

CONVERGENCE OF A FULLY DISCRETE FINITE DIFFERENCE SCHEME FOR THE KORTEWEG–DE VRIES EQUATION

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ABSTRACT. We prove convergence of a fully discrete finite difference scheme for the Korteweg–de Vries equation. Both the decaying case on the full line and the periodic case are considered. If the initial data is of high regularity, $u(x, 0) \in H^6(\mathbb{R})$ the scheme is shown to converge to a classical solution on the whole line, and if the regularity of the initial data is smaller, $u(x, 0) \in L^2(\mathbb{R})$, then the scheme converges in $L^2(0, T, L^2_{\text{loc}}(\mathbb{R}))$ to a weak solution.

1. INTRODUCTION

`sec:intro`

The Korteweg–de Vries (KdV among friends and foes) equation, which reads

`eq:kdv0`

$$(1.1) \quad u_t + uu_x + u_{xxx} = 0,$$

has been studied extensively since its first analysis in 1895 by Korteweg and de Vries. Apart from applications as a model for shallow water waves, the KdV equation has maintained a pivotal role in several branches of mathematics. We here focus on the derivation of convergent numerical methods for the initial value problem where the equation (1.1) is augmented by initial data $u|_{t=0} = u_0$. The problem of analyzing convergent numerical schemes is of course intimately connected with the mathematical properties of the Cauchy problem for the KdV equation, which has undergone a tremendous development the last two decades, see [32, 22]. We will not be able to discuss that literature here, but only refer to the parts that are pertinent to the current paper.

An interesting fact, and rarely referred to in the current literature, is that the first mathematical proof of existence and uniqueness of solutions of the KdV equation, was accomplished by Sjöberg [30] in 1970, using a finite difference approximation very much in the spirit considered here. His proof is valid for initial data that are periodic and with square integrable third derivative, that is, $u_0(x+1) = u_0(x)$ for $x \in [0, 1]$ and $u_0''' \in L^2([0, 1])$. Sjöberg's uniqueness proof still is the standard one, using the Gronwall inequality. His approach is based on a semi-discrete approximation where one discretizes the spatial variable, thereby reducing the equation to a system of ordinary differential equations. However, we stress that for numerical computations also this set of ordinary differential equations will have to be discretized in order to be solved. Thus in order to have a completely satisfactory numerical method, one seeks a fully discrete scheme that reduces the actual computation to a solution of a finite set of algebraic equations. This is accomplished in the present paper, both in the periodic case and on the full line.

There has been a number of papers involving the numerical computation of solutions of the Cauchy problem, starting with the landmark paper by Zabusky and

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Kruskal [38], where they discovered the permanence of solitons (the term “solitons” being coined in the same paper) for the KdV equation using numerical techniques. However, we will here focus on papers that discuss numerical methods *per se*.

A popular numerical approach has been the application of various spectral methods. Little is known rigorously about the convergence of these methods. For a survey and a comparison, see [25]. See also [20]. Multisymplectic schemes have been studied in [4] (see also references therein). There exist convergence proofs for finite element methods for the KdV equation, see [37, 3, 5, 6]. However, the resulting schemes tend to be quite different from finite difference schemes derived *ab initio*.

The numerical computation of solutions of the KdV equation is rather capricious. Two competing effects are involved, namely the nonlinear convective term uu_x , which in the context of the Burgers equation $u_t + uu_x = 0$ yields infinite gradients in finite time even for smooth data, and the linear dispersive term u_{xxx} , which in the Airy equation $u_t + u_{xxx} = 0$ produces hard-to-compute dispersive waves, and these two effects combined makes it difficult to obtain accurate and fast numerical methods. Indeed, any initial data for the Burgers equation that is decreasing in a small neighborhood, will develop infinite gradients in finite time, while the Airy equation preserves all Sobolov norms while creating many oscillatory waves. Most finite difference schemes will consist of a sum of two terms, one discretizing the convective term and one discretizing the dispersive term. These two effects will have to balance each other, as it is known that the KdV equation itself keeps the Sobolov norm H^s bounded; from [7] we know that if $u_0 \in H^s(\mathbb{R})$, with $s \geq 3$, then the solution satisfies $\|u(t)\|_{H^s(\mathbb{R})} \leq C_{T,u_0}$ for $t \in [0, T]$. This dichotomy between these two effects is brought to the forefront in the method of operator splitting. Here the two equations, the Burgers equations and the Airy equation, are solved sequentially for a small time step. This procedure is iterated, and as the time step converges to zero, the approximation converges to the actual solution. In the KdV context operator splitting was introduced by Tappert [33], a Lax–Wendroff theorem was proved in [13], and convergence of the operator splitting technique proved in [14, 17, 15, 16]. Our approach here is a finite difference method which can also be viewed as an operator splitting method.

Recently, a semi-discrete scheme for the generalized KdV equation was shown to converge in L^2_{loc} for initial data in L^2 [2]. We also analyze the scheme presented here, which in contrast to the scheme in [2], does not involve an explicit fourth order stabilizing term, and show similar convergence properties for non-smooth initial data.

The rest of this paper is organized as follows: In Section 2 we present the necessary notation and define the numerical scheme. In Section 3 we show the convergence of the scheme for initial data in $H^3(\mathbb{R})$, while in Section 4 we show the convergence to a weak solution if the initial data is in $L^2(\mathbb{R})$. In Section 5 we exhibit some numerical experiments showing the convergence.

sec:scheme

2. THE SCHEME

We start by introducing the necessary notation. Derivatives will be approximated by finite differences, and the basic quantities are as follows. For any function $p: \mathbb{R} \rightarrow \mathbb{R}$ we set

$$D_{\pm}p(x) = \pm \frac{1}{\Delta x} (p(x \pm \Delta x) - p(x)), \text{ and } D = \frac{1}{2} (D_+ + D_-)$$

for some (small) positive number Δx . If we introduce the average

$$\bar{p}(x) = \frac{1}{2} (p(x + \Delta x) + p(x - \Delta x)),$$

we find the Leibniz rule as

$$D(pq) = \bar{p}Dq + \bar{q}Dp.$$

We discretize the real axis using Δx and set $x_j = j\Delta x$ for $j \in \mathbb{Z}$. For a given function p we define $p_j = p(x_j)$. We will consider functions in ℓ^2 with the usual inner product and norm

$$(p, q) = \Delta x \sum_{j \in \mathbb{Z}} p_j q_j, \quad \|p\| = (p, p)^{1/2}, \quad p, q \in \ell^2.$$

In the periodic case the sum over $\mathbb{Z}$ is replaced by a finite sum $j = 0, \dots, N$. Observe that

$$\|p\|_\infty = \max_{j \in \mathbb{Z}} |p_j| \leq \frac{1}{\Delta x^{1/2}} \|p\|.$$

The various difference operators enjoy the following properties:

$$(p, D_\pm q) = -(D_\mp p, q), \quad (p, Dq) = -(Dp, q), \quad p, q \in \ell^2.$$

Further useful properties include

eq:useful

$$\begin{aligned}
 (u, D_+^2 D_- u) &= \frac{1}{2} (u, D_- D_+^2 u) - \frac{1}{2} (u, D_+ D_-^2 u) \\
 &= \frac{1}{2} (u, (D_- D_+^2 - D_+ D_-^2) u) \\
 &= \frac{1}{2} (u, D_- D_+ (D_- - D_+) u) \\
 &= \frac{\Delta x}{2} \|D_+ D_- u\|^2
 \end{aligned}$$

(2.1)

since $(u, D_+ D_-^2 u) = -(u, D_- D_+^2 u)$ (because $(u, v) = (v, u)$) in the first line, and

$$D_- - D_+ = -\Delta x D_+ D_- = -\Delta x D_- D_+.$$

Similarly we find that

$$(u, D_+ D_-^2 u) = -\frac{\Delta x}{2} \|D_+ D_- u\|^2. \text{ [bruker vi denne?]}$$

We also need to discretize in the time direction. Introduce (a small) time step $\Delta t > 0$, and use the notation

$$D_+^t p(t) = \frac{1}{\Delta t} (p(t + \Delta t) - p(t)),$$

for any function $p: [0, T] \rightarrow \mathbb{R}$. Write $t^n = n\Delta t$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A fully discrete grid function is a function $u_{\Delta x}: \Delta t \mathbb{N}_0 \rightarrow \mathbb{R}^\mathbb{Z}$, and we write $u_{\Delta x}(x_j, t^n) = u_j^n$. (A CFL condition will enforce a relationship between Δx and Δt , and hence we only use Δx in the notation.)

The KdV partial differential equation reads

eq:kdv

$$(2.2) \quad u_t + uu_x + u_{xxx} = 0.$$

We will study the initial value problem for (2.2), that is, we are given a function u_0 such that $u|_{t=0} = u_0$. Both the periodic case and the full line will be treated, and we will focus on the full line. The main idea is that $u(x_j, t^n) \approx u_{\Delta x}(x_j, t^n) = u_j^n$ for $\Delta x, \Delta t \approx 0$.

We propose the following implicit scheme

eq:sjobergexp1

$$(2.3) \quad u_j^{n+1} = \bar{u}_j^n - \Delta t \bar{u}_j^n D u_j^n - \Delta t D_+^2 D_- u_j^{n+1}, \quad n \in \mathbb{N}, j \in \mathbb{Z}.$$

For the initial data we have

$$u_j^0 = u_0(x_j), \quad j \in \mathbb{Z}.$$

Remark 2.1. *This scheme can be reformulated as an operator splitting scheme as follows. Set*

$$u_j^{n+1/2} = \frac{1}{2} (u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x} \left(\frac{1}{2} (u_{j+1}^n)^2 - \frac{1}{2} (u_{j-1}^n)^2 \right),$$

i.e., $u^{n+1/2}$ is solution operator of the Lax-Friedrichs scheme for Burgers' equation, applied to u^n . Then

$$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} = -D_+^2 D_- u^{n+1},$$

i.e., u^{n+1} is the approximate solution operator of a first order implicit scheme for Airy's equation $u_t + u_{xxx} = 0$. If we write these two approximate solution operators as $S_{\Delta t}^B$ and $S_{\Delta t}^A$ respectively, the update formula (2.3) reads

$$u^{n+1} = (S_{\Delta t}^A \circ S_{\Delta t}^B) u^n.$$

The convergence of this type of operator splitting using exact solution operators have been shown in [14, 17], with severe restrictions on the initial data. The results in this paper can be viewed as a convergence result for operator splitting using approximate operators with less restrictions on the initial data, but with specified ratios between the temporal and spatial discretizations (CFL-like conditions).

sec:smoothdata

3. CONVERGENCE FOR SMOOTH INITIAL DATA

To show that the implicit scheme can be solved with respect to u_j^{n+1} , we proceed as follows: Write the scheme as

$$(1 + \Delta t D_+^2 D_-) u_j^{n+1} = \bar{u}_j^n - \Delta t \bar{u}_j^n D u_j^n$$

and hence

$$\begin{aligned} ((1 + \Delta t D_+^2 D_-) u^{n+1}, u^{n+1}) &= \|u^{n+1}\|^2 + \Delta t (D_+^2 D_- u^{n+1}, u^{n+1}) \\ &= \|u^{n+1}\|^2 + \frac{1}{2} \Delta t \Delta x \|D_+ D_- u^{n+1}\|^2 \\ &\geq \|u^{n+1}\|^2, \end{aligned}$$

thus

$$\|u^{n+1}\| \leq \|(1 + \Delta t D_+^2 D_-) u^{n+1}\| = \|\bar{u}^n - \Delta t \bar{u}^n D u^n\|.$$

The fundamental stability lemma reads as follows.

lem:stability1

Lemma 3.1. *Let u_j^n be a solution of the difference scheme (2.3). Then the following estimate holds*

eq:L2

$$(3.1) \quad \|u^{n+1}\|^2 + \Delta t \Delta x^{1/2} \left(\Delta x \lambda \|D_+^2 D_- u^{n+1}\|^2 + \Delta x^{1/2} \|D_+ D_- u^{n+1}\|^2 + \frac{\delta}{2\lambda} \|D u^n\|^2 \right) \leq \|u^n\|^2,$$

provided the CFL condition

eq:CFL1a

$$(3.2) \quad \lambda \|u^n\| \left(\frac{1}{3} + \frac{1}{2} \lambda \|u^n\| \right) < \frac{1-\delta}{2}, \quad \delta \in (0, 1),$$

holds where $\lambda = \Delta t / \Delta x^{3/2}$.

Proof. For the moment we drop the indices j and n from our notation, and use the notation u for u_j^n where j and n are fixed. We first study the ‘‘Burgers’’ term $\Delta t \bar{u} D u$. Let u be a gridfunction and set

$$w = \bar{u} - \Delta t \bar{u} D u.$$

If the timestep Δt satisfies (3.2) then we have the following “cell entropy” inequality

$$\frac{1}{2}w^2 \leq \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 - \delta \frac{\Delta x^2}{2}(Du)^2, \quad \delta \in (0, 1).$$

To prove this we multiply (3) by $\bar{u}$ to find

$$\begin{aligned} \frac{1}{2}w^2 &= \frac{1}{2}\bar{u}^2 - \Delta t \frac{1}{2}\bar{u}Du^2 + \frac{1}{2}(w - \bar{u})^2 \\ &= \frac{1}{2}\bar{u}^2 - \Delta t \frac{1}{2}\bar{u}Du^2 + \frac{1}{2}\Delta t^2 \bar{u}^2 (Du)^2 + \frac{1}{2}(\bar{u}^2 - u^2). \end{aligned}$$

Now we have that

$$\frac{1}{4}(a+b)(a^2-b^2) = \frac{1}{3}(a^3-b^3) - \frac{1}{12}(a-b)^3,$$

and

$$\frac{1}{4}(a+b)^2 - \frac{1}{2}(a^2+b^2) = -\frac{1}{4}(a-b)^2.$$

For a gridfunction, this implies

$$\begin{aligned} \bar{u}Du^2 &= \frac{2}{3}Du^3 - \frac{2\Delta x^2}{3}(Du)^3, \\ \bar{u}^2 - u^2 &= -\Delta x^2 (Du)^2. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2}w^2 &= \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 + \frac{1}{2}\Delta t^2 \bar{u}^2 (Du)^2 + \frac{\Delta t \Delta x^2}{3}(Du)^3 - \frac{\Delta x^2}{2}(Du)^2 \\ &= \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 - \delta \frac{\Delta x^2}{2}(Du)^2 \\ &\quad + \Delta x^2 (Du)^2 \left(\frac{\lambda}{3} \Delta x^{3/2} Du + \frac{1}{2} \Delta x \lambda^2 \bar{u}^2 - \frac{1-\delta}{2} \right) \\ &\leq \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 - \delta \frac{\Delta x^2}{2}(Du)^2 \\ &\quad + \Delta x^2 (Du)^2 \left(\frac{\lambda}{3} \Delta x^{1/2} \|u\|_\infty + \frac{1}{2} \Delta x \lambda^2 \bar{u}^2 - \frac{1-\delta}{2} \right) \\ &\leq \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 - \delta \frac{\Delta x^2}{2}(Du)^2 \\ &\quad + \underbrace{\Delta x^2 (Du)^2 \left(\frac{\lambda}{3} \|u\| + \frac{1}{2} \lambda^2 \|u\|^2 - \frac{1-\delta}{2} \right)}_A \\ &\leq \frac{1}{2}\bar{u}^2 - \frac{\Delta t}{3}Du^3 - \delta \frac{\Delta x^2}{2}(Du)^2, \quad \delta \in (0, 1), \end{aligned}$$

where we have employed that $A < 0$ since λ satisfies the CFL condition (3.2). Estimate (3) follows.

Summing (3) over j we get

eq:L2burg

$$(3.3) \quad \|w\|^2 + \delta \frac{\Delta x^2}{2} \|Du\|^2 \leq \|u\|^2.$$

Next we study the full difference scheme by adding the “Airy term” $\Delta t D_+^2 D_- u_j^{n+1}$. Thus the full difference scheme (2.3) can be written

$$v = w - \Delta t D_+^2 D_- v.$$

Writing this as $w = v + \Delta t D_+^2 D_- v$, we square it and sum over j to get

$$\begin{aligned} \text{eq:345} \quad (3.4) \quad \|w\|^2 &= \|v\|^2 + 2\Delta t(v, D_+^2 D_- v) + \Delta t^2 \|D_+^2 D_- v\|^2 \\ &= \|v\|^2 + \Delta t \Delta x \|D_- D_+ v\|^2 + \Delta t^2 \|D_+^2 D_- v\|^2, \end{aligned}$$

using the identity (2.1).

For the function u^n this means that

$$\begin{aligned} \text{eq:unL2*} \quad (3.5) \quad \|w\|^2 &= \|u^{n+1}\|^2 \\ &\quad + \Delta t \Delta x^{1/2} \left(\Delta x \lambda \|D_+^2 D_- u^{n+1}\|^2 + \Delta x^{1/2} \|D_+ D_- u^{n+1}\|^2 \right) \\ &\leq \|u^n\|^2 - \delta \frac{\Delta x^2}{2} \|Du^n\|^2, \end{aligned}$$

using (3.3). This implies

$$\begin{aligned} \text{eq:unL2} \quad (3.6) \quad \|u^{n+1}\|^2 &+ \Delta t \Delta x^{1/2} \left(\Delta x \lambda \|D_+^2 D_- u^{n+1}\|^2 + \Delta x^{1/2} \|D_+ D_- u^{n+1}\|^2 + \frac{\delta}{2\lambda} \|Du^n\|^2 \right) \\ &\leq \|u^n\|^2. \end{aligned}$$

□

Next, we consider what corresponds to the finite difference scheme satisfied by the time derivative of the original scheme.

Lemma 3.2. *Let u_j^n be a solution of the difference scheme (2.3). Then the following estimate holds*

$$\begin{aligned} \text{eq:vnest_lemma} \quad (3.7) \quad &\|D_+^t u^n\|^2 \\ &+ \Delta t^2 \|D_+^2 D_- D_+^t u^n\|^2 + \Delta t \Delta x \|D_+ D_- D_+^t u^n\|^2 + \tilde{\delta} \frac{\Delta x^2}{2} \|DD_+^t u^{n-1}\|^2 \\ &\leq \frac{1}{2} \|D_+^t u^{n-1}\|^2 \left(1 + \frac{17}{8} \Delta t \|Du^n\|_\infty \right). \end{aligned}$$

provided Δt is chosen such that

$$\text{eq:cf12_lemma} \quad (3.8) \quad 6 \|u_0\|^2 \lambda^2 + 2 \|u_0\| \lambda < \frac{1 - \tilde{\delta}}{4}, \quad \tilde{\delta} \in (0, 1).$$

Proof. Introduce

$$\alpha^n = D_+^t u^{n-1} = \frac{1}{\Delta t} (u^n - u^{n-1}), \quad n \in \mathbb{N}_0.$$

Using (2.3) we see that this gridfunction satisfies

$$\begin{aligned} \text{eq:vndef} \quad (3.9) \quad \alpha^{n+1} &= \bar{\alpha}^n - \Delta t (\bar{\alpha}^n Du^n + \bar{u}^n D \alpha^n) + \Delta t^2 \bar{\alpha}^n D \alpha^n - \Delta t D_+^2 D_- \alpha^{n+1} \\ &= \bar{\alpha}^n - \Delta t D(u^n \alpha^n) + \Delta t^2 \bar{\alpha}^n D \alpha^n - \Delta t D_+^2 D_- \alpha^{n+1}, \quad n \in \mathbb{N}. \end{aligned}$$

Initial data are given as follows

$$\begin{aligned} \text{eq:vndef_init} \quad (3.10) \quad \alpha^0 &= -\bar{u}^0 Du^0 - D_+^2 D_- u^0, \\ \alpha^1 &= \frac{\Delta x^{3/2}}{2\lambda} D_+ D_- u^0 - \bar{u}^0 Du^0 - D_+^2 D_- u^1, \end{aligned}$$

where $u^1 = \bar{u}^0 - \Delta t \bar{u}^0 Du^0 - \Delta t D_+^2 D_- u^1$ from the definition (2.3) of the finite difference scheme, and we set $u^{-1} = 0$. Introduce

$$\text{eq:vndef10} \quad (3.11) \quad \beta = \bar{\alpha} - \Delta t D(u\alpha) + \frac{\Delta t^2}{2} D\alpha^2,$$

which means that (3.9) can be written as

$$\boxed{\text{eq:vndef1}} \quad (3.12) \quad \alpha^{n+1} = \beta - \Delta t D_+^2 D_- \alpha^{n+1}.$$

We proceed as before and square (3.11) to find

$$\begin{aligned} \frac{1}{2}\beta^2 &= \frac{1}{2}\bar{\alpha}^2 + \frac{\Delta t^2}{2} \left(D(u\alpha) - \frac{\Delta t}{2} D\alpha^2 \right)^2 \\ &\quad - \Delta t (\bar{u}\bar{\alpha}D\alpha + \bar{\alpha}^2 Du) + \frac{\Delta t^2}{2} \bar{\alpha}^2 D\alpha - \frac{\Delta x^2}{4} (D\alpha)^2. \end{aligned}$$

We have that

$$\begin{aligned} \frac{1}{2} \left(D(u\alpha) - \frac{\Delta t}{2} D\alpha^2 \right)^2 &\leq (D(u\alpha))^2 + \Delta t^2 \bar{\alpha}^2 (D\alpha)^2 \\ &\leq 2\bar{u}^2 (D\alpha)^2 + 2\bar{\alpha}^2 (Du)^2 + \Delta t^2 \bar{\alpha}^2 (D\alpha)^2, \\ \bar{\alpha}^2 D\alpha &= \frac{1}{3} D\alpha^3 - \frac{\Delta x^2}{3} (D\alpha)^3, \\ \bar{u}\bar{\alpha}D\alpha + \bar{\alpha}^2 Du &= \frac{1}{2} D(u\alpha^2) + \frac{1}{2} \bar{\alpha}^2 Du - \frac{\Delta x^2}{2} (D\alpha)^2 Du. \end{aligned}$$

Using this

$$\begin{aligned} \frac{1}{2}\beta^2 &\leq \frac{1}{2}\bar{\alpha}^2 - \frac{\Delta t}{2} D \left(u\alpha^2 - \frac{\Delta t}{3} \alpha^3 \right) - \frac{\Delta t}{2} \bar{\alpha}^2 Du + \frac{\Delta t \Delta x^2}{2} (D\alpha)^2 Du \\ \boxed{\text{eq:est1}} \quad (3.13) \quad &+ \Delta t^2 \left(2\bar{u}^2 (D\alpha)^2 + 2\bar{\alpha}^2 (Du)^2 + \Delta t^2 \bar{\alpha}^2 (D\alpha)^2 - \frac{\Delta x^2}{6} (D\alpha)^3 \right) \\ &- \frac{\Delta x^2}{4} (D\alpha)^2. \end{aligned}$$

Now we must balance the positive terms with $\Delta x^2 (D\alpha)^2$. To this end we compute

$$\begin{aligned} \Delta x^2 (D\alpha)^2 &\leq \bar{\alpha}^2, \\ \Delta t^2 \bar{\alpha}^2 (Du)^2 &\leq \lambda \Delta x^{1/2} \Delta t \|u^n\|_\infty \bar{\alpha}^2 |Du^n|, \\ \Delta t^2 \bar{\alpha}^2 &\leq 2 \|u^n\|_\infty^2 + 2 \|u^{n-1}\|_\infty^2, \\ \Delta x^2 |D\alpha| &\leq \frac{1}{\lambda \Delta x^{1/2}} (\|u^n\|_\infty + \|u^{n-1}\|_\infty). \end{aligned}$$

Using these in (3.13) we find

$$\begin{aligned} \frac{1}{2}\beta^2 &\leq \frac{1}{2}\bar{\alpha}^2 - \frac{\Delta t}{2} D \left(u\alpha^2 - \frac{\Delta t}{3} \alpha^3 \right) + \left(\frac{\Delta t}{2} + \frac{\Delta t}{2} \right) \bar{\alpha}^2 |Du| + 2\Delta t^2 \bar{\alpha}^2 (Du)^2 \\ &\quad + \lambda^2 \Delta x^3 (D\alpha)^2 \left(2\bar{u}^2 + 2 \|u^n\|_\infty^2 + 2 \|u^{n-1}\|_\infty^2 + \frac{1}{6\lambda \Delta x^{1/2}} (\|u^n\|_\infty + \|u^{n-1}\|_\infty) \right) \\ &\quad - \frac{\Delta x^2}{4} (D\alpha)^2 \\ &\leq \frac{1}{2}\bar{\alpha}^2 - \frac{\Delta t}{2} D \left(u\alpha^2 - \frac{\Delta t}{3} \alpha^3 \right) + \Delta t \left(1 + \lambda \Delta x^{1/2} \|u^n\|_\infty \right) \bar{\alpha}^2 |Du^n| \\ &\quad + \lambda^2 \Delta x^2 (D\alpha)^2 \left(4\Delta x \left(\|u^n\|_\infty^2 + 2 \|u^{n-1}\|_\infty^2 \right) + \frac{\Delta x^{1/2}}{6\lambda} (\|u^n\|_\infty + \|u^{n-1}\|_\infty) \right) \\ &\quad - \frac{\Delta x^2}{4} (D\alpha)^2 \\ &\leq \frac{1}{2}\bar{\alpha}^2 - \frac{\Delta t}{2} D \left(u\alpha^2 - \frac{\Delta t}{3} \alpha^3 \right) + \Delta t (1 + \lambda \|u_0\|) \bar{\alpha}^2 |Du^n| \end{aligned}$$

$$\begin{aligned}
& + \lambda^2 \Delta x^2 (D\alpha)^2 \left(6 \|u_0\|^2 + \frac{1}{3\lambda} \|u_0\| - \frac{1-\tilde{\delta}}{4\lambda^2} \right) - \tilde{\delta} \frac{\Delta x^2}{4} (D\alpha)^2 \\
& \leq \frac{1}{2} \overline{\alpha^2} - \frac{\Delta t}{2} D \left(u\alpha^2 - \frac{\Delta t}{3} \alpha^3 \right) + \Delta t \frac{17-\tilde{\delta}}{16} \overline{\alpha^2} |Du^n| \\
& \quad + \Delta x^2 (D\alpha)^2 \left(6 \|u_0\|^2 \lambda^2 + 4 \|u_0\| \lambda - \frac{1-\tilde{\delta}}{4} \right) - \tilde{\delta} \frac{\Delta x^2}{4} (D\alpha)^2, \quad \tilde{\delta} \in (0, 1).
\end{aligned}$$

Now we choose Δt according to (3.8), multiply with Δx and sum over j to obtain

$$\boxed{\text{eq:vnest1}} \quad (3.14) \quad \frac{1}{2} \|\beta\|^2 + \tilde{\delta} \frac{\Delta x^2}{4} \|D\alpha\|^2 \leq \frac{1}{2} \|\alpha\|^2 + \frac{17-\tilde{\delta}}{16} \Delta t \|Du^n\|_\infty \|\alpha\|^2.$$

Writing equation (3.12) as

$$\beta^2 = (\alpha^{n+1} + \Delta t D_+^2 D_- \alpha^{n+1})^2,$$

we find

$$\begin{aligned}
\|\beta\|^2 &= \|\alpha^{n+1}\|^2 + 2\Delta t (\alpha^{n+1}, D_+^2 D_- \alpha^{n+1}) + \Delta t^2 \|D_+^2 D_- \alpha^{n+1}\|^2 \\
&= \|\alpha^{n+1}\|^2 + \Delta t \Delta x \|D_+ D_- \alpha^{n+1}\|^2 + \Delta t^2 \|D_+^2 D_- \alpha^{n+1}\|^2.
\end{aligned}$$

Combining this with (3.14) we find

$$\boxed{\text{eq:vnest}} \quad (3.15) \quad \|\alpha^{n+1}\|^2 + \Delta t \Delta x \|D_+ D_- \alpha^{n+1}\|^2 + \Delta t^2 \|D_+^2 D_- \alpha^{n+1}\|^2 \\
+ \tilde{\delta} \frac{\Delta x^2}{2} \|D\alpha^n\|^2 \leq \frac{1}{2} \left(1 + \frac{17}{8} \Delta t \|Du^n\|_\infty \right) \|\alpha^n\|^2.$$

□

At this point we recall the inequality (cf. Lemma A.2):

$$\boxed{\text{eq:sjo1}} \quad (3.16) \quad \|Du\|_\infty \leq \epsilon \|D_+^2 D_- u\| + c(\epsilon) \|u\|,$$

where ϵ is any constant, and $c(\epsilon)$ is another constant depending on ϵ .

The definition of u^n , (2.3) can be rewritten

$$\boxed{\text{eq:sjobergexp2}} \quad (3.17) \quad \alpha^{n+1} = D_+^t u^n = \frac{1}{2\mu} D_+ D_- u^n - \bar{u}^n Du^n - D_+^2 D_- u^{n+1},$$

where $\mu = \Delta t / \Delta x^2 = \lambda / \Delta x^{1/2}$. Therefore (using Lemma 3.1 in the second estimate)

$$\begin{aligned}
\|D_+^2 D_- u^{n+1}\| &\leq \|\alpha^{n+1}\| + \|\bar{u}^n Du^n\| + \frac{1}{2\mu} \|D_+ D_- u^n\| \\
&\leq \|\alpha^{n+1}\| + \|Du^n\|_\infty \|u_0\| + \frac{1}{2\mu} (\epsilon \|D_+^2 D_- u^n\| + c(\epsilon) \|u_0\|) \\
&\leq \|\alpha^{n+1}\| + \|u_0\| (\epsilon \|D_+^2 D_- u^n\| + c(\epsilon) \|u_0\|) \\
&\quad + \frac{1}{2\mu} \epsilon \|D_+^2 D_- u^n\| + \frac{1}{2\mu} c(\epsilon) \|u_0\| \\
&\leq \|\alpha^{n+1}\| + \left(\epsilon \|u_0\| + \frac{\Delta x^{1/2}}{2\lambda} \epsilon \right) \|D_+^2 D_- u^n\| \\
&\quad + \underbrace{c(\epsilon) \|u_0\|^2 + \frac{\Delta x^{1/2}}{2\lambda} c(\epsilon) \|u_0\|}_{A(\epsilon, \epsilon)} \\
&= \|\alpha^{n+1}\| + \frac{1}{2} \|D_+^2 D_- u^n\| + A \quad (\text{choosing } \epsilon \text{ and } \epsilon \text{ such that this holds})
\end{aligned}$$

$$\begin{aligned}
&= \|\alpha^{n+1}\| + \frac{1}{2} \|D_+^2 D_- (u^{n+1} - \Delta t \alpha^{n+1})\| + A \\
&\leq \|\alpha^{n+1}\| + \frac{1}{2} \|D_+^2 D_- u^{n+1}\| + \frac{1}{2} \Delta t \|D_+^2 D_- \alpha^{n+1}\| + A \\
&\leq \|\alpha^{n+1}\| + \frac{1}{2} \|D_+^2 D_- u^{n+1}\| + \frac{1}{\sqrt{2}} \|\alpha^n\| \left(1 + \frac{17}{8} \Delta t \|Du^n\|\right)^{1/2} + A \\
&\leq \|\alpha^{n+1}\| + \frac{1}{2} \|D_+^2 D_- u^{n+1}\| + \frac{1}{\sqrt{2}} \|\alpha^n\| \left(1 + \frac{17}{8} \lambda \|u_0\|\right)^{1/2} + A,
\end{aligned}$$

where we have used (3.15) to estimate $\Delta t \|D_+^2 D_- \alpha^{n+1}\|$. Hence

$$(3.18) \quad \|D_+^2 D_- u^{n+1}\| \leq c_0 + c_1 \|\alpha^{n+1}\| + c_2 \|\alpha^n\|,$$

for some constants c_0, c_1 and c_2 that are independent of Δx . Exploiting this and the inequality (cf. Lemma A.2) in (3.15), we get

$$\begin{aligned}
\|\alpha^{n+1}\|^2 &\leq \|\alpha^n\|^2 + \Delta t (\epsilon \|D_+^2 D_- u^n\| + c(\epsilon) \|u^n\|) \|\alpha^n\|^2 \\
&\leq \|\alpha^n\|^2 + C \Delta t (\epsilon (c_0 + c_1 \|\alpha^n\| + c_2 \|\alpha^{n-1}\|) + c(\epsilon) \|u_0\|) \|\alpha^n\|^2.
\end{aligned}$$

Since $\|u^n\|$ is bounded by $\|u_0\|$,

$$(3.19) \quad \|\alpha^{n+1}\|^2 \leq \|\alpha^n\|^2 + \Delta t \left(d_1 \|\alpha^n\|^2 + d_2 \left(\|\alpha^n\|^3 + \|\alpha^n\|^2 \|\alpha^{n-1}\| \right) \right),$$

for constants d_1 and d_2 which only depend on $\|u_0\|$ and λ . Set $a_n = \|\alpha^n\|^2$, so that

$$a_{n+1} \leq a_n + \Delta t \left(d_1 a_n + d_2 \left(a_n^{3/2} + a_n a_{n-1}^{1/2} \right) \right).$$

Now let $A = A(t)$ be the solution of the differential equation

$$\frac{dA}{dt} = d_1 A + 2d_2 A^{3/2}, \quad A(0) = a_0 > 0.$$

This solution has blow-up time

$$T^\infty = T^\infty = t_0 + \frac{2}{d_1} \ln \left(1 + \frac{d_1}{d_2 \sqrt{a_0}} \right).$$

Furthermore, for $t < T^\infty$, A is a convex function of t (since the second derivative clearly is non-negative). We now claim that for $n\Delta t < T^\infty$, we have

$$a_n \leq A(n\Delta t), \quad n \in \mathbb{N}_0.$$

This holds for $n = 0$ and $n = 1$ (recall (3.10)):

$$\begin{aligned}
\|\alpha^0\| &\leq \|u^0\| \|Du^0\|_\infty + \|D_+^2 D_- u^0\| \leq C, \\
\|\alpha^1\| &\leq \frac{2\Delta x}{\lambda} \|D_+ D_- u^0\| + \|\bar{u}^0 Du^0\| + \|D_+^2 D_- u^1\|.
\end{aligned}$$

Concerning the term $\|D_+^2 D_- u^1\|$ we find that

$$(I + \Delta t D_+^2 D_-) D_+^2 D_- u^1 = D_+^2 D_- (\bar{u}^0 - \Delta t \bar{u}^0 Du^0)$$

which implies

$$\begin{aligned}
\|D_+^2 D_- u^1\| &\leq \|D_+^2 D_- (\bar{u}^0 - \Delta t \bar{u}^0 Du^0)\| \\
&\leq \|D_+^2 D_- \bar{u}^0\| + \Delta t \|D_+^2 D_- (\bar{u}^0 Du^0)\| \leq C
\end{aligned}$$

by assumptions on u_0 . Thus

$$\|\alpha^0\| \leq C, \quad \|\alpha^1\| \leq C$$

for a redefined constant C .

Assuming that the claim holds for integers up to n , we get

$$a_{n+1} \leq A(n\Delta t)$$

$$\begin{aligned}
& + \Delta t \left(d_1 A(n\Delta t) \right. \\
& \quad \left. + d_2 \left(A(n\Delta t)^{3/2} + A(n\Delta t) A((n-1)\Delta t)^{1/2} \right) \right) \\
& \leq A(n\Delta t) + \Delta t \left(d_1 A(n\Delta t) + 2d_2 A(n\Delta t)^{3/2} \right) \\
& \leq A((n+1)\Delta t).
\end{aligned}$$

Hence, for $t \leq \bar{T} = T^\infty/2$, $\|\alpha^n\| \leq C$ for some constant independent of Δx .

Therefore, we can follow Sjöberg [30] to prove convergence of the scheme for $t < \bar{T}$. We reason as follows: Let $u_{\Delta x}(x, t)$ be the piecewise bilinear continuous interpolation such that

$$u_{\Delta x}(x_j, t_n) = u_j^n.$$

We have shown that the function $u_{\Delta x}$ satisfies the bounds

$$\boxed{\text{eq:udxL2}} \quad (3.20) \quad \|u_{\Delta x}(\cdot, t)\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})},$$

$$\boxed{\text{eq:udxH1}} \quad (3.21) \quad \|\partial_x u_{\Delta x}(\cdot, t)\|_{L^2(\mathbb{R})} \leq C,$$

$$\boxed{\text{eq:udxtL2}} \quad (3.22) \quad \|\partial_t u_{\Delta x}(\cdot, t)\|_{L^2(\mathbb{R})} \leq C,$$

$$\boxed{\text{eq:udxH3}} \quad (3.23) \quad \|\partial_{xxx} u_{\Delta x}(\cdot, t)\|_{L^2(\mathbb{R})} \leq C,$$

for $t \leq \bar{T}$ and for a constant C which is independent of Δx . The first three bounds have already been shown, to show the last bound notice that

$$\|D_+^2 D_- u^n\| \leq \|D_+^t u^n\| + \|\bar{u}^n\|_\infty \|Du^n\| \leq C.$$

The inequality (3.23) follows readily from this.

The bound on $\partial_t u_{\Delta x}$ also implies that $u_{\Delta x} \in \text{Lip}([0, \bar{T}], L^2(\mathbb{R}))$. Then an application of the Arzelà-Ascoli theorem using (3.20) shows that the set $\{u_{\Delta x}\}_{\Delta x > 0}$ is sequentially compact in $C([0, \bar{T}], L^2(\mathbb{R}))$, such that there exist a sequence $\{u_{\Delta x_j}\}_{j \in \mathbb{N}}$ which converges uniformly in $C([0, \bar{T}]; L^2(\mathbb{R}))$ to some function u . Then we can apply the Lax-Wendroff like result from [13] to conclude that u is a weak solution.

The bounds (3.21), (3.22) and (3.23) means that u is actually a strong solution such that (1.1) holds as an L^2 identity. Thus the limit u is the unique solution to the KdV equation taking the initial data u_0 .

Summing up, we have proved the following theorem:

thm:H3convergence

Theorem 3.3. *Assume that $u_0 \in H^3(\mathbb{R})$, then there exist a finite time $\bar{T}$, depending only on $\|u_0\|_{H^3(\mathbb{R})}$, such that for $t \leq \bar{T}$, the difference approximations defined by (2.3) converges uniformly in $C(\mathbb{R} \times [0, T])$ to the unique solution of the KdV equation (1.1).*

Remark 3.1. Helge: *Describe how the periodic case subsumes the full line (with small modifications).*

sec:L2

4. CONVERGENCE WITH L^2 INITIAL DATA

Let the function p be defined as $p = \hat{p} * \omega$, where

$$\hat{p}(x) = \max \{1, \min \{(1+x+R, 1+2R)\}\},$$

and ω is a symmetric positive function with integral one and support in $[-1, 1]$. We have that all the derivatives of p are bounded, to be concrete we assume that

$$\left| \frac{d^k}{dx^k} p(x) \right| \leq 2^{k-1}, \quad k = 1, 2, 3, \dots \text{ and for all } x.$$

We shall also use that

$$0 \leq \frac{d}{dx} p(x) \leq 1.$$

Since p is positive we can define the weighted inner product and corresponding norms by

$$(u, v)_p = (u, pv) = \Delta x \sum_i p_i u_i v_i, \quad \|u\|_p^2 = (u, u)_p,$$

where $p_i = p(x_i)$. Note that $\|u\|_p^2 \leq (1 + 2R) \|u\|^2$.

Using summation by parts, and writing u_{+1} for the sequence such that $(u_{+1})_j = u_{j+1}$, we have

$$\begin{aligned} (D_- D_+^2 u, u)_p &= (D_- D_+^2 u, up) \\ &= -(D_+^2 u, p D_+ u + u_{+1} D_+ p) \\ &= -(D_+ (D_+ u) D_+ u, p) - (D_+^2 u, u_{+1} D_+ p) \\ &= -\frac{1}{2} (D_+ (D_+ u)^2, p) + \frac{\Delta x}{2} ((D_+^2 u)^2, p) - (D_+^2 u, u_{+1} D_+ p) \\ &= \frac{1}{2} ((D_+^2 u)^2, D_- p) + \frac{\Delta x}{2} ((D_+^2 u)^2, p) + (D_+ u, D_- (u_{+1} D_+ p)) \\ &= \frac{1}{2} ((D_+^2 u)^2, D_- p) + \frac{\Delta x}{2} ((D_+^2 u)^2, p) + (D_+ u, D_- u_{+1} D_+ p + u D_- D_+ p) \\ &= \left((D_+^2 u)^2, \frac{1}{2} D_- p + D_+ p \right) + \frac{\Delta x}{2} ((D_+^2 u)^2, p) + (u D_+ u, D_- D_+ p) \\ &= \left((D_+^2 u)^2, \frac{1}{2} D_- p + D_+ p \right) + \frac{\Delta x}{2} ((D_+^2 u)^2, p) + \frac{1}{2} (u^2, D_- D_+ p) \\ &\quad - \frac{\Delta x}{2} ((D_+ u)^2, D_+ D_- p) \\ &= \left((D_+ u)^2, D_- p + \frac{1}{2} D_+ p \right) + \Delta x ((D_+^2 u)^2, p) - \frac{1}{2} (u^2, D_+ D_- p). \end{aligned}$$

So we have

eq:innerprodrule

$$(4.1) \quad \begin{aligned} (D_- D_+^2 u, u)_p &= \left((D_+ u)^2, D_- p \right) + \frac{1}{2} \left((D_+ u)^2, D_+ p \right) \\ &\quad + \frac{\Delta x}{2} \|D_+^2 u\|_p^2 - \frac{1}{2} (u^2, D_+ D_- p). \end{aligned}$$

lem:h1stability

Lemma 4.1. *Let u_j^n be a solution of the difference scheme (2.3). Let N be such that $N\Delta t = T$, and assume that the CFL-condition (3.2) holds. Then*

eq:uboundh1

$$(4.2) \quad \begin{aligned} \|u^N\|_p^2 + \Delta t \Delta x \sum_{n=0}^{N-1} \sum_{|j\Delta x| \leq R-1} (D_+ u_j^{n+1})^2 \\ \leq \|u^0\|_p^2 + \frac{2\Delta t}{\Delta x^2} (\|u^N\|^2 + \|u^0\|^2) + C, \end{aligned}$$

where the constant C depends only on T and u_0 . In particular, for any finite number R , we have that

eq:uderivRbound

$$(4.3) \quad \Delta t \Delta x \sum_{n=0}^N \sum_{|i\Delta x| \leq R} (D_+ u_j^n)^2 \leq C_R,$$

where $C_R = C(R, \|u_0\|, T)$.

Remark 4.1. *We shall see that this CFL condition is not sufficient to conclude convergence of the scheme. For that we need $\Delta t = \mathcal{O}(\Delta x^2)$.*

Proof. As before we set

$$w = \bar{u} - \Delta t \bar{u} D u.$$

Set $\lambda = \Delta t / \Delta x^{3/2}$. If the timestep Δt satisfies the following CFL condition (3.2) then we can multiply (3) by p to get the “cell entropy” inequality

$$\boxed{\text{eq:H1cell}} \quad (4.4) \quad \frac{1}{2} p w^2 \leq \frac{1}{2} p \bar{u}^2 - \frac{\Delta t}{3} p D u^3 - \delta \frac{\Delta x^2}{2} p (D u)^2, \quad \delta \in (0, 1).$$

Summing (4.4) over j we get

$$\boxed{\text{eq:H1burg}} \quad (4.5) \quad \frac{1}{2} \|w\|_p^2 + \delta \frac{\Delta x^2}{2} \|D u\|_p^2 \leq \frac{1}{2} \|u\|_p^2 - \frac{\Delta t}{3} (p, D u^3) + \frac{\Delta x^2}{2} (u^2, D_+ D_- p).$$

By (A.30) we have

$$\begin{aligned} \|u D_+ p\|_\infty &\leq \varepsilon \|D_- (u D_+ p)\| + c(\varepsilon) \|u D_+ p\| \\ &\leq \varepsilon (\|D_+ u D_+ p\| + \|u D_- D_+ p\|) + c(\varepsilon) \|u D_+ p\|, \end{aligned}$$

and similarly

$$\|u D_- p\|_\infty \leq \varepsilon (\|D_+ u D_- p\| + \|u_{+1} D_- D_+ p\|) + c(\varepsilon) \|u D_- p\|.$$

We use this to estimate

$$\begin{aligned} |(p, D u^3)| &= |(D p, u^3)| \\ &\leq \|u D p\|_\infty \|u\|^2 \\ &\leq \frac{1}{2} (\|u D_+ p\|_\infty + \|u D_- p\|_\infty) \|u\|^2 \\ &\leq \left(\varepsilon (\|D_+ u D_- p\| + \|D_+ u D_+ p\|) + \frac{c(\varepsilon)}{2} (\|u D_+ p\| + \|u D_- p\|) \right) \|u\|^2 \\ &\leq \varepsilon (\|D_+ u D_+ p\|^2 + \|D_+ u D_- p\|^2) + A(\varepsilon, \|u\|) \\ &\leq \varepsilon \left(((D_+ u)^2, D_+ p) + ((D_+ u)^2, D_- p) \right) + A(\varepsilon, \|u_0\|) \end{aligned}$$

where the locally bounded function A now depends on the first and second derivatives of p . Hence,

$$\boxed{\text{eq:wbdnew}} \quad (4.6) \quad \|w\|_p^2 + \delta \frac{\Delta x^2}{2} \|D u\|_p^2 \leq \|u\|_p^2 + A(\varepsilon, \|u_0\|) \Delta t + \varepsilon \Delta t \left(((D_+ u)^2, D_+ p) + ((D_+ u)^2, D_- p) \right)$$

where A is a locally bounded function. Next we study the full difference scheme by adding the “Airy term” $\Delta t D_+^2 D_- u_j^{n+1}$. Thus the full difference scheme (2.3) can be written

$$v = w - \Delta t D_+^2 D_- v.$$

Writing this as $w = v + \Delta t D_+^2 D_- v$, we square it, multiply by p and sum over j to get

$$\begin{aligned} \|w\|_p^2 &= \|v\|_p^2 + 2\Delta t (v, D_+^2 D_- v)_p + \Delta t^2 \|D_+^2 D_- v\|_p^2 \\ &= \|v\|_p^2 + \Delta t^2 \|D_+^2 D_- v\|_p^2 \\ &\quad + 2\Delta t \left((D_+ v)^2, D_- p \right) + \Delta t \left((D_+ v)^2, D_+ p \right) \\ &\quad + \Delta t \Delta x \|D_+^2 v\|_p^2 - \Delta t (v^2, D_+ D_-^2 p). \end{aligned}$$

Combining this with (4.6) we get

$$\begin{aligned} \|v\|_p^2 + 2\Delta t \left((D_+ v)^2, D_+ p \right) + \Delta t \left((D_+ v)^2, D_- p \right) \\ + \Delta t^2 \|D_+^2 D_- v\|_p^2 + \delta \frac{\Delta x^2}{2} \|D u\|_p^2 + \Delta t \Delta x \|D_+^2 v\|_p^2 \end{aligned}$$

$$\begin{aligned} &\leq \|u\|_p^2 + \varepsilon \Delta t \left((D_+ u)^2, D_- p \right) + \varepsilon \Delta t \left((D_+ u)^2, D_+ p \right) \\ &\quad + \Delta t A(\varepsilon, \|u_0\|) + \Delta t (v, D_+ D_-^2 p). \end{aligned}$$

Rearranging and dropping some terms “with the right sign” we obtain

$$\begin{aligned} &\|v\|_p^2 + \Delta t(2 - \varepsilon) \left((D_+ v)^2, D_- p \right) + \Delta t(1 - \varepsilon) \left((D_+ v)^2, D_+ p \right) \\ &\leq \|u\|_p^2 + 2\Delta t \varepsilon \left((D_+ u)^2 - (D_+ v)^2, Dp \right) \\ &\quad + \Delta t \left(A(\varepsilon, \|u_0\|) + \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \|D_+ D_-^2 p\|^2 \right) \end{aligned}$$

Next, observe that

$$\left((D_+ u)^2, D_{\pm} p \right) \geq \Delta x \sum_{|j\Delta x| \leq R-1} (D_+ u_j)^2 \geq 0.$$

Let the locally bounded function $B(\varepsilon, z) = A(\varepsilon, z) + z^2/2 + \|D_+ D_-^2 p\|^2$. We choose $\varepsilon = 1/2$ and recall that $v = u^{n+1}$ and $u = u^n$. Then we get

eq:telescope1

$$\begin{aligned} &\|u^{n+1}\|_p^2 + \Delta t \Delta x \sum_{|j\Delta x| \leq R-1} (D_+ u_j^{n+1})^2 \\ &\leq \|u^n\|_p^2 + \Delta t \left((D_+ u^n)^2 - (D_+ u^{n+1})^2, Dp \right) + \Delta t B(\varepsilon, \|u_0\|). \end{aligned} \quad (4.7)$$

This is a telescoping sum, and we choose N such that $N\Delta t = T$ to find

eq:ubound1

$$\begin{aligned} &\|u^N\|_p^2 + \Delta t \Delta x \sum_{n=0}^{N-1} \sum_{\substack{|j\Delta x| \\ \leq R-1}} (D_+ u_j^{n+1})^2 \\ &\leq \|u^0\|_p^2 + \Delta t \left((D_+ u^0)^2 - (D_+ u^N)^2, Dp \right) + TB(\varepsilon, \|u_0\|). \end{aligned} \quad (4.8)$$

From this we can easily conclude the proof of the lemma. $\square$

Now we define the bilinear interpolation of u_i^n by

eq:bilinearinterp

$$\begin{aligned} u_{\Delta x}(x, t) &= u_i^n + (x - x_i) D_+ u_i^n + (t - t_n) \Delta t u_i^n \\ &\quad + (x - x_i) (t - t_n) D_+^t D_+ u_i^n \end{aligned} \quad (4.9)$$

for $(x, t) \in [x_i, x_{i+1}] \times [t_n, t_{n+1}]$. Note that $u_{\Delta x}$ is continuous everywhere and differentiable almost everywhere.

theo:main_fully

Theorem 4.2. *Let $\{u_i^n\}$ be a sequence defined by the numerical scheme (2.3), and assume that there is a constant K such that $\Delta t = K\Delta x^2$. Assume furthermore that $\|u_0\|_{L^2(\mathbb{R})}$ is finite, then there exist constants C_i , $i = 1, 2, 3$, such that*

eq:ubound1

$$\|u_{\Delta x}\|_{L^\infty(0, T; L^2(-Q, Q))} \leq C_1, \quad (4.10)$$

eq:ubound2

$$\|u_{\Delta x}\|_{L^2(0, T; H^1(-Q, Q))} \leq C_2, \quad (4.11)$$

eq:ubound3

$$\|\partial_t u_{\Delta x}\|_{L^{4/3}(0, T; H^{-2}(-Q, Q))} \leq C_3, \quad (4.12)$$

where $Q = R - 1$ and $u_{\Delta x}$ is defined by bilinear interpolation from $\{u_i^n\}$, cf., (4.9). Moreover, there exists a sequence of $\{\Delta x_j\}_{j=1}^\infty$, $\lim_j \Delta x_j = 0$, and a function $u \in L^2(0, T; L^2(-Q, Q))$ such that

eq:uconverges

$$u_{\Delta x_j} \rightarrow u \text{ strongly in } L^2(0, T; L^2(-Q, Q)), \quad (4.13)$$

as j goes to infinity. The function u is a weak solution of (1.1).

Proof. First recall that $\|u_{\Delta x}\| \leq \|u_0\|$ so that (4.10) holds.

To show (4.11) we calculate that for $(x, t) \in [x_i, x_{i+1}) \times [t_n, t_{n+1})$

$$\begin{aligned}\partial_x u_{\Delta x} &= D_+ u_j^n + (t - t_n) D_+^t D_+ u_j^n \\ &= \alpha_n(t) D_+ u_j^n + (1 - \alpha_n(t)) D_+ u_j^{n+1},\end{aligned}$$

where $\alpha_n(t) = (t - t_n)/\Delta t \in [0, 1)$. Using this, we find

$$\begin{aligned}\|\partial_x u_{\Delta x}\|_{L^2(0,T;L^2(-Q,Q))}^2 &= \int_0^T \|\partial_x u_{\Delta x}(\cdot, t)\|_{L^2(-Q,Q)}^2 dt \\ &\leq 2 \sum_n \sum_{|j\Delta x| \leq Q} (D_+ u_j^n)^2 \frac{1}{\Delta t^2} \int_{t_n}^{t_{n+1}} (t - t_n)^2 dt \\ &\quad + (D_+ u_j^{n+1})^2 \frac{1}{\Delta t^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 dt \\ &\leq \frac{2}{3} \Delta t \sum_n \sum_{|j\Delta x| \leq Q} (D_+ u_j^n)^2 + (D_+ u_j^{n+1})^2 \\ &\leq CT,\end{aligned}$$

by Lemma 4.1. This, and the fact that $\|u_{\Delta x}(\cdot, t)\|_{L^2(-Q,Q)} \leq \|u_0\|$, proves (4.11).

Next, observe that in each cell $[x_j, x_{j+1}) \times [t_n, t_{n+1})$

eq:timederiv2

$$(4.14) \quad \partial_t u^{\Delta t} = D_t^+ u_j^n + (x - x_j) D_+ D_t^+ u_j^n,$$

and from the scheme we have

eq:timederiv1

$$(4.15) \quad D_t^+ u_i^n = \frac{\Delta x^2}{\Delta t} D_+ D_- u_i^n - \bar{u}_i^n D_0 u_i^n - D_- D_+^2 u_i^{n+1}.$$

We claim that for all sufficiently small Δx (actually for $\Delta x < 1/3$):

a) For all $n \geq 0$,

$$\|D_- D_+^2 u^n\|_{H^{-3}(-Q,Q)} \leq 3 \|D_+ u^n\|_{L^2(-Q,Q)}.$$

b) For all $n \geq 0$,

$$\|D_+ D_- u^n\|_{H^{-2}(-Q,Q)} \leq 3 \|D_+ u^n\|_{L^2(-Q,Q)}.$$

c) The piecewise constant function $\bar{u}_i^n D u_i^n$ satisfies

$$\|\bar{u} D u\|_{L^{4/3}(0,T;L^2(-Q,Q))} \leq C,$$

for some constant which only depends on Q, T and u_0 .

To prove the first part of the claim, let $\phi \in H_0^3(-Q, Q)$ be any test function

$$\begin{aligned}\left| \int_{-Q}^Q (D_- D_+^2 u^n) \phi(x) dx \right| &= \left| \sum_{|j\Delta x| \leq Q} D_- D_+^2 u_j^n \int_{x_j}^{x_{j+1}} \phi(x) dx \right| \\ &= \left| \sum_{|j\Delta x| \leq Q} D_+ u_j^n \int_{x_j}^{x_{j+1}} D_+ D_- \phi(x) dx \right| \\ &\leq \underbrace{\sum_{|j\Delta x| \leq Q} |D_+ u_j^n| \int_{x_j}^{x_{j+1}} |\phi''(x)| dx}_a \\ &\quad + \underbrace{\sum_{|j\Delta x| \leq Q} |D_+ u_j^n| \int_{x_j}^{x_{j+1}} |D_+ D_- \phi(x) - \phi''(x)| dx}_b.\end{aligned}$$

We start by estimating b , to that end

$$\begin{aligned} \int_{x_j}^{x_{j+1}} |D_+ D_- \phi(x) - \phi''(x)| \, dx &\leq \frac{1}{\Delta x^2} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} \int_{z-\Delta x}^z \int_x^\tau |\phi'''(\theta)| \, d\theta \, d\tau \, dz \, dx \\ &\leq \frac{1}{\Delta x^2} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} \int_{z-\Delta x}^z \sqrt{\tau-x} \|\phi'''\|_{L^2(x,\tau)} \, d\tau \, dz \, dx \\ &\leq \Delta x \sqrt{3\Delta x} \|\phi'''\|_{L^2(x_{j-1}, x_{j+2})} \end{aligned}$$

Thus

$$\begin{aligned} b &\leq \Delta x \left(\sum_{|j\Delta x| \leq Q} 3\Delta x |D_+ u_j^n|^2 \right)^{1/2} \left(\sum_{|j\Delta x| \leq Q} \|\phi'''\|_{L^2(x_{j-1}, x_{j+2})}^2 \right)^{1/2} \\ &\leq 3\Delta x \|D_+ u^n\|_{L^2(-Q, Q)} \|\phi'''\|_{L^2(-Q, Q)}. \end{aligned}$$

As to a , we calculate

$$\begin{aligned} a &= \sum_{|j\Delta x| \leq Q} |D_+ u_j^n| \int_{x_j}^{x_{j+1}} |\phi''(x)| \, dx \\ &\leq \sum_{|j\Delta x| \leq Q} |D_+ u_j^n| \sqrt{\Delta x} \|\phi''\|_{L^2(x_j, x_{j+1})} \\ &\leq \left(\sum_{|j| \leq Q} \Delta x |D_+ u_j^n|^2 \right)^{1/2} \left(\sum_{|j\Delta x| \leq Q} \|\phi''\|_{L^2(x_j, x_{j+1})}^2 \right)^{1/2} \\ &= \|D_+ u^n\|_{L^2(-Q, Q)} \|\phi''\|_{L^2(-Q, Q)}. \end{aligned}$$

Therefore **a)** follows. Claim **b)** is proved similarly.

To prove **c)** we first define the cut off function $\eta(x)$ as

$$\eta(x) = \begin{cases} 1 & |x| \leq Q, \\ 0 & |x| \geq Q+1, \\ x+Q+1 & x \in [-(Q+1), -Q], \\ Q+1-x & x \in [Q, Q+1], \end{cases}$$

and set $\eta_j = \eta(x_j)$. Then we have that

$$\begin{aligned} \Delta t \sum_{n=0}^{N-1} \left(\sum_j |\eta_j \bar{u}_j^n D u_j^n|^2 \right)^{2/3} &\leq \Delta t \sum_{n=0}^{N-1} \|\eta u^n\|_\infty^{4/3} \left(\sum_{|j\Delta x| \leq R} (D u_j^n)^2 \right)^{2/3} \\ &\leq \left(\Delta t \sum_n \|\eta u^n\|_\infty^4 \right)^{1/3} \left(\Delta t \Delta x \sum_n \sum_{|j\Delta x| \leq R} (D u_j^n)^2 \right)^{2/3} \\ &\leq \left(\Delta t \sum_n \|\eta u^n\|_\infty^4 \right)^{1/3} \left(\Delta t \Delta x \sum_n \sum_{|j\Delta x| \leq R} (D_+ u_j^n)^2 \right)^{2/3} \\ &\leq \left(\Delta t \sum_n \|\eta u^n\|_\infty^4 \right)^{1/3} C_R^{2/3}, \end{aligned}$$

by Lemma 4.1. To proceed we use the inequality

$$\|v\|_\infty \leq C \left(\Delta x \sum_{|j\Delta x| \leq R} v_j^2 \right)^{1/4} \left(\Delta x \sum_{|j\Delta x| \leq R} (D_+ v_j)^2 \right)^{1/4},$$

which holds for any grid function v such that $v_j = 0$ for $|j\Delta x| \geq R$. We shall use this for $v = \eta u^n$, to that end observe

$$\begin{aligned} \Delta x \sum_{|j\Delta x| \leq R} (\eta_j u_j^n)^2 &\leq \|u^n\|^2, \\ \Delta x \sum_{|j\Delta x| \leq R} (D_+ \eta_j u_j^n)^2 &\leq 2\Delta x \sum_{|j\Delta x| \leq R} (u_j^n D_+ \eta_j)^2 + (\eta_{j+1} D_+ u_j^n)^2 \\ &\leq 2\|u^n\|^2 + 2\Delta x \sum_{|j\Delta x| \leq R} (D_+ u_j^n)^2. \end{aligned}$$

Hence

$$\|\eta u^n\|_\infty^4 \leq C \|u^n\|^2 \left(\|u^n\|^2 + \Delta x \sum_{|j\Delta x| \leq R} (D_+ u_j^n)^2 \right).$$

Thus

$$\begin{aligned} \Delta t \sum_{n=0}^{N-1} \left(\sum_{|j\Delta x| \leq Q} |\bar{u}_j^n D u_j^n|^2 \right)^{2/3} &\leq \Delta t \sum_{n=0}^{N-1} \left(\sum_j |\eta_j \bar{u}_j^n D u_j^n|^2 \right)^{2/3} \\ &\leq C_R \left(\Delta t \sum_n \|u^n\|^2 \left(\|u^n\|^2 + \Delta x \sum_{|j\Delta x| \leq R} (D_+ u_j^n)^2 \right) \right)^{1/3} \\ &\leq C, \end{aligned}$$

for a constant C depending only on $\|u_0\|$, R and T . This proves **c**).

Now **a**), **b**) and Lemma 4.1 means that

$$\begin{aligned} \|D_- D_+^2 u^n\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))}^{4/3} &= \Delta t \sum_{n=0}^N \|D_- D_+ u^n\|_{H^{-3}(-Q,Q)}^{4/3} \\ &\leq CT^{1/3} \left(\Delta t \sum_{n=0}^N \|D_+ u^n\|_{L^2(-Q,Q)}^2 \right)^{2/3} \\ &\leq C, \end{aligned}$$

and that

$$\|D_- D_+ u^n\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))}^{4/3} \leq C.$$

Similarly, $\bar{u}^n D u^n \in L^{4/3}(0,T,L^2(-Q,Q)) \subset L^{4/3}(0,T,H^{-3}(-Q,Q))$. Therefore, by (4.15), $D_+^t u_j^n \in L^{4/3}(0,T,H^{-3}(-Q,Q))$. Next, let

$$\alpha(x) = \frac{x - x_j}{\Delta x} \chi_{\{x_j, x_{j+1}\}}(x).$$

Then (4.14) reads

$$\partial_t u_{\Delta x} = \alpha D_+^t u^n + (1 - \alpha) D_+^t u_{+1}^n.$$

Therefore

$$\begin{aligned} \|\partial_t u_{\Delta x}\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))} &\leq \|\alpha\|_{L^\infty(\mathbb{R})} \|D_+^t u^n\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))} \\ &\quad + \|1 - \alpha\|_{L^\infty(\mathbb{R})} \|D_+^t u_{+1}^n\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))} \\ &\leq 2 \|D_+^t u^n\|_{L^{4/3}(0,T,H^{-3}(-Q,Q))} \leq C, \end{aligned}$$

which is (4.12).

Using (4.10), (4.11) and (4.12) we can apply the Aubin–Simon compactness lemma to conclude that $u_{\Delta x}$ has a subsequence which converges strongly in $L^2(0,T;L^2(-Q,Q))$, i.e., (4.13) holds.

Note that this is enough to pass to the limit in the nonlinearity. This means we can apply the Lax–Wendroff like result of [13] to conclude that the limit is a weak solution. $\square$

lemma:simon

Lemma 4.3 (Aubin–Simon). *Let X, B, Y are three Banach spaces such that $X \subset B$ with compact embedding and $B \subset Y$ with continuous embedding. Let $T > 0$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence such that $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^p(0, T, X)$ and $(\partial_t u_n)_{n \in \mathbb{N}}$ is bounded in $L^q(0, T, Y)$, for any $1 \leq p, q \leq \infty$. Then there exists $u \in L^p(0, T, B)$ such that, up to a subsequence,*

$$u_n \rightarrow u \quad \text{in} \quad L^p(0, T, B).$$

5. NUMERICAL EXAMPLES

sec:numex

We have tested the scheme for two examples where the solution is known, and for one example where the solution is not known, but the initial data has a singularity, and is in L^2 .

5.0.1. *A one-soliton solution.* The KdV equation (1.1) has an exact solution given by

eq:onesol

$$(5.1) \quad w_1(x, t) = 9 \left(1 - \tanh^2 \left(\sqrt{3/2}(x - 3t) \right) \right).$$

This represent a single bump moving to the right with speed 3. We have tested our scheme with initial data $u_0(x) = w_1(x, -1)$ in order to check how fast this scheme converges. In Figure 1 we show the exact solution at $t = 2$ as well as the numerical solution computed using 1000 grid points in the interval $[-10, 10]$, i.e., $\Delta x = 20/1000$. We have also computed numerical the error for a range of Δx ,

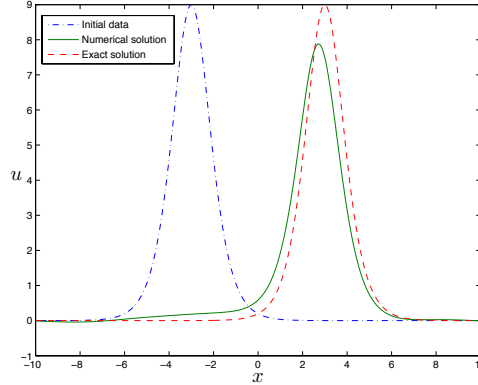


FIGURE 1. Initial data, and the exact and numerical solutions at $t = 2$ with initial data $w_1(x, -1)$ with $N = 1000$ grid points.

fig:1

where the relative error is defined by

$$E = 100 \frac{\sum_{j=1}^N |w_1(x_j, 1) - u_{\Delta x}(x_j, 2)|}{\sum_{j=1}^N w_1(x_j, 1)}.$$

Recall that we are using $w_1(x, -1)$ as initial data, so that $w_1(x, 1)$ represents the solution at $t = 2$. In Table 1 we show the relative errors as well as the numerical convergence rates for this example. The numerical convergence rate indicates that, as expected, the scheme is of first order. Note also that we have to use a rather small Δx in order to get a reasonably small error. Computing soliton solutions is quite hard, since these solutions are close to zero outside a bounded interval, and the speed of the soliton is proportional to its height. Therefore, if a numerical method (due to e.g. numerical diffusion) does not have the correct height, it will

N	E	rate
500	51.2	0.70
1000	31.4	0.83
2000	17.6	0.91
4000	9.4	0.95
8000	4.9	0.96
16000	2.5	

tab:1

TABLE 1. Relative errors for the one-soliton solution.

also have a wrong speed. Thus after some time, it will be in the wrong place and the error is close to 100%.

subsub:twosol

5.0.2. *A two-soliton solution.* Another exact solution of (1.1) is the so-called two-soliton,

eq:twosoliton

(5.2)

$$w_2(x, t) = 6(b - a) \frac{b \operatorname{csch}^2 \left(\sqrt{b/2}(x - 2bt) \right) + a \operatorname{sech}^2 \left(\sqrt{a/2}(x - 2at) \right)}{\left(\sqrt{a} \tanh \left(\sqrt{a/2}(x - 2at) \right) - \sqrt{b} \coth \left(\sqrt{b/2}(x - 2bt) \right) \right)^2},$$

for any real numbers a and b . We have used $a = 0.5$ and $b = 1$. This solution represents two waves which “collide” at $t = 0$ and separate for $t > 0$. For large $|t|$, $w_2(\cdot, t)$ is close to a sum of two one-solitons at different locations.

Computationally, this is a much harder problem than the one-soliton solution. As initial data we have used $u_0(x) = w_2(x, -10)$. In Figure 2 we show the exact and numerical solutions at $t = 20$. Although we have used 4000 grid points, the error

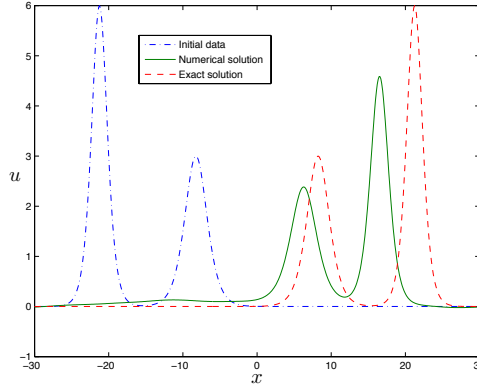


FIGURE 2. Initial data, and the exact and numerical solutions at $t = 20$ with initial data $w_2(x, -10)$ with $N = 4000$ grid points.

fig:2

is a staggering 140%! We see that the qualitative features are “right”, in the sense that the larger soliton has overtaken the slower one, but neither their heights nor their positions are correct. For sufficiently small Δx , the numerical solution will be close to the exact also in this case, but it is impractical to calculate numerical convergence rates since the computations would take too much time.

subsub:12

5.0.3. *Initial data in L^2 .* We have also tried our scheme on an example where the initial data is in L^2 , but not in any Sobolev space with positive index. Furthermore,

note that all the conclusions in Section 4 remain valid if we restrict ourselves to the periodic case. Therefore we have chosen initial data

$$\boxed{\text{eq:l2init}} \quad (5.3) \quad u_0(x) = \begin{cases} 0 & x \leq 0, \\ \frac{3}{2}x^{-1/3} & 0 < x < 1, \\ 0 & x \geq 1, \end{cases}$$

if x is in $[-5, 5]$, and extended periodically outside this interval. In this case we have no exact solution available. Therefore we can determine the convergence by viewing solutions with different Δx . In Figure 3 we have plotted the numerical solutions at $t = 0.5$ using 3750, 7500, 15000 and 30000 grid cells in the interval $[-5, 5]$. From

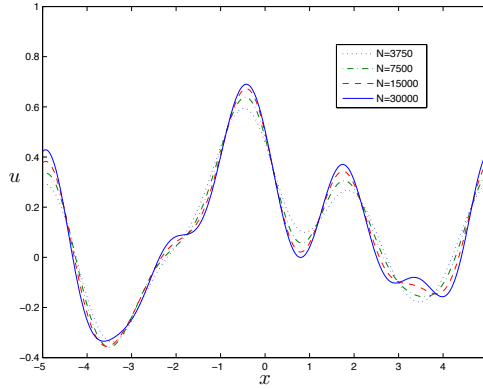


FIGURE 3. The numerical solution $u_{\Delta x}(x, 0.5)$ with initial data (5.3) for various Δx .

fig:3

this figure we can observe that the numerical solutions seems to converge nicely to a (smooth) function. The coarser features are already resolved using 3750 grid cells, and only the finer structures become more apparent for smaller Δx .

APPENDIX A. SOBOLEV INEQUALITIES

For functions of sufficient regularity we have the following estimates.

lem:Sobolev

Lemma A.1. *Let $n, m \in \mathbb{N}_0$ with $m < n$. For each positive ϵ we can choose a positive constant $C(\epsilon)$ such that for any function such that either (i) for $u \in H^n(\mathbb{R})$, or (ii) $u \in H_{\text{per}}^n(\mathbb{R})$, the periodic functions with period 1, the following estimates hold:*

eq:sob1

$$(A.1) \quad \|u^{(m)}\|_2^2 \leq \epsilon \|u^{(n)}\|_2^2 + C(\epsilon) \|u\|_2^2,$$

eq:sob2a

$$(A.2) \quad \|u^{(m)}\|_\infty^2 \leq \epsilon \|u^{(n)}\|_2^2 + C(\epsilon) \|u\|_2.$$

Here the norms are the usual L^2 and L^∞ norms on the whole line in case (i) and on the interval $[0, 1]$ in case (ii).

Proof. The proofs are based on integration by parts in such a way that the boundary terms disappear both in the periodic case and the case of the full line, thus we only provide the details for the full line.

The proof of (A.1) goes as follows. We first establish by induction

eq:malek6

$$(A.3) \quad \|u^{(n)}\|_2^2 \leq \epsilon \|u^{(n+1)}\|_2^2 + C(\epsilon) \|u\|_2^2, \quad n \in \mathbb{N}.$$

For $n = 1$ we find

$$\boxed{\text{eq: malek7}} \quad (\text{A.4}) \quad \int_{\mathbb{R}} u'(x)^2 dx = - \int_{\mathbb{R}} u''(x)u(x) dx \leq \epsilon \|u''\|_2^2 + C(\epsilon) \|u\|_2^2.$$

Assume that (A.3) holds for all $n \leq N$. Then we find

$$\begin{aligned} \boxed{\text{eq: malek8}} \quad (\text{A.5}) \quad \int_{\mathbb{R}} u^{(N+1)}(x)^2 dx &= - \int_{\mathbb{R}} u^{(N+2)}(x)u^{(N)}(x) dx \\ &\leq \epsilon \|u^{(N+2)}\|_2^2 + C(\epsilon) \|u^{(N)}\|_2^2 \\ &\leq \epsilon \|u^{(N+2)}\|_2^2 + C(\epsilon)(\tilde{\epsilon} \|u^{(N+1)}\|_2^2 + C(\tilde{\epsilon}) \|u\|_2^2), \end{aligned}$$

which we rewrite as

$$\boxed{\text{eq: malek9}} \quad (\text{A.6}) \quad (1 - C(\epsilon)\tilde{\epsilon}) \|u^{(N+1)}\|_2^2 \leq \epsilon \|u^{(N+2)}\|_2^2 + C(\epsilon)C(\tilde{\epsilon}) \|u\|_2^2,$$

which proves the induction step: Given ϵ , we determine $C(\epsilon)$, and then we choose $\tilde{\epsilon}$ such that $C(\epsilon)\tilde{\epsilon} \leq \frac{1}{2}$. The general case follows:

$$\begin{aligned} \boxed{\text{eq: a1_general}} \quad (\text{A.7}) \quad \|u^{(m)}\|_2^2 &\leq \epsilon \|u^{(m+1)}\|_2^2 + C(\epsilon) \|u\|_2^2 \\ &\leq \epsilon(\epsilon_1 \|u^{(m+2)}\|_2^2 + C(\epsilon_1) \|u\|_2^2) + C(\epsilon) \|u\|_2^2 \\ &\leq \epsilon\epsilon_1 \|u^{(m+2)}\|_2^2 + (\epsilon C(\epsilon_1) + C(\epsilon)) \|u\|_2^2 \\ &\leq \epsilon\epsilon_1(\epsilon_2 \|u^{(m+3)}\|_2^2 + C(\epsilon_2) \|u\|_2^2) + (\epsilon C(\epsilon_1) + C(\epsilon)) \|u\|_2^2 \\ &\leq \epsilon\epsilon_1\epsilon_2 \|u^{(m+3)}\|_2^2 + (\epsilon\epsilon_1 C(\epsilon_2) + \epsilon C(\epsilon_1) + C(\epsilon)) \|u\|_2^2 \\ &\leq \epsilon\epsilon_1 \cdots \epsilon_{k-1} \|u^{(m+k)}\|_2^2 \\ &\quad + (\epsilon\epsilon_1 \cdots \epsilon_{k-2} C(\epsilon_{k-1}) + \cdots + \epsilon C(\epsilon_1) + C(\epsilon)) \|u\|_2^2. \end{aligned}$$

The proof of (A.2) goes as follows. We start by showing

$$\boxed{\text{eq: sob3}} \quad (\text{A.8}) \quad \|u^{(m)}\|_{\infty}^2 \leq \epsilon \|u^{(m+1)}\|_2^2 + C(\epsilon) \|u\|_2^2$$

by induction. Consider first $m = 0$. We have

$$\boxed{\text{eq: sob4a}} \quad (\text{A.9}) \quad u(x)^2 = \int_{-\infty}^x \frac{d}{dx} u(y)^2 dy = 2 \int_{-\infty}^x u(y)u_x(y) dy,$$

or

$$\boxed{\text{eq: sob5a}} \quad (\text{A.10}) \quad \|u\|_{\infty}^2 \leq 2 \int_{\mathbb{R}} |u(y)u_x(y)| dy \leq \epsilon \|u_x\|_2^2 + C(\epsilon) \|u\|_2^2.$$

Assume now that (A.8) holds for all $m \leq n$. Next

$$\boxed{\text{eq: sob4}} \quad (\text{A.11}) \quad u^{(n+1)}(x)^2 = \int_{-\infty}^x \frac{d}{dx} u^{(n+1)}(y)^2 dy = 2 \int_{-\infty}^x u^{(n+1)}(y)u^{(n+2)}(y) dy,$$

or

$$\begin{aligned}
\|u^{(n+1)}\|_\infty^2 &\leq 2 \int_{\mathbb{R}} |u^{(n+1)}(y)u^{(n+2)}(y)| dy \\
&\leq \epsilon \|u^{(n+2)}\|_2^2 + C(\epsilon) \|u^{(n+1)}\|_2^2 \\
&\leq \epsilon \|u^{(n+2)}\|_2^2 + C(\epsilon)(\tilde{\epsilon} \|u^{(n+2)}\|_2^2 + C(\tilde{\epsilon}) \|u\|_2^2) \\
&\leq (\epsilon + C(\epsilon)\tilde{\epsilon}) \|u^{(n+2)}\|_2^2 + C(\epsilon)C(\tilde{\epsilon}) \|u\|_2^2 \\
&\leq \epsilon \|u^{(n+2)}\|_2^2 + C(\epsilon) \|u\|_2^2,
\end{aligned}
\tag{A.12}$$

proving the induction step. The general case follows as the general inequality above. $\square$

Remark A.1. (a) An alternative proof using Fourier transform goes as follows. It has the advantage that it works in multidimensions. Check: In the periodic case one has to apply an argument based on Fourier series. For simplicity we set 2π equal to 1. We start by the elementary inequality (for $x \neq 0$)

$$|x|^{2m} \leq \epsilon |x|^{\alpha p} + C(\epsilon) |x|^{q(2m-\alpha)}, \tag{A.13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in (1, \infty)$, $\alpha \in (2m, 2n)$. Choose $p = 2n/\alpha$, which implies $p \in (1, n/m)$. Thus (for $x \neq 0$)

$$|x|^{2m} \leq \epsilon |x|^{2n} + C(\epsilon) |x|^{q(2m-\alpha)}. \tag{A.14}$$

For $|x| \geq 1$ we see that $|x|^{q(2m-\alpha)} \leq 1$. For $|x| < 1$ we note the elementary inequality

$$|x|^{2m} \leq \epsilon |x|^{2n} + 1. \tag{A.15}$$

Combining the two we see

$$|x|^{2m} \leq \epsilon |x|^{2n} + C(\epsilon). \tag{A.16}$$

Multiplying this inequality with $|\hat{u}(x)|^2$ and integrating over $\mathbb{R}$ proves (A.1).

(b) Alternatively, one can use Lemma 2.18 in Malek et al. [23], which states

$$\|u\|_{k_2, p} \leq c \|u\|_{k_1, p}^{k_2/k_1} \|u\|_p^{1-(k_2/k_1)}, \quad k_1 \geq k_2 > 0, \tag{A.17}$$

where $\|u\|_{k, p}$ is the usual Sobolev norm. Assuming (A.17) we find (with $p = 2$)

$$\begin{aligned}
\|u\|_{k_2, 2}^2 &\leq c^2 \|u\|_{k_1, 2}^{2k_2/k_1} \|u\|_2^{2(1-(k_2/k_1))} \\
&\leq \epsilon \|u\|_{k_1, 2}^{2qk_2/k_1} + C(\epsilon) \|u\|_2^{2r(1-(k_2/k_1))}
\end{aligned}
\tag{A.18}$$

with $\frac{1}{q} + \frac{1}{r} = 1$. Let $2qk_2/k_1 = 2$ which implies that $2r(1-(k_2/k_1)) = 2$. Thus

$$\begin{aligned}
\|u\|_{k_2, 2}^2 &\leq \epsilon \|u\|_{k_1, 2}^2 + C(\epsilon) \|u\|_2^2 \\
&= \epsilon \sum_{k \leq k_1} \|u^{(k)}\|_2^2 + C(\epsilon) \|u\|_2^2.
\end{aligned}
\tag{A.19}$$

Consider first the case when $k_1 = k_2 + 1$. Then we find

$$(1 - \epsilon) \|u^{(k_2)}\|_2^2 \leq (1 - \epsilon) \sum_{k \leq k_2} \|u^{(k)}\|_2^2 \leq \epsilon \|u^{(k_2+1)}\|_2^2 + C(\epsilon) \|u\|_2^2. \tag{A.20}$$

We can rewrite this as

$$\|u^{(k_2)}\|_2^2 \leq \epsilon \|u^{(k_2+1)}\|_2^2 + C(\epsilon) \|u\|_2^2. \tag{A.21}$$

Inductively, we find

$$\boxed{\text{eq:malek5}} \quad (\text{A.22}) \quad \left\| u^{(k_2)} \right\|_2^2 \leq \epsilon \left\| u^{(k_1)} \right\|_2^2 + C(\epsilon) \|u\|_2^2,$$

which is what we want.

The discrete analog reads as follows.

$\boxed{\text{lem:SobolevDisc}}$

Lemma A.2. *Let $m < n$, and consider the two cases of functions in $\ell^n(\mathbb{R})$ or $\ell^\infty(\mathbb{R})$. The following estimates hold*

$$\boxed{\text{eq:sob1D}} \quad (\text{A.23}) \quad \|D^m u\|_2^2 \leq \epsilon \|D^n u\|_2^2 + C(\epsilon) \|u\|_2^2,$$

$$\boxed{\text{eq:sob2aD}} \quad (\text{A.24}) \quad \|D^m u\|_\infty^2 \leq \epsilon \|D^n u\|_2^2 + C(\epsilon) \|u\|_2.$$

Proof. The proof follows that continuum case closely. We here treat the case of the full line only. Consider first the case with $n = m + 1$. If $n = 1$ we have

$$\boxed{\text{eq:n1D}} \quad (\text{A.25}) \quad \|Du\|_2^2 = (Du, Du) = -(u, D^2 u) \leq \epsilon \|D^2 u\|_2^2 + C(\epsilon) \|u\|_2^2.$$

Assume that (A.23) holds for all $n = m + 1 \leq N$. Then we find

$$\begin{aligned} \boxed{\text{eq:n1DN}} \quad (\text{A.26}) \quad \|D^{N+1} u\|_2^2 &= (D^{N+1} u, D^{N+1} u) = -(D^N u, D^{N+2} u) \leq \epsilon \|D^{N+2} u\|_2^2 + C(\epsilon) \|D^N u\|_2^2 \\ &\leq \epsilon \|D^{N+2} u\|_2^2 + C(\epsilon) (\tilde{\epsilon} \|D^{N+1} u\|_2^2 + C(\tilde{\epsilon}) \|u\|_2^2), \end{aligned}$$

which can be rewritten as

$$\boxed{\text{eq:malek9D}} \quad (\text{A.27}) \quad (1 - C(\epsilon)\tilde{\epsilon}) \|D^{N+1} u\|_2^2 \leq \epsilon \|D^{N+2} u\|_2^2 + C(\epsilon)C(\tilde{\epsilon}) \|u\|_2^2,$$

which proves the induction step. The rest of the proof follows as in the continuum case. The periodic case follows similarly.

As for the (A.24) we note that

$$\boxed{\text{eq:partialD}} \quad (\text{A.28}) \quad u_{j+1}^2 = \Delta x \sum_{k=-\infty}^j (Du^2)_k = 2\Delta x \sum_{k=-\infty}^j \bar{u}_k Du_k,$$

which implies

$$\boxed{\text{eq:partialD1}} \quad (\text{A.29}) \quad \|u\|_\infty^2 \leq 2 |(\bar{u}, Du)| \leq \epsilon \|Du\|_2^2 + C(\epsilon) \|\bar{u}\|_2^2 \leq \epsilon \|Du\|_2^2 + C(\epsilon) \|u\|_2^2.$$

By analogous arguments we can calculate

$$\boxed{\text{eq:partialD1_2}} \quad (\text{A.30}) \quad \|u\|_\infty^2 \leq \epsilon \|D_\pm u\|_2^2 + C(\epsilon) \|u\|_2^2.$$

□

Remark A.2. *According to [30] one can modify the left-hand side of (A.23) as follows: One can replace $D^m u$ by any operator $D_+^{m_1} D_-^{m_2} D_-^{m_3}$ with $m = m_1 + m_2 + m_3$. The right-hand side can likewise be replaced by $D_+^{n_1} D_-^{n_2}$ with $n = n_1 + n_2$. which is what we need! However, the right-hand side cannot contain the operator D .*

old stuff

To introduce the approach by Sjöberg, we first study the periodic case. For simplicity we assume a unit period and that $1/\Delta x \in \mathbb{N}$. In that case $u_{j+N}^0 = u_j^0$ for $j \in \mathbb{Z}$. Let $u_{\Delta x}$ be a grid function, i.e., a function $u_{\Delta x}: [0, T] \rightarrow \mathbb{R}^N$, where $u_{\Delta x}(x_j, t) = u_j(t)$, such that $u_0 = u_N$. Sjöberg proved that the following semi-discrete scheme

$$\boxed{\text{eq:sjoberg}} \quad (\text{A.31}) \quad \frac{d}{dt} u_j = -\hat{u}_j Du_j - D_+^2 D_- u_j, \quad t > 0, \quad u_j(0) = u_0(x_j),$$

produces a sequence $\{u_{\Delta x}\}_{\Delta x > 0}$ which converges to the unique solution of (2.2), thus $u_{\Delta x}(x_j, t) = u_j(t) \approx u(x_j, t)$. Here

$$\hat{u}_j = \frac{1}{3}(u_{j-1} + u_j + u_{j+1}).$$

We shall now consider fully discrete versions of this scheme. We introduce (a small) time step $\Delta t > 0$, and use the notation

$$D_+^t p(t) = \frac{1}{\Delta t}(p(t + \Delta t) - p(t)),$$

for any function $p: [0, T] \rightarrow \mathbb{R}$. Write $t^n = n\Delta t$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A fully discrete grid function is a function $u_{\Delta x}: \{t^n\}_{n \in \mathbb{N}_0} \rightarrow \mathbb{R}^{\mathbb{Z}}$, and we write $u_{\Delta x}(x_j, t^n) = u_j^n$. (A CFL condition will enforce a relationship between Δx and Δt , and hence we only use Δx in the notation.)

An implicit fully discrete version of Sjöberg's scheme is

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$$(A.32) \quad D_+^t u_j^n = -\hat{u}_j^{n+1} D_+ u_j^{n+1} - D_+^2 D_- u_j^{n+1}, \quad n \in \mathbb{N}_0, \quad u_j^0 = u_0(x_j).$$

Now we must prove the following: (a) that this has a unique solution (at least for small Δt), (b) that we get the right estimates (should not be too difficult). Here one can use [19], specifically, we introduce

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$$(A.33) \quad \frac{1}{\Delta t}(u_j^{n+1-(m+1)^{-1}} - u_j^n) = -\hat{u}_j^{n+1-m^{-1}} D_+ u_j^{n+1-m^{-1}} - D_+^2 D_- u_j^{n+1-(m+1)^{-1}}, \quad m \in \mathbb{N}.$$

Consider $\{u_j^n\}$ given, and use (A.33) iteratively: For any $u_j^{n+1-m^{-1}}$, we apply (A.33) to compute $u_j^{n+1-(m+1)^{-1}}$. We prove that

$$\lim_{m \rightarrow \infty} u_j^{n+1-m^{-1}} = u_j^{n+1}.$$

Provided the limit exists, it is clear that it satisfies (A.32). The drawback with this scheme is that it entails solving a non-linear equation at each timestep.

lem:iterate

Lemma A.3. *Let $\lambda = \Delta t/Dx$ and the function F be given by*

$$F(a) = (I + \Delta t D_+^2 D_-)^{-1} (a^n - \Delta t \hat{a} D a).$$

If $(\|a\|_{\infty} + \|b\|_{\infty})2\lambda = \mu < 1$, then

$$\|F(a) - F(b)\| \leq \mu \|a - b\|.$$

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