

Modules and slices in logarithmic motivic homotopy theory

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Abstract

For perfect fields, we construct an adjunction between the ∞ -category of log motivic spectra and ∞ -category of log motives and identify the category of modules over the associated monad. Assuming a description of the dualizing objects in $\log\mathcal{SH}(k)$, we refine this result to give a symmetric monoidal equivalence between the category of modules over the log motivic ring spectrum representing log motivic cohomology and the ∞ -category of log motives. For fields admitting resolution of singularities, we compute the log slices of the log cobordism spectrum and the log motivic sphere spectrum. This generalizes the slice computation of the motivic sphere to the logarithmic motivic setting. As a consequence, we identify log slices of log motivic spectra as Eilenberg–MacLane spectra of logarithmic motives.

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I.1 Introduction

This paper aims to generalize the classification of modules over motivic cohomology [RØ06] [RØ08] and the slice computation of the motivic sphere spectrum [Voe04] [RSØ16] to the logarithmic motivic setting. Logarithmic motivic homotopy theory, recently developed in [BPØ22a], [BPØ22b], [BPØ23], extends Voevodsky's motivic homotopy theory to accommodate non- $\mathbb{A}^1$ -invariant cohomology theories of schemes, such as topological Hochschild homology THH and topological cyclic homology TC. Since such cohomology theories are not representable in the stable motivic homotopy category $\mathcal{SH}(S)$ - indeed the $\mathbb{A}^1$ -localization of THH is trivial [Elm21] - they cannot be studied by standard motivic techniques. Therefore, one of the motivations for developing logarithmic motivic homotopy theory is to broaden motivic theory to include such objects and to study them using motivic techniques. Achieving this goal requires generalizing well-established results from motivic homotopy theory to the logarithmic motivic setting.

Our starting point is the classical correspondence between the category of modules over the integral Eilenberg–MacLane ring spectrum and the derived category of abelian groups. This correspondence allows for turning questions in homological algebra into questions in stable homotopy theory. For fields of characteristic 0, [RØ06] and [RØ08] used motivic functors [DRØ03] to establish an equivalence between the category of modules over the integral motivic spectrum $\mathbb{M}\mathbb{Z}$, which represents motivic cohomology, and Voevodsky's derived category of triangulated motives $\mathcal{DM}(k)$ introduced in [Voe00]. This result is a motivic analogue of the classical result from [SS03], which states that the category of modules over the Eilenberg–MacLane spectrum $\mathbb{M}\mathbb{Z}$ is equivalent to the category of chain complexes over $\mathbb{Z}$. Voevodsky had already suggested the existence of such an equivalence for rational coefficients in the concluding remarks of his ICM address [Voe98, §7]. The result has since been generalized to regular schemes over a field in [CD15] and to general categories of correspondences in [Gar19].

In the logarithmic situation, we construct an adjunction

$$\gamma_{\sharp} : \log\mathcal{SH}(k, \Lambda) \rightleftarrows \log\mathcal{DM}(k, \Lambda) : \gamma^*$$

between the stable category of logarithmic motivic spectra [BPØ23] and the category of logarithmic motives [BPØ22b]. We then apply the strategy from [EK20b] to establish an equivalence between the category of modules over the monad $\gamma_{\sharp}\gamma^*$ with the category of logarithmic motives, leading to the following theorem.

Theorem I.1.1 (Theorem I.3.7). *Let k be a perfect field and Λ be any ring. Let $\text{Mod}_{\gamma_{\sharp}\gamma^*}(\log\mathcal{SH}(k, \Lambda))$ denote the ∞ -category of modules over the monad $\gamma_{\sharp}\gamma^*$. There is an equivalence of ∞ -categories*

$$\text{Mod}_{\gamma_{\sharp}\gamma^*}(\log\mathcal{SH}(k, \Lambda)) \simeq \log\mathcal{DM}(k, \Lambda)$$

The approach in [EK20b] builds on the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5], which provides two criteria for when an ∞ -category can

be described as the category of modules over a monad. While this method works well in many cases, both $\log \mathcal{DM}(k)$ and $\log \mathcal{SH}(k)$ are equipped with symmetric monoidal structures, and the framework of [EK20b] does not fully capture an equivalence of *symmetric monoidal* ∞ -categories. To address this, we instead turn to a more refined version of the Barr–Beck–Lurie theorem, specifically [Lur17, Corollary 4.8.5.21], to establish the desired equivalence in the symmetric monoidal setting. This result requires three additional conditions for an ∞ -category to characterize modules, one being that $\log \mathcal{SH}(k)$ satisfies a certain logarithmic projection formula. For $\mathcal{SH}(k)[1/e]$, when k is a field with exponential characteristic e , this is a consequence of the fact that $\mathcal{SH}(k)[1/e]$ is strongly generated by dualizable objects (combine [LYZ19, Corollary B.2] with [EK20a]) and the standard computation in [Lemma I.3.3]. However, it is unknown if $\log \mathcal{SH}(k)$ is generated by strongly dualizable objects. We state this as [Conjecture I.3.8] and prove it for the full subcategory of $\log \mathcal{SH}(k)$ that is generated by proper log schemes. Our main result is the following theorem.

Theorem I.1.2 ([Corollary I.3.15]). *Let k be a perfect field and assume that $\log \mathcal{SH}(k)$ is generated by strongly dualizable objects. Then there is an equivalence of presentably stable symmetric monoidal ∞ -categories*

$$\mathrm{Mod}_{\log \mathrm{MZ}}(\log \mathcal{SH}(k)) \simeq \log \mathcal{DM}(k)$$

One of the main sources of examples of modules over motivic cohomology MZ are the slices $s_i(E)$ coming from the slice filtration [Voe02, [SØ10] in motivic homotopy theory. For every motivic spectrum $E \in \mathcal{SH}(k)$, Voevodsky constructed a sequence of spectra

$$f_{n+1}E \rightarrow f_nE \rightarrow f_{n-1}E \rightarrow \cdots \rightarrow E$$

known as the *slice tower*, which is the motivic analogue of the Postnikov tower. The n -th slice $s_i(E)$ of a motivic spectrum E is defined as the cofiber of the morphism $f_{n+1}E \rightarrow f_nE$, and represents the motivic version of the Eilenberg–MacLane spectrum $\Sigma^n H\pi_n^s(E)$, where $\pi_n^s(E)$ denotes the n -th stable homotopy group of E . Under such an analogy, slices represent stable homotopy groups in motivic homotopy theory. In particular, Voevodsky showed in [Voe04] that, in characteristic 0, the zero slice of the sphere spectrum $s_0(\mathbb{1}) \cong \mathrm{MZ}$, which is analogous to the classical topological computation that $\pi_n^s(S^n) \cong \mathbb{Z}$. From the morphism $E \rightarrow \mathbb{1}$, this equips all slices $s_i(E)$ with a natural MZ -module structure.

Very recently, [BPØ24] introduced the *logarithmic* slice filtration, which is the generalization of the slice filtration to logarithmic motivic homotopy theory. They showed that the logarithmic slice filtration admits many of the same nice properties as the slice filtration [BPØ24, §2]. This made it possible to apply the slice tower to the logarithmic motivic spectra THH and TC , and they showed that the filtration from the logarithmic slice tower compared nicely with the motivic filtration on THH , TC and TP introduced by Bhatt–Morrow–Scholze in [BMS18, §1.4]. They also showed that the logarithmic trace morphisms in

$\log\mathcal{SH}(k)$ are compatible with the logarithmic slice filtration and identified the graded pieces.

In [Corollary I.4.3](#), we show that the functor $\omega^*: \mathcal{SH}(k) \rightarrow \log\mathcal{SH}(k)$ commutes with taking slices. We then prove that $\omega^* \text{MGL} \cong \log\text{MGL}$, and use the fact that ω^* is monoidal [[Par23](#), Theorem 1.1.(4)] to prove that $\log\text{MGL}$ admits an E^∞ -log structure in [Corollary I.4.7](#) and determine all its log slices in [Corollary I.4.8](#). Furthermore, using the results of [[Par23](#)], we generalize the main theorem of [[RSØ16](#)], that is, we compute all the logarithmic slices of the logarithmic sphere spectrum $\mathbb{1}^{\log}$:

Theorem I.1.3 ([Theorem I.4.9](#)). *Assume that k admits resolution of singularities and let Λ be any ring. Then there are isomorphisms of $\log\text{M}\Lambda$ -modules*

$$s_q^{\log}(\mathbb{1}^{\log}) \cong \bigvee_{p \geq 0} \Sigma^{2q,q} \log\text{M}\Lambda \otimes \text{Ext}_{MU_* MU}^{p,2q}(MU_*, MU_*)[-p]$$

in $\log\mathcal{SH}(k)$.

As a special case, this shows that $s_0(\mathbb{1}^{\log}) \cong \log\text{M}\mathbb{Z}$, which is the logarithmic analogue of the main result of [[Voe02](#)]. This implies that each logarithmic slice $s_i(E)$ has the canonical structure of a $\log\text{M}\mathbb{Z}$ -module [Proposition I.4.12](#). Assuming that $\log\mathcal{SH}(k)$ is generated by strongly dualizable objects, [Theorem I.1.2](#) implies that the log slices are naturally Eilenberg–MacLane spectra of logarithmic motives [Corollary I.4.13](#). This generalizes [[Lev08](#), Corollary 10.5.2] to the logarithmic setting.

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I.2 Preliminaries

I.2.1 Modules over a monad

One of the basic facts in abstract algebra is that the forgetful functor of modules over $\mathbb{Z}$ to abelian groups is an equivalence of categories. A homotopical version of this statement is that modules over the Eilenberg–MacLane ring spectrum HR are equivalent to the category of chain complexes of R -modules $C_\bullet(R\text{-mod})$ [[Rob87](#)]. Such perspectives have proven useful, as they allow for turning problems from homological algebra into problems in stable homotopy theory. The concept of a monad generalizes the theory of modules to category theory, and gives a tool for studying commutative algebra objects in more general settings.

Let $\mathcal{C}$ denote a 1-category. Recall that a *monad* on $\mathcal{C}$ consists of a functor $T : \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\eta : 1_{\mathcal{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ such that the two squares

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ T\eta \downarrow & \searrow & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

commute. For ∞ -categories $\mathcal{C}$, a monad on $\mathcal{C}$ is an algebra object in the monoidal category $\text{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors on $\mathcal{C}$ [Lur17, Definition 4.7.0.1].

Given an adjunction of ∞ -categories $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$, the composition $T = R \circ L$ defines a monad over $\mathcal{C}$, where the unit morphism $\text{id} \rightarrow T$ is given by the adjunction between L and R , and the product given by composition

$$T \circ T = R \circ (L \circ R) \circ L \rightarrow R \circ \text{id} \circ L = T$$

Much like how algebras and modules relate, one may also define an algebra over a monad. Namely, if (T, μ, η) is a monad on $\mathcal{C}$, then a *module* (or *algebra*) over T is an object $X \in \mathcal{C}$ with a *multiplication morphism* $m : TX \rightarrow X$ such that the diagrams

$$\begin{array}{ccc} T^2 X & \xrightarrow{Tm} & TX \\ \mu X \downarrow & & \downarrow m \\ TX & \xrightarrow{m} & X \end{array} \quad \begin{array}{ccc} TX & \xleftarrow{\eta X} & X \\ m \downarrow & \swarrow & \\ X & & \end{array}$$

commute.

For an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$, every object $R(D)$ for all $D \in \mathcal{D}$ has the structure of a module over the monad $T = R \circ L$ given by

$$TR(D) = ((R \circ L) \circ R)(D) = (R \circ (L \circ R))(D) \rightarrow R(D)$$

This determines a functor from $\mathcal{D}$ to the category of RL -modules over $\mathcal{C}$. Furthermore, the functor R factors through the category of RL -modules $\text{Mod}_{RL}(\mathcal{C})$, meaning that we have a factorization

$$\begin{array}{ccc} \mathcal{D} & \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{\bar{R}} \end{array} & \mathcal{C} \\ & \searrow \bar{L} \quad \swarrow u & \\ & \text{Mod}_{RL}(\mathcal{C}) & \end{array}$$

where u denotes the forgetful functor, Free_{RL} the free functor and $\bar{L}$ is the left adjoint of $\bar{R}$. We call the functor R *monadic* if $\bar{R}$ is an equivalence.

Example I.2.1. There is a free-forgetful adjunction

$$Ab \begin{array}{c} \xrightarrow{\otimes R} \\ \xleftarrow{u} \end{array} R\text{-Mod}$$

and $T = u(- \otimes R)$ acts on Ab . The modules over T are precisely the R -modules.

I.2.2 The Barr–Beck–Lurie theorem

The classification of modules over a monad is due to the Barr–Beck monadicity theorem [Bar69] [Bec68], which gives sufficient and necessary conditions for a category to be equivalent to being an algebra over a monad. Its generalization to ∞ -categories, due to Lurie [Lur17, Theorem 4.7.3.5], is the following theorem.

Theorem I.2.2 (Barr–Beck–Lurie theorem). *Let $\mathcal{C}$ be a ∞ -category and $\mathcal{D}$ be a ∞ -category which admits small colimits. Suppose that $L : \mathcal{C} \rightarrow \mathcal{D}$ is a functor that admits a left adjoint $R : \mathcal{D} \rightarrow \mathcal{C}$. Then there is an equivalence*

$$\bar{R} : \mathcal{D} \xrightarrow{\simeq} \mathrm{Mod}_{RL}(\mathcal{C})$$

if and only if

- (i) *R is conservative, and*
- (ii) *R preserves small colimits.*

By applying the Barr–Beck–Lurie theorem, Elmanto–Kolderup provides an axiomatic framework to characterize stable ∞ -categories that are modules over a motivic ring spectrum.

Theorem I.2.3 ([EK20b, Theorem 3.5]). *Suppose that S is a scheme such that $\mathcal{SH}(S)$ admits the projection formula. Then there is an equivalence of modules over the motivic cohomology spectrum MZ and Voevodsky’s category of motives $\mathcal{DM}(S)$.*

Their result is stated even more generally for general correspondence categories. The stable motivic homotopy category $\mathcal{SH}(S)$ admits the projection formula [EK20b, Definition 3.4] if a certain transformation in $\mathcal{SH}(S)$ is an equivalence. The transformation is induced by a general adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ of symmetric monoidal categories. Namely, if $\mathbb{1}_{\mathcal{C}}$ denote the tensor-unit of $\mathcal{C}$, then the transformation in [Theorem I.2.3] is the transformation

$$RL(\mathbb{1}_{\mathcal{C}}) \otimes (-) \longrightarrow RL(-) \tag{I.1}$$

Let $\mathrm{Map}_{\mathcal{C}}(-, -)$ denote the mapping space in an ∞ -category $\mathcal{C}$. Recall that an object X is called *strongly dualizable* in $\mathcal{C}$ if the canonical morphism $\mathrm{Map}_{\mathcal{C}}(M, \mathbb{1}) \otimes X \rightarrow \mathrm{Map}_{\mathcal{C}}(M, X)$ is an equivalence for all objects M . We say that $\mathcal{C}$ is *generated by strongly dualizable objects* if $\mathcal{C}$ is generated under colimits by a set of strongly dualizable objects. A standard computation provides a condition for when the transformation in (I.1) is an equivalence.

Lemma I.2.4. *Let*

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

be an adjunction of symmetric monoidal categories and let $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ and $\mathbb{1}_{\mathcal{D}}$ denote the respective units, and suppose that $\mathbb{1}_{\mathcal{C}} = R(\mathbb{1}_{\mathcal{D}})$. If $\mathcal{C}$ is generated by strongly dualizable objects, then the morphism in (I.1) is an equivalence.

Proof. Suppose that $X \in \mathcal{C}$ is strongly dualizable, and let DX denote its dual. Then a standard computation shows

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(T, RL(X)) &= \mathrm{Map}_{\mathcal{C}}(L(T), L(X)) \\ &= \mathrm{Map}_{\mathcal{C}}(L(T) \otimes DX, \mathbb{1}_{\mathcal{D}}) \\ &= \mathrm{Map}_{\mathcal{C}}(L(T \otimes DX), \mathbb{1}_{\mathcal{C}}) \\ &= \mathrm{Map}_{\mathcal{C}}(T \otimes DX, \mathbb{1}_{\mathcal{C}}) \\ &= \mathrm{Map}_{\mathcal{C}}(T, X) \end{aligned}$$

for which it implies that $X \cong RL(X)$ by the Yoneda embedding.

The assumption that $\mathcal{C}$ is generated by strongly dualizable objects and R preserves colimits, enables us to conclude the equivalence for all objects $X \in \mathcal{C}$. $\square$

In the motivic case, we replace $\mathcal{C}$ with the motivic stable homotopy category $\mathcal{SH}(S)$ and $\mathcal{D}$ with Voevodsky's big category of motives $\mathcal{DM}(S)$. Then the condition of [Lemma I.2.4](#) holds in characteristic 0 by [LYZ19](#), Corollary B.2], or more generally for $\mathcal{SH}(k, \Lambda)[1/p]$ for perfect fields k of exponential characteristic k . The perfection hypothesis was removed in [EK20a](#), Theorem 3.2.1]. This gives another proof of [RØ08](#), Theorem 1] that avoids having to deal with a Zig-Zag of Quillen equivalences between various model categories. However, if the ∞ -categories $\mathcal{C}$ and $\mathcal{D}$ admit a symmetric monoidal product, then the equivalence in [Theorem I.2.2](#) is not sufficient to give a equivalence of symmetric monoidal ∞ -categories. Instead we will rely on a stronger result.

Recall the following notions: Let $\mathcal{C}$ be a ∞ -category and $\mathrm{CAlg}(\mathcal{C})$ the ∞ -category of commutative algebra objects of $\mathcal{C}$. Let Cat_{∞} denote the $(\infty, 1)$ -category of all small ∞ -categories, and we let $(\mathrm{Cat}_{\infty})_{\mathcal{C}/}$ denote the full subcategory of Cat_{∞} spanned by ∞ -categories that admits small colimits and a functor which morphisms to $\mathcal{C}$. Then the stronger version of [Theorem I.2.2](#), which also captures the symmetric monoidal structure, is the following result due to J. Lurie:

Theorem I.2.5 ([Lur17](#), Corollary 4.8.5.21]). *Let $\mathcal{C}$ be a symmetric monoidal ∞ -category which admits small colimits, and suppose that the tensor product $\mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$ preserves small colimits separably in each variable. Then the functor $A \mapsto \mathrm{Mod}_A(\mathcal{C})$ determines a fully faithful embedding*

$$\theta : \mathrm{CAlg}(\mathcal{C}) \rightarrow \mathrm{CAlg}(\mathrm{Cat}_{\infty})_{\mathcal{C}/}$$

Moreover, a functor of symmetric monoidal ∞ -categories $L : \mathcal{C} \rightarrow \mathcal{D}$ belongs to the essential image of θ if and only if

- (i) *The functor R is conservative.*
- (ii) *The functor R preserves small colimits.*
- (iii) *The ∞ -category $\mathcal{D}$ admits small colimits and the tensor product $\mathcal{D} \otimes \mathcal{D} \rightarrow \mathcal{D}$ preserves small colimits separably in each variable.*

(iv) *The functor L admits a right adjoint R .*

(v) *For every objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the canonical morphism*

$$C \otimes R(D) \longrightarrow R(L(C) \otimes D)$$

is an equivalence.

We note that condition (i) and (ii) of [Theorem I.2.2](#) also appear as (i) and (ii) in the 5 conditions of [Theorem I.2.5](#). However, to obtain an equivalence of symmetric monoidal ∞ -categories, we also need to prove the three additional conditions (iii),(iv) and (v). We will place special emphasis on condition (v), which requires knowledge of the strongly dualizable objects in $\log\mathcal{SH}(k)$.

I.3 Modules over log motivic cohomology

Fix a quasi-compact quasi-separated base scheme S equipped with the trivial log structure. Let lSm_S denote the category of fine and saturated log schemes that are log smooth over S , and let SmlSm_S denote its full subcategory of log schemes X whose underlying scheme $\underline{X}$ is smooth. By [\[BPØ22b, A.5.10\]](#), objects $X \in \mathrm{SmlSm}_S$ can be described as pairs $(\underline{X}, D)$ where $\underline{X}$ is a smooth scheme and $D \subset \underline{X}$ is a strict normal crossing divisor. The idea of applying motivic homotopy theory in logarithmic geometry is based on the idea to parameterize homotopies by the log scheme

$$\square := (\mathbb{P}^1, \infty)$$

which can be considered as the compactification of the affine line $\mathbb{A}^1$ in logarithmic geometry. Specifically, the construction of the unstable logarithmic motivic homotopy category $\mathrm{lSpc}(S)_*$, introduced in [\[BPØ23, Definition 2.4.12\]](#), uses three ingredients:

- (i) Simplicial presheaves on lSm_S .
- (ii) Dividing Nisnevich topology.
- (iii) $\square$ -localization.

For every E_∞ -ring Λ , this gives a functor

$$S \longrightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{Mod}_\Lambda}^L), \quad S \mapsto \mathrm{lSpc}(S, \Lambda)_*$$

where $\mathrm{lSpc}(S, \Lambda)_*$ is the full subcategory of the ∞ -category spanned by pointed dividing Nisnevich sheaves on lSm_S with values in Mod_Λ that are $\square$ -invariant. Here Mod_Λ denotes the ∞ -category of Λ -modules in Sp . Generalizing the fundamental motivic equivalence

$$\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$$

of [MV99, Lemma 2.15], there is an equivalence

$$\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m^{\log}$$

in $\mathrm{ISpc}(S, \Lambda)_*$ by [BPØ23, Proposition 2.5.14], where S^1 denotes the constant log motivic space and $\mathbb{G}_m^{\log} := (\mathbb{P}^1, 0 + \infty)$. From the ∞ -category $\mathrm{ISpc}(S, \Lambda)_*$, one obtains the stable ∞ -category of logarithmic motivic spectra $\log\mathcal{SH}(S, \Lambda)$ by inverting the object $\mathbb{P}^1$ pointed at 1, a process described in [Rob15]. This describes $\log\mathcal{SH}(S)$ in terms of a limit

$$\lim \left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \mathrm{ISpc}(S, \Lambda)_* \xrightarrow{\Omega_{\mathbb{P}^1}} \mathrm{ISpc}(S, \Lambda)_* \right)$$

in Pr^R , or alternatively as a colimit

$$\mathrm{colim} \left(\mathrm{ISpc}(S, \Lambda)_* \xrightarrow{\Sigma_{\mathbb{P}^1}} \mathrm{ISpc}(S, \Lambda)_* \xrightarrow{\Sigma_{\mathbb{P}^1}} \cdots \right)$$

in the category Pr^L of pointed presentable monoidal ∞ -categories and left exact functors between them (i.e. functors which admit right adjoints) [Rob15, Corollary 2.22].

The functor that forgets the log structure

$$\omega : \mathrm{ISm} \longrightarrow \mathrm{Sm}, \quad X \longmapsto \underline{X}$$

induces an adjoint pair of functors

$$\omega_{\sharp} : \log\mathcal{SH}(S) \rightleftarrows \mathcal{SH}(S) : \omega^*$$

that allows us to compare the theory of logarithmic motivic spectra to the classical theory of motivic spectra. This adjunction was recently studied by D. Park, who showed that after assuming resolution of singularities, the functors $\omega_{\sharp}$ and ω^* preserve colimits and are monoidal [Par23, Theorem 1.1]. Moreover, the functor ω^* is fully faithful, with the essential image being the subcategory spanned by $\mathbb{A}^1$ -local objects.

There is also the logarithmic analogue of Voevodsky's category of motives $\mathcal{DM}(k, \Lambda)$ that was introduced in [BPØ22b] for a field k with trivial log structure, which we denote by $\log\mathcal{DM}(k, \Lambda)$. This ∞ -category is constructed as follows: Let $\mathrm{PSh}^{\mathrm{ltr}}(k, \Lambda)$ denote the ∞ -category of complexes of presheaves with log transfers, and $\mathrm{Shv}^{\mathrm{ltr}}(k, \Lambda)$ full subcategory spanned by complexes of dividing Nisnevich sheaves; see [BPØ22b, §4.3] for an approach using model categories. Then we define the ∞ -category of *effective* log motives $\log\mathcal{DM}^{\mathrm{eff}}(k, \Lambda)$ as the full subcategory of $\mathrm{Shv}^{\mathrm{ltr}}(k, \Lambda)$ spanned by $\square$ -local objects. The ∞ -category of *logarithmic motives* $\log\mathcal{DM}(k, \Lambda)$ is defined as the stable ∞ -category of $\mathbb{G}_m^{\log}$ -symmetric spectra in $\log\mathcal{DM}^{\mathrm{eff}}(k, \Lambda)$. There are adjoint functor pairs

$$\omega_{\sharp} : \log\mathcal{DM}^{\mathrm{eff}}(k, \Lambda) \rightleftarrows \mathcal{DM}^{\mathrm{eff}}(k, \Lambda) : \omega^*$$

and ω^* is fully faithful if k admits resolution of singularities [BPØ22b, Theorem 8.2.13]. Its essential image can be characterized as the full subcategory of $\log\mathcal{DM}^{\mathrm{eff}}(k, \Lambda)$ consisting of motives of log schemes whose underlying scheme is proper.

I.3.1 Universally constructing the adjunction

Our goal in this section is to relate the two symmetric monoidal ∞ -categories $\log\mathcal{SH}(S, \Lambda)$ and $\log\mathcal{DM}(S, \Lambda)$. For a perfect field k , we achieve this by constructing a pair of adjoint functors

$$\gamma_{\sharp} : \log\mathcal{SH}(k, \Lambda) \rightleftarrows \log\mathcal{DM}(k, \Lambda) : \gamma^*$$

that is characterized by preserving colimits and sending suspension spectra $\Sigma^{\infty} X_+$ to motives $M(X)$. This will be a generalization of the Eilenberg–MacLane functor

$$\mathcal{H} := \gamma^* : \mathcal{DM}(k, \Lambda) \longrightarrow \mathcal{SH}(k, \Lambda)$$

and the “Suslin” homology functor

$$h_s := \gamma_{\sharp} : \mathcal{SH}(k, \Lambda) \longrightarrow \mathcal{DM}(k, \Lambda)$$

The ∞ -category of motivic spectra $\log\mathcal{SH}(S)$ exhibits the following universal property.

Proposition I.3.1 ([BPØ23, Proposition 3.4.3]). *For every $S \in \mathbf{lSch}$ and pointed presentable symmetric monoidal ∞ -category $\mathcal{D}$, there is a naturally induced fully faithful functor*

$$\mathrm{Fun}^L(\log\mathcal{SH}(S, \Lambda), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathbf{lSm}_S, \mathcal{D})$$

The essential image consists of functors $\mathcal{F}$ satisfying $\square$ -invariance and dNis -descent such that the cofiber of $\mathcal{F}(1) \xrightarrow{\mathcal{F}(i_1)} \mathcal{F}(\mathbb{P}^1)$ is an invertible object and i_1 is the 1-section.

From this universal property, we get the adjunction.

Corollary I.3.2. *Let k be a perfect field and Λ be any ring. Then there is a unique adjunction of pointed presentably symmetric monoidal ∞ -categories*

$$\gamma_{\sharp} : \log\mathcal{SH}(k, \Lambda) \rightleftarrows \log\mathcal{DM}(k, \Lambda) : \gamma^* \quad (\text{I.2})$$

Proof. Let $X \in \mathbf{lSm}_k$. Then there is a functor $\mathbf{lSm}_k \rightarrow \log\mathcal{DM}(k, \Lambda)$ defined by sending $X \mapsto M(X)$. Letting $D := \log\mathcal{DM}(k, \Lambda)$ in Proposition I.3.1, the universal property this gives us a functor

$$\gamma_{\sharp} : \log\mathcal{SH}(k, \Lambda) \longrightarrow \log\mathcal{DM}(k, \Lambda) \quad (\text{I.3})$$

Since it preserves colimits, it has a right adjoint which we denote by γ^* . $\square$

Lemma I.3.3. *Let k be a perfect field and Λ be any ring. Consider the set of compact generators*

$$\{\Sigma^{p,q}\Sigma^{\infty} X_+ \mid X \in \mathbf{lSm}_k, p, q \in \mathbb{Z}\}$$

for $\log\mathcal{SH}(k, \Lambda)$. The image of this set under $\gamma_{\sharp}$ is a set of compact generators for $\log\mathcal{DM}(k, \Lambda)$.

Proof. The functor $\gamma_{\sharp}$ is characterized by sending

$$\Sigma^{p,q}\Sigma_+^{\infty}X \mapsto M(X)(q)[p]$$

and the set

$$\{M(X)(q)[p] \mid X \in \mathbf{ISm}_k, p, q \in \mathbb{Z}\}$$

forms a set of compact generators for $\log\mathcal{DM}(k, \Lambda)$ by [BPØ22b, Proposition 5.2.5]. $\square$

Lemma I.3.4. *Let k be a perfect field and Λ be any ring. Then the functor*

$$\gamma^* : \log\mathcal{DM}(k, \Lambda) \longrightarrow \log\mathcal{SH}(k, \Lambda)$$

preserves colimits.

Proof. By [Lur17, Proposition 1.4.4.1.(2)], it suffices to show that γ^* preserves coproducts. We therefore need to show that we have an equivalence

$$\mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \gamma^* \coprod Y) \cong \coprod \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \coprod \gamma^* Y)$$

for all sets of generators.

Let $C_{C \in \log\mathcal{SH}(k, \Lambda)}$ be a set of generators of $\log\mathcal{SH}(k, \Lambda)$. Then each object $\gamma_{\sharp}C$ is compact by [Lemma I.3.3]. Hence we have that

$$\begin{aligned} \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(\gamma_{\sharp}C, \coprod Y) &\cong \coprod \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(\gamma_{\sharp}C, Y) \\ &\cong \coprod \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \gamma^* Y) \\ &\cong \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \coprod \gamma^* Y) \end{aligned}$$

which completes the proof. $\square$

Lemma I.3.5. *Let k be a perfect field and Λ be any ring. Then the functor*

$$\gamma^* : \log\mathcal{DM}(k, \Lambda) \longrightarrow \log\mathcal{SH}(k, \Lambda)$$

is conservative.

Proof. Since $\log\mathcal{SH}(k, \Lambda)$ is compactly generated, it suffices to show that

$$\mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \gamma^* X) \longrightarrow \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(C, \gamma^* Y)$$

is an equivalence for every compact object $C \in \log\mathcal{SH}(k, \Lambda)$. By adjunction, this is equivalent to showing that

$$\mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(\gamma_{\sharp}C, X) \longrightarrow \mathrm{Map}_{\log\mathcal{SH}(k, \Lambda)}(\gamma_{\sharp}C, Y)$$

is an equivalence. But this follows since $\gamma_{\sharp}C$ is compact by [Lemma I.3.3]. $\square$

Proposition I.3.6. *Let k be a perfect field and Λ be any ring. Consider the diagram*

$$\begin{array}{ccc}
 \log\mathcal{SH}(k, \Lambda) & \xrightleftharpoons[\gamma^*]{\gamma_\#} & \log\mathcal{DM}(k, \Lambda) \\
 \omega_\# \downarrow \uparrow \omega^* & & \omega_\# \downarrow \uparrow \omega^* \\
 \mathcal{SH}(k, \Lambda) & \xrightleftharpoons[\gamma^*]{\gamma_\#} & \mathcal{DM}(k, \Lambda)
 \end{array} \tag{I.4}$$

Then there is an equivalence

$$\gamma_\# \omega_\# \cong \omega_\# \gamma_\#$$

Proof. Since the functors preserves colimits by [Lemma I.3.4](#), and $\log\mathcal{SH}(k, \Lambda)$ is generated under colimits by suspension spectra $\Sigma_+^\infty X$ for $X \in \mathbf{lSm}_k$, it suffices to check the equivalence

$$\omega_\# \gamma_\#(\Sigma_+^\infty X) \cong \gamma_\# \omega_\#(\Sigma_+^\infty X)$$

We explicitly compute both sides: The functor

$$\gamma_\# : \log\mathcal{SH}(k, \Lambda) \longrightarrow \log\mathcal{DM}(k, \Lambda)$$

is characterized by sending

$$\Sigma_+^\infty X \mapsto M(X)$$

and $\omega_\# : \log\mathcal{SH}(k, \Lambda) \rightarrow \mathcal{SH}(k, \Lambda)$ by sending $\Sigma_+^\infty X \mapsto \Sigma_+^\infty(X - \partial X)$. Hence on the one side we have

$$\omega_\# \gamma_\#(\Sigma_+^\infty X) \cong \omega_\# M(X) \cong M(X - \partial X)$$

On the other side we get

$$\gamma_\# \omega_\#(\Sigma_+^\infty X) \cong \gamma_\#(\Sigma_+^\infty(X - \partial X)) \cong M(X - \partial X)$$

Thus both sides coincide, giving the equivalence. $\square$

The endofunctor

$$T = \gamma^* \gamma_\# : \log\mathcal{SH}(k, \Lambda) \longrightarrow \log\mathcal{SH}(k, \Lambda)$$

is a monad, hence we can consider the ∞ -category $\mathrm{Mod}_{\gamma^* \gamma_\#}(\log\mathcal{SH}(k, \Lambda))$ of modules over the monad T . This admits the factorization

$$\begin{array}{ccccc}
 & & \gamma_\# & & \\
 \log\mathcal{SH}(k, \Lambda) & \xleftarrow{\quad} & & \xrightarrow{\quad} & \log\mathcal{DM}(k, \Lambda) \\
 & \searrow \text{free} & \gamma^* & \nearrow \gamma_\#^{\mathrm{enh}} & \\
 & \text{forget} \searrow & & \nearrow & (\gamma^*)^{\mathrm{enh}} \\
 & & \mathrm{Mod}_{\gamma_\# \gamma^*}(\log\mathcal{SH}(k, \Lambda)) & &
 \end{array} \tag{I.5}$$

Let $\mathrm{Mod}_{\gamma_\# \gamma^*}(\log\mathcal{SH}(k, \Lambda))$ denote the ∞ -category of modules over the monad $\gamma_\# \gamma^*$. We have now proven the conditions in the Barr–Beck–Lurie theorem [Theorem I.2.2](#), and as a result we get the following theorem.

Theorem I.3.7. *Let k be a perfect field and Λ be any ring. There is an equivalence of ∞ -categories*

$$\mathrm{Mod}_{\gamma_{\sharp}^*}(\log\mathcal{SH}(k, \Lambda)) \cong \log\mathcal{DM}(k, \Lambda)$$

Proof. Lemma I.3.5 shows that γ^* is conservative and preserves all colimits by Lemma I.3.4. Thus the conditions in (2) of Theorem I.2.2 are satisfied, which shows that the adjunction given by $\gamma_{\sharp}^{\mathrm{enh}}$ and $(\gamma^*)^{\mathrm{enh}}$ in (I.5) are inverse equivalences. $\square$

I.3.2 An explicit construction of the adjunction

In Section I.3.1 we used the universal property of $\log\mathcal{SH}(k, \Lambda)$ to construct the adjunction

$$\gamma_{\sharp} : \log\mathcal{SH}(k, \Lambda) \rightleftarrows \log\mathcal{DM}(k, \Lambda) : \gamma^*$$

However, it is also possible to provide an explicit construction of the adjunction, which we do in this section. Since it also satisfies the universal property of Proposition I.3.1 this construction will agree with the construction provided in the section above.

Let $\mathrm{lSpc}(k)_*$ denote the ∞ -category of log motivic spaces. Let Λ be a commutative unital ring, and consider the free Λ -presheaf functor

$$\mathrm{lSpc}(k)_* \longrightarrow \mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_* \quad (\text{I.6})$$

to the ∞ -category of Λ -modules, and the normalized chain complex functor that appeared in the Dold-Kan correspondence [Wei94, Theorem 8.4.1]

$$\mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_* \longrightarrow C^{\leq 0}(\mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_*) \quad (\text{I.7})$$

Then composing (I.6) and (I.7) with the canonical inclusion

$$C^{\leq 0}(\mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_*) \rightarrow C(\mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_*)$$

naturally induces a left Quillen functor

$$\mathrm{lSpc}(k)_* \longrightarrow C(\mathrm{lSpc}(\mathrm{lSm}_k, \Lambda)_*)$$

Taking symmetric S^1 -spectra on both sides, this induces a functor

$$\log\mathcal{SH}_{S^1}(k, \Lambda) \longrightarrow \log\mathcal{DA}^{\mathrm{eff}}(k, \Lambda)$$

where $\log\mathcal{DA}^{\mathrm{eff}}(k, \Lambda)$ is ∞ -category of effective log motives without transfers, consider notably in [BLPØ22] and [BLMP24]. Then taking $\mathbb{G}_m^{\log}$ -spectra on both sides, this induces a functor

$$\beta_{\sharp} : \log\mathcal{SH}(k, \Lambda) \longrightarrow \log\mathcal{DA}(k, \Lambda)$$

see [BPØ23, §2.8] for a more detailed explanation. Since $\beta_{\sharp}$ preserves small colimits, it has a right adjoint, which we denote by β^* . Furthermore, according to [BPØ22b, §4.3] there is a colimit preserving symmetric monoidal functor

$$\alpha_{\sharp} : \log\mathcal{DA}(k, \Lambda) \longrightarrow \log\mathcal{DM}(k, \Lambda)$$

Since it preserves colimits, it has a right adjoint, which we denote by α^* . This produces the adjunctions

$$\log\mathcal{SH}(k, \Lambda) \xrightleftharpoons[\beta^*]{\beta^\sharp} \log\mathcal{DA}(k, \Lambda) \xrightleftharpoons[\alpha^*]{\alpha^\sharp} \log\mathcal{DM}(k, \Lambda)$$

Composing the two adjunctions gives us the desired adjunction

$$\gamma_\sharp : \log\mathcal{SH}(k, \Lambda) \rightleftarrows \log\mathcal{DM}(k, \Lambda) : \gamma^*$$

of presentably symmetric monoidal stable ∞ -categories, where γ^* is right adjoint to γ_* , and γ_* preserves small colimits. Since it satisfies the universal property of [Proposition I.3.1](#), this construction agrees with the adjoint pair of functors in [Corollary I.3.2](#).

I.3.3 The logarithmic projection formula

In order to invoke [Theorem I.2.5](#), we especially need to show that the transformation

$$\gamma_\sharp \gamma^*(1) \otimes (-) \longrightarrow \gamma_\sharp \gamma^*(-) \quad (\text{I.8})$$

is an equivalence in $\log\mathcal{SH}(k, \Lambda)$. We call this transformation the *logarithmic projection formula*, which we will study in this section.

Let k be a perfect field and $M(k)$ the logarithmic motive of k in $\mathcal{DM}(k, \Lambda)$. The *logarithmic motivic Eilenberg–MacLane spectra* over k with Λ -coefficients is the pullback

$$\log\mathbf{M}\Lambda := \gamma^* M(k, \Lambda) \in \log\mathcal{DM}(k, \Lambda)$$

By [\[BPØ23 Proposition 7.7.8\]](#), $\log\mathbf{M}\Lambda$ is an $\mathbb{E}_\infty$ -ring spectrum, and if k admits resolution of singularities it is an oriented $\mathbb{E}_\infty$ -ring spectrum by [\[BPØ23 Proposition 7.7.9\]](#). Then [\(I.8\)](#) becomes the statement that

$$\gamma^* \gamma_\sharp(\Sigma_+^\infty X) \cong \Sigma_+^\infty X \wedge \log\mathbf{M}\Lambda \quad (\text{I.9})$$

should be an equivalence in $\log\mathcal{SH}(k, \Lambda)$.

For the stable motivic homotopy category, this is known for general fields when the exponential characteristic e is inverted by using [Lemma I.2.4](#) and the fact that $\mathcal{SH}(k)[\frac{1}{e}]$ is generated under sifted colimits by strongly dualizable objects [\[EK20a\]](#). However, we do not yet have a classification of the strongly dualizable objects in $\log\mathcal{SH}(k, \Lambda)$, and the equivalence of [\(I.9\)](#) is therefore unknown. However, by analogy with the classical situation, we also expect it to be true for logarithmic schemes, so we make the following conjecture:

Conjecture I.3.8. *Let k be a field and Λ be any ring. Suppose that the exponential characteristic e of k is invertible in Λ . Then for every $X \in \mathbf{LSm}_k$ there is an equivalence*

$$\gamma^* \gamma_\sharp(\Sigma_+^\infty X) \cong \Sigma_+^\infty X \wedge \log\mathbf{M}\Lambda$$

in $\log\mathcal{SH}(k)$.

We provide some evidence for [Conjecture I.3.8](#). First, recall from [\[BPØ22b, Definition 7.6.3\]](#) that a perfect field k admits *resolution of singularities* if it satisfies the following two conditions:

- (i) For any integral scheme X of finite type over k , there is a proper birational morphism $Y \rightarrow X$ of schemes over k such that Y is smooth over k .
- (ii) If $f: Y \rightarrow X$ is a proper birational morphism of integral schemes over k , such that X is smooth over k and where $Z_1, \dots, Z_n$ are smooth divisors forming a strict normal crossing divisor where

$$f^{-1}(X - (Z_1 \cup \dots \cup Z_n)) \rightarrow X - (Z_1 \cup \dots \cup Z_n)$$

is an isomorphism, then the following conditions holds: There is a sequence of blow-ups

$$X_n \xrightarrow{f_{n-1}^{-1}} X_{n-1} \xrightarrow{f_{n-2}^{-1}} \dots \xrightarrow{f_0} X_0 = X$$

along smooth center $W_i \subset X_i$ such that

- a) The composition $X_n \rightarrow X$ factors f .
- b) The subschemes W_i is contained in the preimage of $Z_1 \cup \dots \cup Z_n$ in X_n .
- c) The subscheme W_i has strict normal crossing with the sum of the reduced strict transforms of

$$Z_1, \dots, Z_r, f_0^{-1}(W_0), \dots, f_{i-1}^{-1}(W_{i-1})$$

in X_i .

We then prove the following special case of [Conjecture I.3.8](#).

Proposition I.3.9. *Assume that k is a perfect field that admits resolution of singularities. Let X be an fs log scheme log smooth over k , and assume that the underlying scheme $\underline{X}$ is proper over k . Then there is an equivalence*

$$\gamma^* \gamma_\#(\Sigma_+^\infty X) \cong \Sigma_+^\infty X \wedge \log \mathbf{MA}$$

in $\log \mathcal{SH}(k, \Lambda)$.

Proof. Since $\mathcal{SH}(k, \Lambda)$ is generated by strongly dualizable objects [\[LYZ19, Corollary B.2\]](#), [Lemma I.2.4](#) gives an equivalence

$$\gamma^* \gamma_\#(\Sigma_+^\infty(X - \partial X)) \cong \Sigma_+^\infty(X - \partial X) \wedge \mathbf{MZ}$$

for $X \in \mathbf{lSm}_k$. Applying ω^* , and using that it is monoidal by [\[Par23, Theorem 1.1.\(4\)\]](#), we get

$$\begin{aligned} \omega^* \gamma^* \gamma_\#(\Sigma_+^\infty(X - \partial X)) &\cong \omega^*(\Sigma_+^\infty(X - \partial X) \wedge \mathbf{MZ}) \\ &\cong \omega^*(\Sigma_+^\infty(X - \partial X)) \wedge \omega^*(\mathbf{MZ}) \end{aligned}$$

Assuming that X is proper, we have $\omega^*(\Sigma_+^\infty(X - \partial X)) \cong \Sigma_+^\infty X$ by [Par23, Theorem 1.1.(2)] and assuming resolution of singularities we have $\omega^* \mathrm{M}\mathbb{Z} \cong \log \mathrm{M}\mathbb{Z}$ by [BPØ23, Proposition 7.7.6]. Finally, since ω^* commutes with $\gamma^* \gamma_\#$, and since X is proper, the left-hand side equals $\gamma^* \gamma_\#(\Sigma_+^\infty X)$, which gives the equivalence

$$\gamma^* \gamma_\#(\Sigma_+^\infty X) \cong \Sigma_+^\infty X \wedge \log \mathrm{M}\mathbb{Z}$$

in $\log \mathcal{SH}(k, \Lambda)$. $\square$

Remark I.3.10. The results of [Par23] are expected to hold without assuming resolution of singularities by instead inverting the exponential characteristic and using Gabber's modification theorem [LM14a] and [LM14b]. This would immediately generalize [Proposition I.3.9] to all fields where the exponential characteristic is invertible in the coefficients.

Remark I.3.11. Very recently, Atiyah duality was proven for non- $\mathbb{A}^1$ -invariant proper motivic spectra in [AHI24] using the newly constructed (unpublished) Gysin morphism of Tang [Tan24]. The author expects that their approach, combined with the newly constructed Gysin morphism for logarithmic spectra by D. Park [Par24], would give a proof of [Proposition I.3.9] that avoids resolution of singularities.

Lemma I.3.12. *Assume [Conjecture I.3.8] and let $C \in \log \mathcal{SH}(k, \Lambda)$ and $D \in \log \mathcal{DM}(k, \Lambda)$ be any objects. Then the canonical morphism*

$$C \wedge \gamma^*(D) \xrightarrow{\sim} \gamma^*(\gamma_\#(C) \otimes D)$$

is an equivalence, where $\wedge$ denotes the smash product in $\log \mathcal{SH}(k, \Lambda)$ and $\otimes$ denotes the tensor product in $\log \mathcal{DM}(k, \Lambda)$.

Proof. Since γ^* and $\gamma_\#$ preserves colimits by [Lemma I.3.4], it is enough to check the equivalence for representatives $C = \Sigma^\infty X_+$ and $D = M(Y)$ for $X, Y \in \mathrm{ISm}_k$. By [Conjecture I.3.8], we have

$$\gamma^*(M(Y)) \cong \Sigma^\infty Y \wedge \log \mathrm{M}\Lambda$$

Hence the left-hand side equals $\Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \wedge \log \mathrm{M}\Lambda$. Moreover, the right-hand side equals

$$\begin{aligned} \gamma^*(\gamma_\#(\Sigma^\infty X_+) \otimes M(Y)) &\cong \gamma^*(M(X) \otimes M(Y)) \\ &\cong \gamma^*(M(X \times Y)) \\ &\cong \Sigma^\infty (X \times Y)_+ \wedge \log \mathrm{M}\Lambda \\ &\cong \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+ \wedge \log \mathrm{M}\Lambda \end{aligned}$$

in $\log \mathcal{SH}(k, \Lambda)$. Therefore both sides are equal, so the canonical morphism must be an equivalence. $\square$

I.3.4 The main theorem

Let $\log\mathcal{SH}_{\text{prop}}(k)$ denote the full subcategory of $\log\mathcal{SH}(k)$ that is generated by $\Sigma_+^\infty X$ where $X \in \text{ISm}_k$ is proper over k . Similarly, we define $\log\mathcal{DM}_{\text{prop}}(k)$ denote the full subcategory of $\log\mathcal{DM}(k)$ that is generated by $M(X)$ where $X \in \text{ISm}_k$ is proper over k . The adjunction from [Corollary I.3.2](#) restricts to an adjunction

$$\gamma_\# : \log\mathcal{SH}_{\text{prop}}(k, \Lambda) \rightleftarrows \log\mathcal{DM}_{\text{prop}}(k, \Lambda) : \gamma^*$$

We are now ready to prove our first main result:

Theorem I.3.13. *Assume that k is a perfect field that admits resolution of singularities and that Λ is any ring. Then there is an equivalence of presentably stable symmetric monoidal ∞ -categories*

$$\text{Mod}_{\gamma_\# \gamma^*(\mathbb{1})}(\log\mathcal{SH}_{\text{prop}}(k, \Lambda)) \simeq \log\mathcal{DM}_{\text{prop}}(k, \Lambda)$$

Proof. We have the symmetric monoidal functor

$$\gamma_\# : \log\mathcal{SH}(k, \Lambda) \rightarrow \log\mathcal{DM}(k, \Lambda)$$

and a fully faithful embedding

$$\theta : \text{CAlg}(\log\mathcal{SH}(k, \Lambda)) \hookrightarrow \text{CAlg}(\text{Cat}_\infty)_{\log\mathcal{SH}(k, \Lambda)}/$$

from commutative algebra objects in $\log\mathcal{SH}(k, \Lambda)$ to commutative algebra objects in $\text{CAlg}(\text{Cat}_\infty)$ which admits small colimits over $\log\mathcal{SH}(k, \Lambda)$ given by $A \mapsto \text{Mod}_A(\log\mathcal{SH}(k, \Lambda))$. We need to show that $\gamma_\#$ belongs to the essential image of θ , which will do by checking the five conditions of [Theorem I.2.5](#):

- (i) Condition (i) follows since the bifunctor $\otimes : \text{Pr}^{\text{L}} \times \text{Pr}^{\text{L}} \rightarrow \text{Pr}^{\text{L}}$ preserves small colimits in each variable by combining [Lur17](#), Proposition 4.8.1.17 and [Lur09](#), Theorem 5.5.3.18].
- (ii) Condition (ii) is by definition of the adjunction (F preserves colimits).
- (iii) Condition (iii) follows from the fact that γ^* preserves all colimits by [Lemma I.3.4](#)
- (iv) Condition (iv) is satisfied by [Lemma I.3.5](#)
- (v) By [Proposition I.3.9](#) and [Lemma I.3.12](#) we have that condition (v) is satisfied for objects $C \in \log\mathcal{SH}_{\text{prop}}(k, \Lambda)$ and $D \in \log\mathcal{DM}_{\text{prop}}(k, \Lambda)$. Therefore the restriction of the adjunction ([I.2](#)) to the adjunction

$$\gamma_\# : \log\mathcal{SH}_{\text{prop}}(k, \Lambda) \rightleftarrows \log\mathcal{DM}_{\text{prop}}(k, \Lambda) : \gamma^*$$

satisfies condition (v).

Hence, the restriction of $\gamma_\#$ to $\log\mathcal{SH}_{\text{prop}}(k, \Lambda)$ belongs to the essential image of θ , which by [Theorem I.2.5](#) identifies $\log\mathcal{DM}_{\text{prop}}(k, \Lambda)$ as the ∞ -category of modules over a commutative algebra object in $\log\mathcal{SH}_{\text{prop}}(k, \Lambda)$. $\square$

Remark I.3.14. Under resolution of singularities, it was proven in [BPØ22b, Theorem 8.2.17] that there is an equivalence of triangulated categories

$$\log \mathcal{DM}_{\text{prop}}^{\text{eff}}(k) \simeq \mathcal{DM}^{\text{eff}}(k)$$

and we know that

$$\log \mathcal{SH}_{\text{prop}, \text{ét}}(S) \simeq \mathcal{SH}_{\text{ét}}(S)$$

It is therefore expected that

$$\log \mathcal{SH}_{\text{prop}}(S) \simeq \mathcal{SH}(S)$$

Assuming [Conjecture I.3.8], we have the following corollary.

Corollary I.3.15. *Let k be a perfect field and Λ be any ring. Suppose that the exponential characteristic e of k is invertible in Λ and assume [Conjecture I.3.8]. Then there is an equivalence of presentably stable symmetric monoidal ∞ -categories*

$$\text{Mod}_{\gamma_{\#}\gamma^*(\mathbb{1})}(\log \mathcal{SH}(k, \Lambda)) \simeq \log \mathcal{DM}(k, \Lambda)$$

Proof. It remains to show condition (v) in [Theorem I.2.5], but this is a direct consequence of [Lemma I.3.12] by assuming [Conjecture I.3.8]. $\square$

Remark I.3.16. The first version of this theorem appeared in work of Röndigs–Østvær [RØ08] over fields of characteristic 0, and was later generalized to fields of characteristic $p > 0$, up to inverting p in work of Hoyois–Kelly–Østvær in [MSP17]. Cisinski–Déglise proved in [CD15] that the equivalence holds more generally for noetherian k -schemes of finite dimension provided one uses the cdh-topology and inverts the exponential characteristic. Furthermore, the equivalence has further been extended to general correspondences in [Gar19], and was placed in an axiomatic framework in [EK20b].

Applications of this theorem have been considered in [Lev08], which applies this characterization to show that each layer $s_n E$ in Voevodsky’s slice tower for E are naturally the Eilenberg–MacLane spectra of objects of $\mathbf{DM}(k, \Lambda)$. We give an analog for logarithmic motivic spectra in the next section. Other applications regarding zeta functions have been considered in [Gul06], but we do not discuss this further.

I.4 Log slices of log motivic spectra

I.4.1 The logarithmic slice filtration

In this section we introduce the logarithmic slice filtration developed in [BPØ24]. The logarithmic slice filtration is the generalization of the slice filtration for motivic spectra, introduced in [Voe04] and [SØ10], to the stable logarithmic motivic homotopy theory developed in [BPØ23].

The slice filtration has led to remarkable computations in motivic homotopy theory, most notably the computations of the first [RSØ19], and recently the

second stable homotopy groups of motivic spheres [RSØ24]. It has also notably produced a proof of the Milnor conjecture for quadratic forms [RØ16]. The paper [BPØ24] compared the logarithmic slice filtration with the Bhatt–Morrow–Scholze on topological Hochschild homology.

Voevodsky originally introduced this filtration as a means of establishing an analogue of the Postnikov tower in the motivic setting. For every motivic spectrum E , the slice filtration produces a sequence $s_n(E)$ of motivic spectra, where each level is known as the n -th slice of E . The slices $s_n(E)$ are analogous to the topological spectrum $\Sigma^n H\pi_n^s(E)$, where HA denotes the Eilenberg–MacLane spectrum corresponding to the group A and $\pi_n^s(E)$ denotes the n -th stable homotopy group of E . In this correspondence, slices may be view as replacement for stable homotopy groups for motivic spectra. Our main application of the previous section is that this correspondence extends to the logarithmic setting, that is, the slices that appear in the logarithmic slice filtration can naturally be identified with Eilenberg–MacLane spectra of logarithmic motives.

Let $\log\mathcal{SH}(S, \lambda)^{\text{eff}}$ denote the full subcategory of $\log\mathcal{SH}(S, \Lambda)$ spanned by objects $\Sigma_+^\infty X$ for $X \in \text{ISm}_S$. For every $q \in \mathbb{Z}$, the categories $\Sigma_{\mathbb{P}^1}^q \log\mathcal{SH}(k, \Lambda)^{\text{eff}}$ form a sequence of full embeddings

$$\dots \subset \Sigma_{\mathbb{P}^1}^{q+1} \log\mathcal{SH}(S, \lambda)^{\text{eff}} \subset \Sigma_{\mathbb{P}^1}^q \log\mathcal{SH}(S, \lambda)^{\text{eff}} \subset \Sigma_{\mathbb{P}^1}^{q-1} \log\mathcal{SH}(S, \lambda)^{\text{eff}} \subset \dots$$

Work of Neeman ([Nee96, Theorem 4.1]) generalizes to show that the inclusions

$$i_q^{\log}: \Sigma_{\mathbb{P}^1}^q \log\mathcal{SH}^{\text{eff}}(S, \lambda) \rightarrow \log\mathcal{SH}(S, \lambda)$$

have right adjoints $r_q^{\log}$. Since $i_q^{\log}$ is a full embedding, the adjunction $\text{id} \rightarrow r_q^{\log} i_q^{\log}$ is an isomorphism. Define the q -th effective log cover as

$$f_q^{\log} := i_q^{\log} r_q^{\log}$$

and the $q - 1$ -th *coeffective* cover as the cofiber

$$f_{\geq q}^{\log} E \rightarrow E \rightarrow f_{\leq q-1}^{\log} E$$

We define the q -th log slice of E as the cofiber

$$s_q^{\log} E := \text{cofib} \left(f_{q+1}^{\log} E \rightarrow f_q^{\log} E \right)$$

This gives a canonical functor

$$s_q^{\log}: \log\mathcal{SH}(S, \lambda) \longrightarrow \Sigma_{\mathbb{P}^1}^q \log\mathcal{SH}(S, \lambda) \subset \log\mathcal{SH}(S, \lambda)$$

that is uniquely determined up to isomorphism, and a cofiber sequence

$$s_q E \rightarrow f_{\leq q} E \rightarrow f_{\leq q-1} E \rightarrow \Sigma_{\mathbb{P}^1}^{1,0} s_q E$$

Assembling the various cofiber sequences for various q 's into a tower gives an exact couple, and the associated spectral sequence is a trigraded spectral sequence

called the *logarithmic slice spectral sequence* that generalizes the classical slice spectral sequence [Voe02] to logarithmic motivic spectra: For every logarithmic spectrum $E \in \log\mathcal{SH}(S)$, the E_1 -page is

$$E_{p,q,w}^1(E) = \pi_{p,w}^{\log}(s_q^{\log} E) \implies \pi_{p,w}^{\log} \left(\lim_{q \rightarrow \infty} (E/(f_q E)) \right) = \pi_{p,w}^{\log} sc(E)$$

with d^r -differentials

$$d^r : E_{p,q,w}^r \rightarrow E_{p-1,q+r,w}^r.$$

It converges conditionally to the logarithmic motivic homotopy groups of the slice completion $sc(E)$. There is also a very effective version of the slice spectral sequence, but we do not discuss this here; see [BPØ24] for further details.

Proposition I.4.1. *The logarithmic slice filtration is exhaustive, that is, for every logarithmic spectrum E there is an isomorphism*

$$\operatorname{colim}_{q \rightarrow -\infty} f_q^{\log}(E) \xrightarrow{\cong} E$$

Proof. This follows the same strategy as the classical case:

- (i) Each generator $\Sigma_{\mathbb{P}^1}^{p,q} X \in \log\mathcal{SH}(S)$ is contained in $\Sigma_{\mathbb{P}^1}^q \log\mathcal{SH}(S)$ for some $q \in \mathbb{Z}$.
- (ii) Each $f_q^{\log}$ preserves homotopy colimits.
- (iii) By the universal property of the log slices there is an isomorphism

$$\operatorname{Map}_{\log\mathcal{SH}(k,\Lambda)}(\Sigma_{\mathbb{P}^1}^{p,q} X, \operatorname{colim}_{q' < q} f_{q'}^{\log} E) \xrightarrow{\cong} \operatorname{Map}_{\log\mathcal{SH}(k,\Lambda)}(\Sigma_{\mathbb{P}^1}^{p,q} X, f_q^{\log} E)$$

and then using that $f_q^{\log} f_{q'}^{\log} \cong f_{q'}^{\log}$ for $q' < q$.

□

Proposition I.4.2 ([BPØ24, Proposition 3.3]). *Assume that k is a perfect field that admits resolution of singularities. Then for every $E \in \log\mathcal{SH}(k)$ and $n \in \mathbb{Z}$ there is an equivalence*

$$\omega^* f_n(E) \cong f_n^{\log} \omega^*(E)$$

Proof. By [Par23, Theorem 1.1.(4)], ω^* sends $\mathcal{SH}_{\leq 0}(k)$ to $\log\mathcal{SH}_{\leq 0}(k)$, or equivalently, $\omega_{\sharp}$ sends $\log\mathcal{SH}^{\text{eff}}(k)$ to $\mathcal{SH}^{\text{eff}}(k)$. Hence for every $E \in \mathcal{SH}(k, \Lambda)$ there is an morphism

$$\omega_{\sharp} f_{\leq n-1}^{\log} \omega^* E \longrightarrow \omega_{\sharp} \omega^* f_{\leq n-1} \cong f_{\leq n-1} E$$

which from the equivalence $\text{id} \xrightarrow{\cong} \omega_{\sharp} \omega^*$ induces an induces an isomorphism $f_{\leq n-1} \omega^* E \xrightarrow{\cong} \omega^* f_{\leq n-1}$. Thus the two right vertical morphisms in the morphism of cofiber sequences

$$\begin{array}{ccccc}
 f_n^{\log} \omega^* E & \longrightarrow & \omega^* E & \longrightarrow & f_{\leq n-1}^{\log} \omega^* E \\
 \downarrow & & \downarrow \cong & & \downarrow \cong \\
 \omega^* f_n E & \longrightarrow & \omega^* E & \longrightarrow & \omega^* f_{\leq n-1} E
 \end{array}$$

are isomorphisms, and we conclude by the five lemma. $\square$

The main result in this section is that, under resolution of singularities, the functor $\omega^*: \mathcal{SH}(k) \rightarrow \log \mathcal{SH}(k)$ commutes with taking slices.

Corollary I.4.3. *Assume that k is a perfect field that admits resolution of singularities. Then for every $E \in \log \mathcal{SH}(k, \Lambda)$ and integral n there is an equivalence*

$$\omega^* s_n(E) \cong s_n^{\log} \omega^*(E).$$

in $\log \mathcal{SH}(k, \Lambda)$

Proof. By applying ω^* to the cofiber sequence

$$f_n E \longrightarrow f_{n+1} E \longrightarrow s_n E \longrightarrow \Sigma_{\mathbb{P}^1}^{1,0} f_n E$$

we get a sequence

$$\omega^* f_n E \longrightarrow \omega^* f_{n+1} E \longrightarrow \omega^* s_n E \longrightarrow \Sigma_{\mathbb{P}^1}^{1,0} \omega^* f_n E$$

Applying [Proposition I.4.2](#) three times we then find

$$\begin{array}{ccccccc}
 \omega^* f_n E & \longrightarrow & \omega^* f_{n+1} E & \longrightarrow & \omega^* s_n E & \longrightarrow & \Sigma_{\mathbb{P}^1}^{1,0} \omega^* f_n E \\
 \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
 f_n^{\log} \omega^* E & \longrightarrow & f_{n+1}^{\log} \omega^* E & \longrightarrow & s_n^{\log} \omega^* E & \longrightarrow & \Sigma_{\mathbb{P}^1}^{1,0} f_n^{\log} \omega^* E
 \end{array}$$

By the five lemma we conclude that $\omega^* s_n(E) \rightarrow s_n^{\log} \omega^*(E)$ is an equivalence. $\square$

Corollary I.4.4. *Assume that k is a perfect field that admits resolution of singularities. Then there is an equivalence*

$$s_0^{\log}(\log M\Lambda) \cong \log M\Lambda$$

in $\log \mathcal{SH}(k)$.

Proof. By definition we have $\log M\Lambda := \omega^* M\Lambda$. Thus

$$s_0^{\log}(\log M\Lambda) \cong s_0^{\log} \omega^* M\Lambda \cong \omega^* s_0(M\Lambda)$$

It then is a classical fact by [Lev08](#) and [Voe04](#) that motivic cohomology agrees with its zero slice, since the morphisms

$$M\Lambda \leftarrow f_0 M\Lambda \rightarrow s_0 M\Lambda$$

are isomorphisms due to the fact that $M\Lambda$ is effective after assuming resolution of singularities. This gives the last equivalence

$$\omega^* s_0(M\Lambda) \cong \omega^* M\Lambda = \log M\Lambda$$

that completes the proof. $\square$

The following result gives a proof for the logarithmic version of [Voe04, Conjecture 7], which is also proved independently in [BPØ24, Theorem 3.4].

Proposition I.4.5. *Assume that k is a perfect field that admits resolution of singularities. The log slices of $\log KGL$ is given by*

$$s_q^{\log}(\log KGL) \cong \Sigma_{\mathbb{P}^1}^{2q,q} \log M\Lambda$$

Proof. We have $\omega^* KGL \cong \log KGL$ over fields that admit resolution of singularities by [BPØ24, Corollary 4.5]. Then $s_q^{\log}(\log KGL) \cong s_q^{\log} \omega^* KGL$ and using [Corollary I.4.3] we have the $s_q^{\log} \omega^* KGL \cong \omega^* s_q KGL$. By [RSØ19, Theorem 2.19] we have $s_q(KGL) \cong \Sigma_{\mathbb{P}^1}^{2q,q} M\Lambda$. Thus

$$\omega^* s_q(KGL) \cong \omega^* (\Sigma_{\mathbb{P}^1}^{2q,q} M\Lambda) = \Sigma_{\mathbb{P}^1}^{2q,q} \log M\Lambda$$

Combining these equivalences gives the result. $\square$

I.4.2 Log slices of the log algebraic cobordism spectrum

To define the logarithmic analogue of the algebraic cobordism spectrum, we follow [BPØ23, §7.6], which mimics Voevodsky's classical approach to construct the algebraic cobordism spectrum [Voe98, Section 6.3].

The *logarithmic Thom space* $Th(\mathcal{E}/X)$ of a rank d vector bundle $\mathcal{E} \rightarrow X$ with 0-section $Z \rightarrow X$ is the pointed log motivic space

$$Th(\mathcal{E}/X) := \frac{\mathcal{E}}{(Bl_Z \mathcal{E}, E)}$$

in $\mathbf{lSpc}(S)_*$, where E is the exceptional divisor on the blow-up $Bl_Z(\mathcal{E})$. Let $\mathcal{T}_{m,n}$ be the tautological bundle over the Grassmannian $Gr(m, n)$ with trivial log structure and let

$$\log MGL_m := L_{\square, dNis} \operatorname{colim}_n Th(\mathcal{T}_{m,n})$$

denote the m -th constituent space of log cobordism spectra $\log MGL \in \log \mathcal{SH}(S)$, where the structure morphisms are determined the morphisms

$$Th(\mathcal{T}_{m,n}) \otimes Th(\mathcal{T}_{r,s}) \longrightarrow Th(\mathcal{T}_{m+r, n+s})$$

in $\mathbf{lSpc}(S)_*$. However, in the case of resolution of singularities, we have the following simpler description.

Proposition I.4.6. *Assume that k is a perfect field that admits resolution of singularities. Then there is an equivalence*

$$\omega^* \text{MGL} \cong \log \text{MGL}.$$

in $\log \mathcal{SH}(k, \Lambda)$

Proof. Since we have that $\omega^* \Sigma_{\mathbb{P}^1}^\infty \cong \Sigma_{\mathbb{P}^1}^\infty \omega^*$, and $Th(\mathcal{T}_{m,n})$ is proper, the result follows from the identification

$$\omega^* L_{\mathbb{A}^1, Nis} Th(\mathcal{T}_{m,n}) \cong L_{\square, dNis} Th(\mathcal{T}_{m,n})$$

by using [Par23, Theorem 1.1.(2)]. $\square$

Corollary I.4.7. *Assume that k is a perfect field that admits resolution of singularities. Then $\log \text{MGL}$ admits an E_∞ -structure in $\log \mathcal{SH}(k, \Lambda)$.*

Proof. By [BPØ23, Proposition 7.6.2], $\log \text{MGL}$ admits a commutative monoid structure in $\log \mathcal{SH}(k, \Lambda)$. Then since MGL has an E_∞ -structure, we may use [Proposition I.4.6] and [Par23, Theorem 1.1.(4)] to upgrade the monoid structure to an E_∞ -structure. $\square$

Corollary I.4.8. *Assume that k is a perfect field that admits resolution of singularities. Then there is an isomorphism*

$$s_q^{\log}(\log \text{MGL}) \cong \Sigma^{2q,q} \log \text{MA} \otimes MU_{2q} \cong \bigvee_{I=(i_1, \dots, i_r)} \Sigma^{2q,q} \log \text{MA}$$

where the indices $i_j \geq 0$ is such that $\Sigma_{j=1}^r 2j \cdot i_j = 2q$.

Proof. The slices of MGL as known by combining the result of [Spi08, Corollary 4.7] with [Hoy15, Theorem 7.12], which gives

$$s_q(\text{MGL}) \cong \Sigma^{2q,q} \mathbb{M}\mathbb{Z} \otimes MU_{2q} \cong \bigvee_{I=(i_1, \dots, i_r)} \Sigma^{2q,q} \mathbb{M}\mathbb{Z}$$

Then the result follows from [Proposition I.4.6] by applying ω^* and using [Proposition I.4.2]. $\square$

I.4.3 Log slices of the log motivic sphere spectrum

Owing to [RSØ19, Theorem 2.12], the slices of the motivic sphere spectrum are given by a finite wedge product of suspensions of motivic Eilenberg-MacLane spectra

$$s_q(\mathbb{1}) \cong \bigvee_{p \geq 0} \Sigma^{2q,q} \text{MA} \otimes \text{Ext}_{MU_* MU}^{p, 2q}(MU_*, MU_*)[-p] \quad (\text{I.10})$$

Here the extension groups are calculated in comodules over the Hopf algebroid for the cobordism spectrum MU ; these form the E^2 -page of the classical Adams-Novikov spectral sequence [Rav86].

We can now prove the main theorem of [RSØ16] for logarithmic schemes over fields of characteristic 0.

Theorem I.4.9. *Assume that k is a perfect field that admits the resolution of singularities, and let Λ denote any ring. Let $\mathbb{1}^{\log}$ denote the log sphere spectrum in $\log\mathcal{SH}(k, \Lambda)$. Then there are isomorphisms of $\log\mathbf{M}\Lambda$ -modules*

$$\begin{aligned} s_q^{\log}(\mathbb{1}^{\log}) &\cong \Sigma^{2q,q} \log\mathbf{M}\Lambda \otimes \mathrm{Tot}(\pi_{2q} MU^{\bullet+1}) \\ &\cong \bigvee_{p \geq 0} \Sigma^{2q,q} \log\mathbf{M}\Lambda \otimes \mathrm{Ext}_{MU_* MU}^{p,2q}(MU_*, MU_*)[-p] \end{aligned}$$

in $\log\mathcal{SH}(k, \Lambda)$.

Proof. By Proposition I.4.3 we have the isomorphism

$$s_q^{\log} \omega^* \mathbb{1} \cong \omega^* s_q \mathbb{1}$$

in $\log\mathcal{SH}(k)$. By [RSØ19, Theorem 1.12] there is an isomorphism of $\mathbf{M}\Lambda$ -modules

$$\begin{aligned} s_q(\mathbb{1}) &\cong \Sigma^{2q,q} \mathbf{M}\Lambda \otimes \mathrm{Tot}(\pi_{2q} MU^{\bullet+1}) \\ &\cong \bigvee_{p \geq 0} \Sigma^{2q,q} \mathbf{M}\Lambda \otimes \mathrm{Ext}_{MU_* MU}^{p,2q}(MU_*, MU_*)[-p] \end{aligned}$$

in $\mathcal{SH}(k)$. Since $\omega^* \mathbb{1} \cong \mathbb{1}^{\log}$, and ω^* commutes with taking limits, we get identifications

$$\begin{aligned} s_q^{\log}(\mathbb{1}^{\log}) &\cong s_q^{\log} \omega^*(\mathbb{1}) \\ &\cong \omega^* s_q(\mathbb{1}) \\ &\cong \omega^* \Sigma^{2q,q} \mathbf{M}\Lambda \otimes \mathrm{Tot}(\pi_{2q} MU^{\bullet+1}) \\ &\cong \Sigma^{2q,q} \log\mathbf{M}\Lambda \otimes \mathrm{Tot}(\pi_{2q} MU^{\bullet+1}) \\ &\cong \bigvee_{p \geq 0} \Sigma^{2q,q} \log\mathbf{M}\Lambda \otimes \mathrm{Ext}_{MU_* MU}^{p,2q}(MU_*, MU_*)[-p] \end{aligned}$$

□

We state the case $n = 0$ as a corollary, which provides a proof of the main conjecture of [Voe02, Conjecture 10] in the logarithmic motivic setting.

Corollary I.4.10. *Assume that k is a perfect field that admits resolution of singularities, and let Λ be any ring. Then there is an isomorphism*

$$s_0^{\log}(\mathbb{1}^{\log}) = \log\mathbf{M}\Lambda$$

in $\log\mathcal{SH}(k, \Lambda)$.

This also shows that there is a multiplicative structure on the sphere log slices, which is a generalization of [RSØ16, Theorem 2.20].

Corollary I.4.11. *Assume that k is a perfect field that admits resolution of singularities. Then there is a morphism*

$$s_k^{\log}(\mathbb{1}^{\log}) \wedge_{\log\mathbf{M}\Lambda} s_l^{\log}(\mathbb{1}^{\log}) \longrightarrow s_{k+l}^{\log}(\mathbb{1}^{\log})$$

Proof. By [RSØ16, Theorem 2.20] there is a morphism

$$s_k(\mathbb{1}) \wedge_{s_0(\mathbb{1})} s_l(\mathbb{1}) \longrightarrow s_{k+l}(\mathbb{1}) \quad (\text{I.11})$$

Under resolution of singularities, ω^* is monoidal [Par23, Theorem 1.1], so applying ω^* to (I.11) we obtain a morphism

$$\omega^* s_k(\mathbb{1}) \wedge_{\omega^* s_0(\mathbb{1})} \omega^* s_l(\mathbb{1}) \longrightarrow \omega^* s_{k+l}(\mathbb{1}) \quad (\text{I.12})$$

Using that $\omega^* s_n \cong s_n^{\log} \omega^*$ by [Corollary I.4.3] and $\omega^* \mathbb{1} \cong \mathbb{1}^{\log}$, we may rewrite (I.12) as

$$s_k^{\log}(\mathbb{1}^{\log}) \wedge_{s_0^{\log}(\mathbb{1}^{\log})} s_l^{\log}(\mathbb{1}^{\log}) \longrightarrow s_{k+l}^{\log}(\mathbb{1}^{\log})$$

To conclude, we apply [Corollary I.4.10]. $\square$

Proposition I.4.12. *For each log motivic spectrum $X \in \log\mathcal{SH}(k, \Lambda)$, the log slice $s_n^{\log} X$ has a canonical structure of a $\log\mathcal{MA}$ -module.*

Proof. The theory of [GRSØ12] provides an axiomatic framework that describes the algebra and module structure on slices (first studied in [Pel11]), which applies to logarithmic slices. That is, each log slice has a canonical structure of a $s_0 \mathbb{1}^{\log}$ -module. Then the result follows from the identification $s_0^{\log}(\mathbb{1}^{\log}) \cong \log\mathcal{MA}$ in [Corollary I.4.10]. $\square$

The following corollary generalizes [Lev08, Corollary 10.5.2] and demonstrates how the results of [Section I.3], together with those established in this section, can be combined to show that the logarithmic slices of log motivic spectra admit a description as stable logarithmic motives.

Corollary I.4.13. *Let k be a perfect field and Λ be any ring. Suppose that the exponential characteristic e of k is invertible in Λ and assume [Conjecture I.3.8]. Let $E \in \log\mathcal{SH}(k, \Lambda)$ be a logarithmic motivic spectrum. Then for each $n \geq 0$, the logarithmic slice $s_n^{\log} E$ is naturally an Eilenberg–MacLane spectrum of objects from $\log\mathcal{DM}(k, \Lambda)$.*

Proof. The unit morphism $e : \mathbb{1} \rightarrow \log\mathcal{MZ}$ induces an isomorphism on the zeroth slice

$$e_0 : s_0^{\log}(\mathbb{1}) \rightarrow s_0^{\log}(\log\mathcal{MZ}) \cong \log\mathcal{MA}$$

by [Corollary I.4.10]. Then [Proposition I.4.12] shows that $s_n^{\log} E$ has the structure of a $s_0^{\log} \mathbb{1} \cong \log\mathcal{MZ}$ -module. Finally, assuming [Conjecture I.3.8], [Corollary I.3.15] identifies the ∞ -category of modules over $\log\mathcal{MZ}$ with the ∞ -category of stable log motives $\log\mathcal{DM}(k, \Lambda)$. $\square$

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