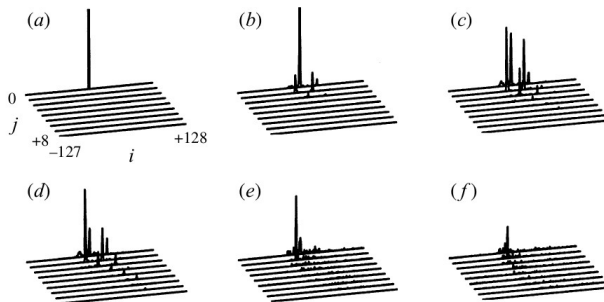


FIGURE 9. Snapshots of the spectrum $|\hat{B}(\lambda_i, \mu_j, \eta)|^2$ for $-127 \leq i \leq 128$ and $0 \leq j \leq 8$ (the spectrum is symmetric for positive and negative j): (a) initial state at $\eta = 0$ (the top of the peak at $i = j = 0$ has been truncated); (b) $\epsilon^2 \eta = 6$ before the downshift; (c) $\epsilon^2 \eta = 7$ during the downshift; (d) $\epsilon^2 \eta = 8$ after the downshift; (e) $\epsilon^2 \eta = 14$; (f) $\epsilon^2 \eta = 22$. The vertical scale has been magnified by a factor 2 in (c–f).



Experimental evidence for transversal modulation leading to downshift — Trulsen, Stansberg & Velarde (1999)

PHYSICS FLUIDS

VOLUME 11, NUMBER 1

JANUARY 1999

BRIEF COMMUNICATIONS

The purpose of this Brief Communications section is to present important research results of more limited scope than regular articles appearing in Physics of Fluids. Submission of material of a peripheral or cursory nature is strongly discouraged. Brief Communications cannot exceed three printed pages in length, including space allowed for title, figures, tables, references, and an abstract limited to about 100 words.

Laboratory evidence of three-dimensional frequency downshift of waves in a long tank

Karsten Trulsen^{a)}

Instituto Pluridisciplinar, Universidad Complutense de Madrid, Paseo Juan XXIII 1, E-28040 Madrid, Spain

Carl Trygve Stansberg

MARINTEK, P.O. Box 4125 Valentinlyst, N-7002 Trondheim, Norway

Manuel G. Velarde

Instituto Pluridisciplinar, Universidad Complutense de Madrid, Paseo Juan XXIII 1, E-28040 Madrid, Spain

(Received 22 July 1998; accepted 2 October 1998)

Experiments show that the evolution of Stokes waves can lead to significant growth of transverse standing wave oscillations across the tank at a large distance from the wave maker. The peak power intensity of narrow-banded transverse standing waves can reach a similar level as that of the collinear waves, and a significant downshift is observed for the transverse waves. Dissipation may be less important for the downshift than previously anticipated. © 1999 American Institute of Physics. [S1070-6631(99)02001-2]



Experimental evidence for transversal modulation leading to downshift — Trulsen, Stansberg & Velarde (1999)

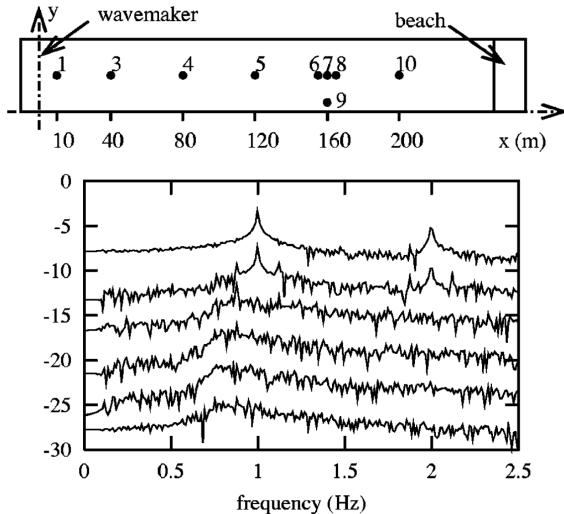
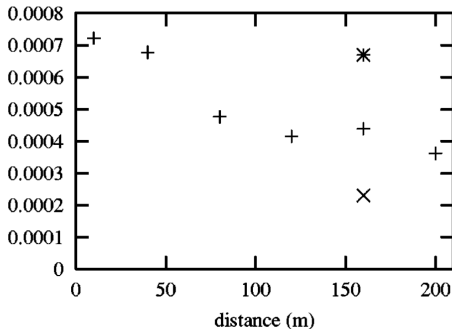
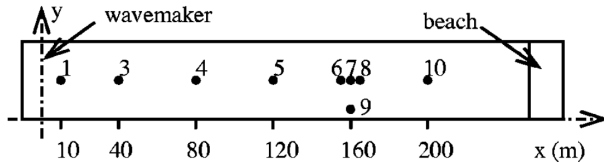


FIG. 2. Collinear power spectrum $\log_{10}|\hat{\xi}_{||}|^2$. From top to bottom $x = 10, 40, 80, 120, 160, 200$ m. For each consecutive value of x , the spectrum has been

Experimental evidence for transversal modulation leading to downshift — Essentially without dissipation!

— Trulsen, Stansberg & Velarde (1999)



Have all the “one-dimensional” experiments tricked us to believe that the downshift is a dissipative process?

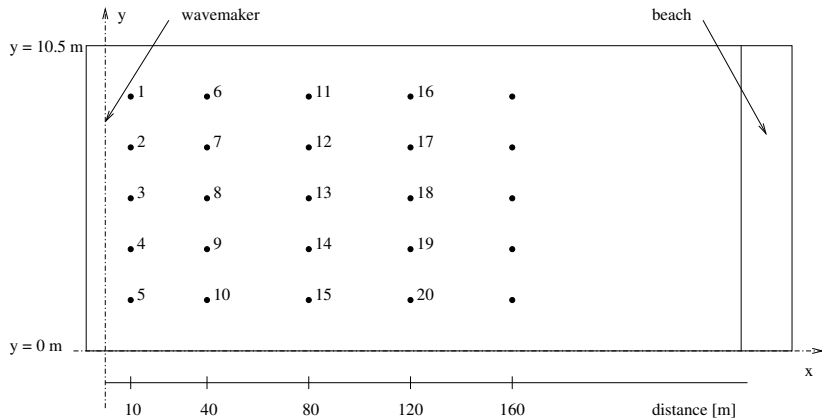
Maybe the “one-dimensional” experiments were actually “two-dimensional”, employing short wavelengths in order to make the wave tanks long?

Is dissipation less important than previously believed due to energy “hiding” in transversal modes?

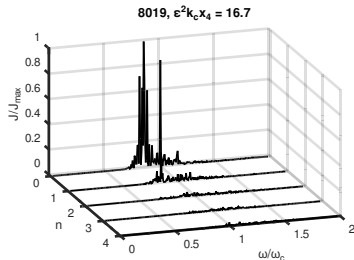
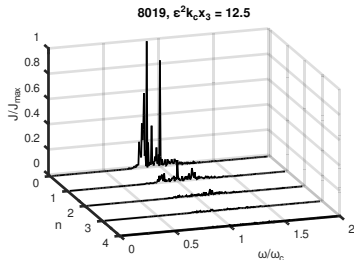
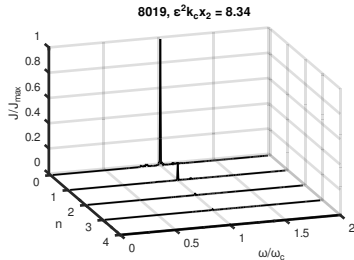
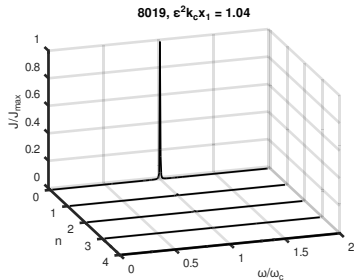
We need new experiments with better resolution of transversal modes!



Design of new experiments at MARINTEK (now renamed SINTEF Ocean)

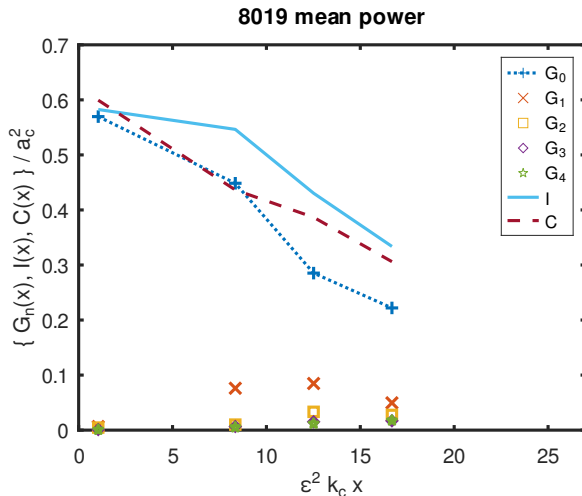


Run	T [s]	H [m]	ϵ	$\Delta\mu$	$\epsilon^2 k_c x_4$	$\frac{\omega_p(x_4)}{\omega_c}$	$\frac{\omega_{p,c}(x_4)}{\omega_c}$	$\frac{\omega_m(x_4)}{\omega_c}$	$\frac{\omega_{m,c}(x_4)}{\omega_c}$
8000	1	0.05	0.1	0.074	4.9	1	1	1	1
8001	1	0.08	0.16	0.074	13	1	0.85	0.99	0.97
8002	1	0.1	0.2	0.074	20	0.81	0.81	0.92	0.91
8003	1	0.13	0.26	0.074	33	0.73	0.75	0.88	0.87
8007	0.82	0.04	0.12	0.05	10	1	1	1	0.99
8008	0.82	0.06	0.18	0.05	23	0.82	0.83	0.92	0.92
8009	1.24	0.11	0.14	0.11	6.5	1	1	1	1
8010	1.24	0.15	0.2	0.11	12	1	1	1	1
8011	1.24	0.19	0.25	0.11	19	0.8	0.8	0.94	0.92
8012	1.24	0.24	0.31	0.11	31	1	1	0.95	0.91
8013	1	0.16	0.32	0.074	50	0.73	0.73	0.87	0.86
8014	0.82	0.03	0.09	0.05	5.8	1	1	1	1
8015	0.82	0.04	0.12	0.05	10	1	0.89	1	1
8016	0.82	0.05	0.15	0.05	16	0.85	0.85	0.97	0.95
8017	0.82	0.08	0.24	0.05	41	0.74	0.74	0.88	0.87
8018	1	0.03	0.06	0.074	2.3	1	1	1	1
8019	1	0.08	0.16	0.074	17	0.85	0.85	0.96	0.94
8020	1.43	0.15	0.15	0.15	6.9	1	1	1	1
8021	1.43	0.2	0.2	0.15	12	1	1	1	1
8022	1.43	0.25	0.25	0.15	19	1	1	0.96	0.94
8023	1.43	0.31	0.31	0.15	29	1	1	0.96	0.95
8024	1.43	0.225	0.22	0.15	15	1	1	1	0.99
8025	1.43	0.31	0.31	0.15	29	1	1	0.96	0.96
8026	1	0.1	0.2	0.074	26	0.8	0.8	0.9	0.9



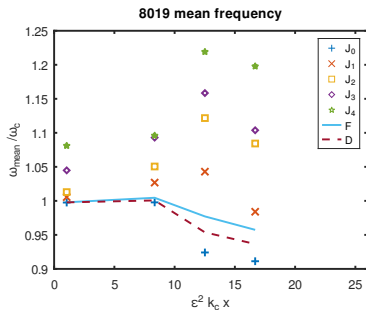
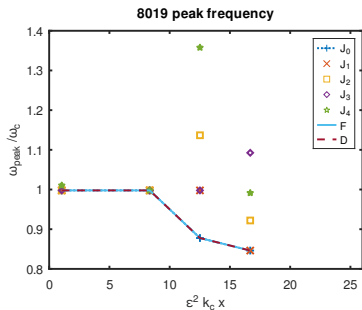
Frequency power spectrum $J_n(x, \omega)$ for each transversal mode number n at each station x , normalized peak value 1 at each station x .





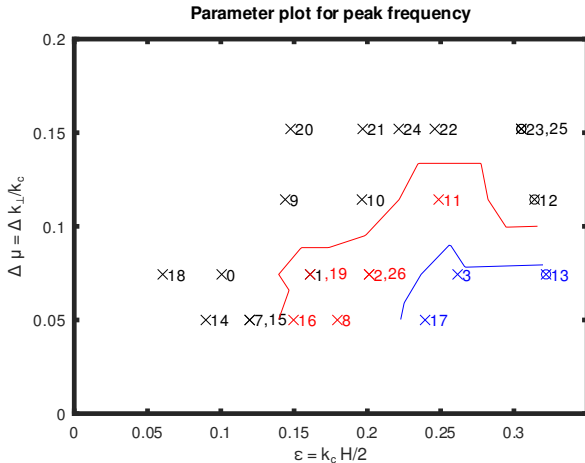
Mean power along the tank for each transversal mode $G_n(x)$, the total mean power $I(x)$, and the mean power along the centerline of the tank $C(x)$.





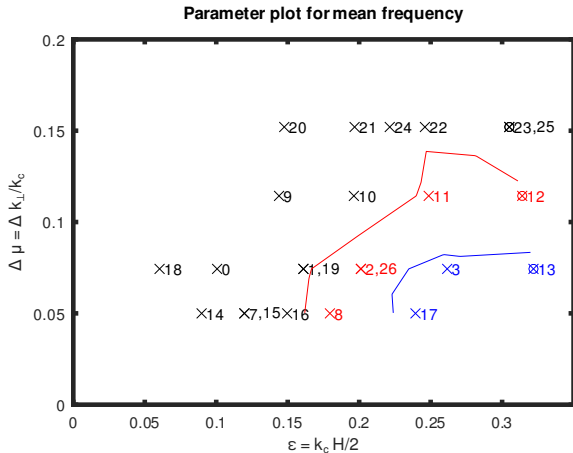
Peak and mean frequencies along the tank, $\omega_{\text{peak},n}(x)$ and $\omega_{\text{mean},n}(x)$ for each transversal mode $J_n(x, \omega)$, $\omega_{\text{peak}}(x)$ and $\omega_{\text{mean}}(x)$ for the combined measurements across the tank $F(x, \omega)$, $\omega_{\text{peak},c}(x)$ and $\omega_{\text{mean},c}(x)$ for the centerline of the tank $D(x, \omega)$.





Parametric plot for peak frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns. Run numbers reduced by 8000. Colors: black, no downshift $\Delta\omega_{\text{peak}}/\omega_c > 0.9$; red, small downshift $0.9 \geq \Delta\omega_{\text{peak}}/\omega_c > 0.76$; blue, clear downshift $\Delta\omega_{\text{peak}}/\omega_c \leq 0.76$.





Parametric plot for mean frequency downshift. Runs without, $\times$, and with, $\otimes$, clear visual observation of type II horseshoe patterns.

Run numbers reduced by 8000. Colors: black, no downshift

$\Delta\omega_{\text{mean}}/\omega_c > 0.95$; red, small downshift

$0.95 \geq \Delta\omega_{\text{mean}}/\omega_c > 0.89$; blue, clear downshift

$\Delta\omega_{\text{mean}}/\omega_c \leq 0.89$.

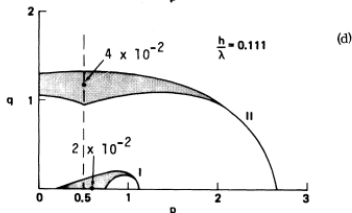
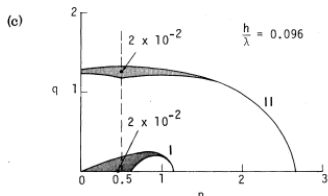
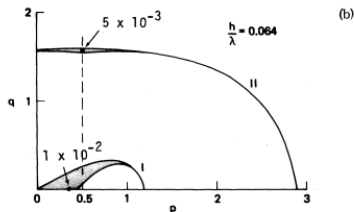
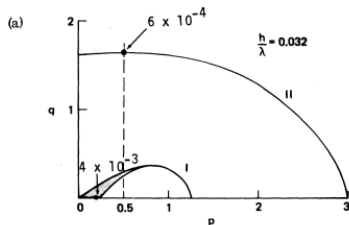


Downshift occurs when transversal type-I instabilities are enabled — McLean et al. (1981)

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- We only observe recurrence with downshift when transversal type-I instabilities are enabled : $q = \mu = k_{\perp}/k_c < 0.15$
- Type-II instabilities do not influence recurrence with downshift
- Dissipation / non-conservative effects less important for the downshift than previously believed?

