

UiO : **Department of Mathematics**
University of Oslo

Nikolai Opdan

Rational and irrational hypersurfaces

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Supervisor: John Christian Ottem



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Abstract

We study the rationality problem for hypersurfaces, that is, when a given hypersurface in projective space is rational. This is a classical problem in algebraic geometry, and as we will see, it is largely unsolved. Nevertheless, we examine some of what is known. This will lead us to survey some of the recent developments in the rationality problem for hypersurfaces.

1 Introduction

Algebraic geometry is essentially the study of varieties, that is, systems of polynomial equations. Polynomials first appear in algebra, but considering their sets of zeroes, it is clear that polynomials also have a geometric structure. This allow us to study polynomials not only by using algebraic methods, but it also allow for a geometric approach. By having, as it were, the «binocular vision», we are able to get geometric results by algebraic means, and visa versa. This interplay is at the heart of algebraic geometry.

General varieties can be difficult to handle: Fermat's last theorem states that the set to zeroes of the polynomial equation $x^n + y^n - 1$ over $\mathbb{Q}$ only contains the trivial solutions for $n > 2$. We therefore often restrict our attention to polynomial equations defined over algebraically closed fields, always having the complex numbers in mind. That being said, most of the definitions, and several of the results, can be modified to make sense over arbitrary fields. However, the problems presented here become much harder when working over non-algebraically closed fields, for example over $\mathbb{R}, \mathbb{Q}$, or $\mathbb{C}(t)$. It therefore makes sense to first understanding these problems over algebraically closed fields first before asking them for more general fields.

We are mostly interested in projective varieties, that is, varieties embedded in some projective space. They correspond to homogeneous polynomials, but locally studying the affine charts one gets the general case. Besides, many techniques are simplified and generalized when working over the projective space.

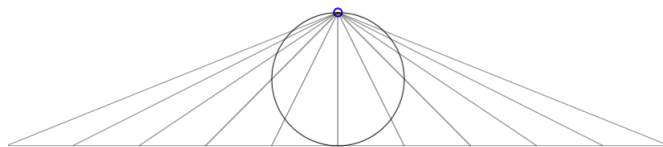


Figure 1: The stereographic projection from the north pole of S^1 to $\mathbb{R}$. This map is not defined at the north pole.

It is important to determine the maps between varieties as they carry a lot of information. One of the more famous maps is the stereographic projection. This map is not defined at the point from which we are projecting. This hints of a more general concept in algebraic geometry; we must consider maps that are not necessarily defined at the whole space. We therefore get the notion of rational maps:

Definition 1.1 ([HS00, p. 16]). A *rational* map $\phi: X \dashrightarrow Y$ (note the use of the dashed arrow) is a map between varieties that is a morphism on some nonempty open subset of X . A rational map is a *birational* map if it has an inverse. If there exists a birational map from X to Y , one says that they are *birationally equivalent*, or simply *birational*.

A rational map $\phi: X \dashrightarrow Y$ induces a homomorphism $\phi^*: K(Y) \rightarrow K(X)$ between the function fields of X and Y defined by $\phi^*: f \mapsto f \circ \phi$ and it is therefore clear that birational maps correspond to isomorphisms of the corresponding function fields.

Our prime example of a projective variety is the projective n -space $\mathbb{P}^n$. It is therefore important to ask which varieties are birational to $\mathbb{P}^n$. We call such varieties rational and it is a classical problem to classify such varieties. This is important because such a variety admits a nice parametrization and contains many rational curves, which gives important information about the variety.

Projecting from a general point on the variety, it can be shown that any variety is birational to a hypersurface. Hypersurfaces being the simplest varieties; varieties defined by a single polynomial. Thus it is enough to study rational hypersurfaces. In particular, our goal will be to try to answer if a generic projective hypersurface of degree d and dimension n is rational. The degree of the hypersurface being the degree of the defining polynomial.

Remark 1.2. A generic hypersurface corresponds to points in the complement of countably many closed hypersurfaces in the space of all hypersurfaces of the given condition.

The rationality problem for hypersurfaces is a classical problem. J. Lüroth showed in 1876 that if a curve C admits a rational dominant parametrization $f: \mathbb{P}^1 \dashrightarrow C$, then one can find another parametrization that is generically one to one [Lür76]. He uses this to show that curves are rational if and only if the genus is 0. The Italian geometers also knew that a quadric is rational and that a surface is rational if and only if it satisfies Castelnuovo's criterion. It is known that a hypersurface could be rational if it is very singular. On the other hand, it is not clear that there should exist irrational hypersurfaces. However, we will show that a smooth hypersurface cannot be rational if its degree is high enough (Theorem 6.1). On the contrary, the case for the generic hypersurface (of degree d and dimension n) was unsolved for a long time. It was not until the nineteen seventies that Clemens and Griffiths proved the irrationality of the cubic threefold [CG72] and Iskovskikh and Manin proved the irrationality of the quartic threefold [IM71]. These proofs developed new tools that enabled us to prove irrationality for a wider range of hypersurfaces. In 2019, Nicaise and Ottem proved the irrationality of the quartic fivefold [NO19], which provided recent process in the rationality problem. That being said, there is still much to be known for rational hypersurfaces, especially for hypersurfaces on low degree in high dimension.

It turns out difficult to determine whether or not a hypersurface is rational. We will therefore also mention some weaker notions than being rational. Expecting that these problems are easier to establish we hope that they may shed some light on the rationality problem.

1.1 Structure of the paper

The structure of this essay is organized as follows:

In section 2 we prove that a curve is rational if and only if its genus is 0. Doing so, we will need to evoke the Riemann-Roch theorem, which will require us to provide a brief introduction to differentials. Differentials will play an important role in the proof of several of the theorems presented here. In section 3 we prove the quadric is rational by generalizing the stereographic projection. Section 4 is devoted to a brief mention of the situation for surfaces, especially focusing on the cubic surface. In section 5 we prove that a cubic fourfold that contains two planes is rational, and discuss the general case (which is unsolved!). Section 6 studies hypersurfaces in high degree, while we in section 7 examine some of what is known about hypersurfaces of low degree in high dimension and present some open problems. Here we also provide a quick investigation on some weaker notions than rationality.

2 Curves

We start out by describing the situation for curves. This is completely settled and effectively characterised by the genus: A smooth projective curve is rational if and only if the genus is 0. Furthermore, two birational curves are actually isomorphic. Thus there exist only one curve in each birational equivalence class.

The following lemma turns out to be very useful.

Lemma 2.1 ([HS00, prop. A.1.4.4]). *Let ϕ be a rational map from a smooth projective variety X to a projective variety. Then*

$$\mathrm{codim}_X(X \setminus \mathrm{dom}(\phi)) \geq 2.$$

In particular, if X is a smooth curve and $\phi : X \dashrightarrow Y$ is a rational map, then ϕ is a morphism. This proves the following proposition.

Proposition 2.2. *Let X and Y be smooth projective curves. Then they are isomorphic if and only if they are birational.*

The classification of each birational equivalence class of curves is thus very trivial, since there is only one element.

One could think that the higher the degree the more complicated the curve. This is not quite true; There is a much more accurate tool for which to determine the complexity of a curve, namely the genus:

Definition 2.3. The *arithmetic genus* of a variety X is

$$p_a(X) := \dim H^1(X, \mathcal{O}_X).$$

The *geometric genus* of a projective curve C , denoted p_g , is the arithmetic genus of its normalization $\tilde{C}$. Moreover, it can be shown that C and $\tilde{C}$ are birational, thus it follows from Proposition 2.2 that the arithmetic genus and the geometric genus coincide. We may therefore refer to the *genus* $g(C)$ in place for both the geometric genus and the arithmetic genus on projective curves. It follows from the definition that the genus is a nonnegative integer. If the curve is smooth projective of $\mathbb{C}$, then its genus is the same as the number of holes in

the corresponding Riemann surface. It follows from Proposition 2.2 that the genus is a birational invariant.

We show that the genus of $\mathbb{P}^1$ is 0. Thus all rational curves have genus equal to 0.

Proposition 2.4. *A rational curve has genus 0.*

Proof. We have the complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \xrightarrow{d_0} \mathcal{O}_{\mathbb{P}^1}(U_1) \times \mathcal{O}_{\mathbb{P}^1}(U_2) \xrightarrow{d_1} \mathcal{O}_{\mathbb{P}^1}(U_1 \cap U_2) \longrightarrow 0,$$

where U_1, U_2 are the two affine charts and $d_0(s) = (s|_{U_1}, s|_{U_2})$ and $d_1(s, t) = s|_{U_2} - t|_{U_1}$. Moreover, we have that $\text{im } d_0 = k$ since the only global sections on $\mathbb{P}^1$ are constants. In addition, we have that $\mathcal{O}_{\mathbb{P}^1}(U_1) = k[x]$ and $\mathcal{O}_{\mathbb{P}^1}(U_2) = k[1/x]$. Thus if $d_1(s, t) = 0$ one would have that $g(x) = f(1/x)$ for some polynomials f and g . The only polynomials that satisfies is the constants. Hence,

$$g(\mathbb{P}^1) = p_a(\mathbb{P}^1) = \dim H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \dim \frac{\ker d_1}{\text{im } d_0} = \dim \frac{k}{k} = \dim 0 = 0.$$

■

To show the other implication we will have to evoke the Riemann-Roch theorem. It is a useful tool to count the dimension of linear systems and embeddings and it is thus very suitable for many applications in algebraic geometry. However, it uses the some theory on divisors and differentials, so we will provide a brief examination of these before presenting the theorem. Especially, we are interested in defining the canonical divisor K_X on a variety X . The definitions here are taken from [HS00, Chapter A.2].

Let $X \subseteq \mathbb{P}^n$ be smooth projective variety.

Definition 2.5. A *Cartier divisor* on X is a collection of pairs $(U_i, f_i)_{i \in I}$ such that

- (i) The U_i is an open set for each i , and these form a cover of X .
- (ii) The $f_i \in K(X)^*$ are nonzero rational functions.
- (iii) $f_i f_j^{-1} \mathcal{O}_X(U_i \cap U_j)^*$.

Two Cartier divisors $(U_i, f_i)_{i \in I}$ and $(V_j, g_j)_{j \in J}$ are considered equal if $f_i g_j^{-1} \in \mathcal{O}_X(U_i \cap V_j)$ for all i and j .

We define the sum of two Cartier divisors by

$$\{(U_i, f_i) : i \in I\} + \{(V_j, g_j) : j \in J\} = \{(U_i \cap V_j, f_i g_j^{-1}) : (i, j) \in I \times J\}.$$

Given a rational nonzero function $f \in K(X)^*$ we can define its Cartier divisor as

$$\text{Div}(f) = \{(X, f)\}.$$

Such a divisor is called a *principal Cartier divisor*. We say that two divisors are *linearly equivalent* if their difference is a principal Cartier divisor.

Now let X be a smooth projective variety of dimension n and let ω be a nonzero differential n -form on X . On any open affine subset U with local coordinates $x_1, \dots, x_n$ we can write

$$\omega = f_U dx_1 \wedge \dots \wedge dx_n$$

for some rational function $f_U \in k(X)$. We can then define a divisor on ω by

$$\text{Div}(\omega) = \{(U, f_U)\},$$

where the pairs (U, f_U) patch together to form a well-defined differential on X since they agree on the intersection $U_i \cap U_j$. Moreover, any nonzero differential n -form ω' has the form $\omega' = f\omega$ for some rational function f . Thus $\text{Div}(\omega') = \text{Div}(\omega) + \text{Div}(f)$. Hence the divisor class is independent of the chosen form given modulo linear equivalence. This divisor class is called the *canonical divisor* of X , denoted K_X , and is an important invariant of X . This shows the important feature of differentials: They can be used to define invariants on X , and we will see more on this later.

We let $\Omega_X^r = \bigwedge^r \Omega_X^1$ denote the $\mathcal{O}_X$ -module that consists of all regular r -forms on X and recall that $H^0(X, \mathcal{O}_X(D)) = \{f \in K(X) : \text{Div}(f) + D \geq 0\}$. We will use the following important fact several times throughout the paper.

Proposition 2.6 ([Kol19, Lemma 39]). *There are no global regular n -forms on $\mathbb{P}^n$.*

Proof. Let

$$U_1 = \mathbb{P}^n - Z(x_0) \simeq (x_1/x_0 : \dots : x_{n+1}/x_0) := (x_1, \dots, x_n)$$

and

$$U_2 = \mathbb{P}^n - Z(x_n) \simeq (x_0/x_n : x_{n-1}/x_n) := (y_0, \dots, y_{n-1})$$

be two affine charts. We can transition between there charts by

$$x_1 = \frac{y_1}{y_0}, \dots, x_{n-1} = \frac{y_{n-1}}{y_0}, x_n = \frac{1}{y_0}.$$

Thus we have that

$$\begin{aligned} dx_1 \wedge \dots \wedge dx_n &= d\frac{y_1}{y_0} \wedge \dots \wedge d\frac{y_{n-1}}{y_0} \wedge d\frac{1}{y_0} \\ &= \frac{(-1)^n}{y_0^{n+1}} \cdot dy_0 \wedge \dots \wedge dy_{n-1}. \end{aligned}$$

Now, any regular n -form on U_1 is of the form

$$f(x) \cdot dx_1 \wedge \dots \wedge dx_n,$$

for a rational function f . On the second chart it becomes

$$\frac{(-1)^n}{y_0^{n+1}} \cdot f\left(\frac{y_1}{y_0}, \dots, \frac{y_{n-1}}{y_0}, \frac{1}{y_0}\right) \cdot dy_0 \wedge \dots \wedge dy_{n-1}.$$

This map has a pole of order at least $n+1$ on $Z(y_0)$, hence it is not defined there. Thus there cannot be any global regular n -forms on $\mathbb{P}^n$. ■

We are now ready to present the Riemann-Roch theorem.

Theorem 2.7 (Riemann-Roch, [EO20, Theorem 19.6]). *Let D be a divisor on a curve C of genus g . Then*

$$\dim H^0(C, \mathcal{O}_C(D)) - \dim H^0(C, \mathcal{O}_C(K_C - D)) = \deg D + 1 - g.$$

Proof. The proof of this theorem uses tools that are not available to us in this paper, thus we only provide a sketch of the the common proof. The first part uses a duality theorem to express the left hand side as an Euler-Poincaré characteristic and the second part is to calculate this Euler-Poincaré characteristic. For example, see [Har77, Theorem IV.1.3]. ■

Having established this theorem we are now ready to prove that a curve of genus 0 is actually rational.

Proposition 2.8. *Let C be a projective curve of genus 0. Then C is isomorphic to $\mathbb{P}^1$. In particular, it is rational.*

Proof. First we apply Riemann-Roch to the divisor $D = 0$. We have that

$$\dim H^0(C, \mathcal{O}_C) = 1,$$

since the only regular functions on a projective variety are the constant functions (See for instance [Ell19, Theorem 4.2]). Thus

$$\dim H^0(X, \mathcal{O}_C(K_C)) = g.$$

Applying Riemann-Roch with $D = K_C$ gives $\deg(K_C) = 2g - 2$.

We now let p be a point on the curve and apply Riemann-Roch to the divisor $D = p$. Since $\text{Div}(f) + K_C - D \simeq K_C - D$, we have that $\deg(\text{Div}(f) + K_C - D) = \deg(K_C - D) = -3$ for all $f \in K(X)$. Hence

$$H^0(C, \mathcal{O}_C(K_C - D)) = \{f \in K(C) : \text{Div}(f) + K_C - D \geq 0\} \cup \{0\} = 0,$$

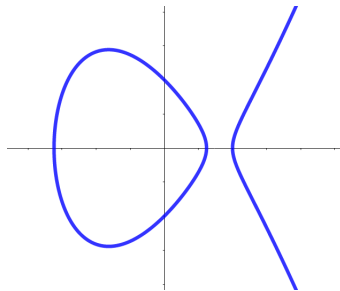
since effective divisors have nonnegative degree. Moreover, $\deg(D) = 1$, and $g = 0$ by hypothesis, thus $\dim H^0(C, \mathcal{O}_C(D)) = 2$. Thus there exists global nonconstant rational function on C . Such a function f defines a rational map $f: C \dashrightarrow \mathbb{P}^1$. The degree of the map equals the degree of the defining divisor, hence $\deg f = \deg D = 1$. Thus each fiber $f^{-1}(y)$ consists of $\deg(f) = 1$ point. Thus we are able to construct an rational inverse, proving that f is a birational map, and hence an isomorphism by Lemma 2.1. ■

Combining Proposition 2.4 and Proposition 2.8, we see that a curve is rational if and only if its genus is 0.

The Riemann-Roch-theorem also yields the following:

Theorem 2.9 (The genus-degree formula). *The genus of a nonsingular projective plane curve $C \subseteq \mathbb{P}^2$ of degree d is given by*

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Figure 2: The elliptic curve $y^2 = x^3 - 2x + 1$.

Example 2.10. A projective *elliptic curve* over k is a plane curve E on the form

$$y^2z = x^3 + axz^2 + bz^3$$

for values $a, b \in k$.

Let $E \subseteq \mathbb{P}^2$ be a projective elliptic curve. Since it is of degree 3 we get that $g(E) = \frac{3-1}{2} = 1$ from Theorem 2.9. We therefore conclude that elliptic curves are not rational.

More generally, by Theorem 2.9, the genus is 0 if and only if the degree is either 1 or 2. We therefore conclude that curves of degree larger than 2 is not rational.

3 Quadrics

The rationality problem for quadrics $X \subseteq \mathbb{P}^{n+1}$ is also an easy case: Every smooth quadric is rational.

For n equal to 1 or 2 one is able to find a classification up to isomorphism: Every smooth conic is isomorphic to $\mathbb{P}^1$ and every smooth quadric surface is isomorphic to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, which is birational to $\mathbb{P}^2$. For larger dimension, we are able to prove its rationality by generalizing the stereographic projection [Mij12], although it is still true that all smooth quadrics (over an algebraically closed field) are isomorphic to $\mathbb{P}^n$. Actually, it is not difficult to write out these maps explicitly.

Theorem 3.1 ([Mij12, Example 1.3]). *Let X be a quadric hypersurface in $\mathbb{P}^{n+1}$. Let $p \in X$ be a smooth point and pick a hyperplane $H \simeq \mathbb{P}^n$ that does not contain p . Then the projection from p to H is a birational map.*

Proof. By a linear change of variables, we may assume that $p = (0 : \dots : 0 : 1)$ and that $H = Z(x_{n+1})$. Then we get a projection

$$\pi: \begin{cases} \mathbb{P}^{n+1} & \dashrightarrow \mathbb{P}^n \\ [x_0, x_1 : \dots : x_{n+1}] & \mapsto [x_0, x_1 : \dots : x_n] \end{cases}$$

and its restriction to X is generically one to one. Moreover, assume that X is given by the zero locus of

$$f = \sum_{i,j=0}^n a_{i,j} x_i x_j + x_{n+1} \sum_{k=0}^n a_k x_k + a x_{n+1}^2,$$

where $p \in X$ implies $a = 0$. Then for $[x_0, x_1 : \dots : x_n] \in \mathbb{P}^n \setminus Z(\sum_{k=1}^n a_k x_k)$ we have the inverse map

$$\pi^{-1}|_X: \begin{cases} \mathbb{P}^n & \dashrightarrow \mathbb{P}^{n+1} \\ [x_1 : \dots : x_n] & \mapsto \left[x_1 : \dots : x_n : \frac{\sum_{i,j=0}^n a_{i,j} x_i x_j}{\sum_{k=0}^n a_k x_k} \right] \end{cases}.$$

Thus $\pi|_X$ is a birational map, proving that X is rational. $\blacksquare$

4 Surfaces

Before we take a look at general surfaces, we take a look at a special family of cubic surfaces; namely those that contains two lines. Projecting from these lines one can show that these are rational.

Theorem 4.1. *Let $S \subseteq \mathbb{P}^3$ be a cubic surface that contains two disjoint lines. Then S is rational.*

Proof. Let L_1 and L_2 be the two lines and pick $p_1 \in L_1$ and $p_2 \in L_2$. The line connecting them $\langle p_1, p_2 \rangle$ generically meets S in three points (by Bézout), and since we already know two of them we call the third $\phi(p_1, p_2)$. This gives a rational map

$$\phi: \begin{cases} L_1 \times L_2 & \dashrightarrow S \\ p_1 \times p_2 & \mapsto \phi(p_1, p_2) \end{cases}.$$

To get its inverse, let $q \in S$ and consider the linear span $\langle q, L_1 \rangle$ of q with L_1 and $\langle q, L_2 \rangle$ of q with L_2 . Let P be the unique intersection of L_2 with $\langle q, L_1 \rangle$ and Q the unique intersection of L_1 with $\langle q, L_2 \rangle$ respectively. We then get a rational map

$$\psi: \begin{cases} S & \dashrightarrow L_1 \times L_2 \\ q & \mapsto (P, Q) \end{cases}$$

which is one to one for all q such that q, P, Q are collinear (a dense open set). We have that ϕ and ψ are rational inverses, since ϕ gives the third intersection point while ψ , given the intersection point and the two lines, produces the two original points. It follows that S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, which is birational to $\mathbb{P}^2$. $\blacksquare$

Due to a well known theorem by Cayley and Salmon [Cay69] from 1849, any smooth projective cubic surface $S \in \mathbb{P}^3$ (over an algebraically closed field) contains precisely 27 lines and at least 2 disjoint lines. Thus any smooth cubic surface is rational from Theorem 4.1.

For general surfaces, things are more subtle. For curves we found that the genus completely characterised whether or not they were rational. It is therefore natural to ask whether a similar statement holds for surfaces: Castelnuovo proposed that the condition

$$H^0(S, \Omega_S^1) = H^0(S, K_S) = 0$$

would be enough to classify rational surfaces, but Enriques proposed a counterexample, now known as Enriques surfaces. Castelnuovo then found the right condition, now known as the Castelnuovo's criterion [Bea16].

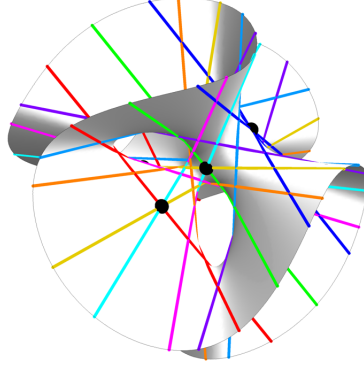


Figure 3: The Clebsch cubic with the 27 lines outlined [Ega16].

Theorem 4.2 (Castelnuovo’s criterion, [Har77, Theorem V.6.2]). *A surface $S \subseteq \mathbb{P}^3$ is rational if and only if $P_2(S) = 0$, where*

$$P_2(S) = \dim H^0(S, K_S^2).$$

In fact, P_2 is the second in a family of birational invariants. They are called *plurigenera* and are defined by

$$P_d(X) = \dim H^0(X, K_X^d)$$

of a variety X for all $d \in \mathbb{N}$. In particular, $P_1 = P_g$ for curves. On $\mathbb{P}^n$ they are all 0, thus they can be used to find irrational hypersurfaces. We will not study these invariants further, but they can be found in several books on algebraic geometry (ex. see [Har77, p. 190]).

5 Cubics

The case for all cubics of dimension more than 3 is an open problem and has been a field of intense study. The cubic threefold is irrational due to a famous result by Clemens and Griffiths from 1972 [CG72]. They proved that the smooth cubic threefold V_3 is irrational by studying the intermediate Jacobian of V_3 [CG72]. This method is quite efficient, but only works in dimension three.

It is not known, but it is widely believed that the generic cubic fourfold is irrational:

Conjecture 5.1. *The generic cubic fourfold is irrational.*

That being said, we are able to say something about some cubic fourfolds: Those containing two planes are rational. This proof resembles the proof given in Theorem 4.1. By projecting from the two planes we obtain a rational map:

Theorem 5.2. *Let X be a cubic fourfold. If X contains two distinct planes, then X is rational.*

d	$\vdots$							
8	X	X	X	X	X	X		
7	X	X	X	X	X	X		
6	X	X	X	X	X	X		
5	X	X	X	X	X	X		
4	X	X	X	X	?	?		
3	X	✓	X	?	?	?		
2	✓	✓	✓	✓	✓	✓	...	n

Figure 4: Given the degree d and the dimension n , is the generic hypersurface rational? «X» denotes «no», «✓» denotes «yes», and «?» denotes «unknown».

Proof. . Let H_1 and H_2 be the two planes and pick points $p_1 \in H_1$ and $p_2 \in H_2$. The line joining them $\langle p_1, p_2 \rangle$ generically meets X in a third point $\phi(p_1, p_2)$. We therefore get a rational map

$$\psi: \begin{cases} H_1 \times H_2 & \dashrightarrow X \\ (p_1, p_2) & \mapsto \phi(p_1, p_2) \end{cases}.$$

On the other hand, the intersection $H_1 \cap H_2$ parametrizes lines in $\mathbb{P}^5$. Thus choosing a point $q \in \mathbb{P}^5$ gives a unique line joining the two planes, giving rise to a rational map. This map is naturally an inverse of ϕ . Hence we have a birational map. ■

In general, it is possible to prove that if a cubic $X_3 \subseteq \mathbb{P}^{2n+1}$ contains two disjoint linear subspaces of dimension n , then X_3 is rational [Kol95] (there exists such smooth cubics), but it is an open problem if there exists smooth rational cubics of odd dimension.

For the general cubic of dimension greater than 3, calculating all known birational invariants turn up zero (in fact they are all zero when $d \leq n+1$) since any cubic fourfold can be deformed to the rational cubic fourfold that contains two planes. In the light of Theorem 5.2, a «useful» birational invariant would have to be very complicated because it would need to see the difference of whether or not X contains two planes.

6 Hypersurfaces in high degree

Reading the section on quartics and cubics one can come under the impression that the rationality problem gets harder and harder the higher the degree. This is not true. In fact, on the contrary, a hypersurface X is irrational if its degree is high enough and it is not difficult to prove so. This follows from the fact that for a given hypersurface of degree at least $n+2$ in $\mathbb{P}^{n+1}$ we have that $H^0(K_X) \neq 0$.

Theorem 6.1 ([Kol19, Theorem 35]). *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d . If $d \geq n+2$, then X is not rational.*

Proof. This will be an intimate consequence of Proposition 6.2 and Proposition 6.3. ■

The following two propositions show the usefulness of differentials.

Proposition 6.2 ([Kol19, prop. 40]). *Let X be a smooth projective hypersurface that has an everywhere defined, nonzero, n -form σ_X . Then X is not rational.*

Proof. If $\phi: \mathbb{P}^n \dashrightarrow X$ is a rational map, there is a closed subset Z of $\mathbb{P}^n$ with codimension 2 such that ϕ is everywhere defined on $\mathbb{P}^n \setminus Z$ by Lemma 2.1. Pulling back σ_X with ϕ one finds that $\phi^*\sigma_X$ is a regular n -form on $\mathbb{P}^n \setminus Z$.

By the nature of differentials, they are either defined at the whole space, or possibly it is undefined exactly along a hypersurface. Thus a differential on $\mathbb{P}^n \setminus Z$ extends to a everywhere defined n -form on $\mathbb{P}^n$, which is impossible by Proposition 2.6. ■

Proposition 6.3 ([Kol19, prop. 42]). *Let X be a smooth hypersurface of degree d . If $d \geq n + 2$, then X has an everywhere defined, nonzero, regular n -form.*

Proof. We provide an explicit construction.

Recall that $\mathbb{P}^{n+1}$ is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by the k^* -action

$$m_\lambda: (x_0, \dots, x_{n+1}) \mapsto (\lambda x_0, \dots, \lambda x_{n+1}).$$

Let $X = Z(H) \subseteq \mathbb{P}^{n+1}$ be the smooth hypersurface of degree d , and let C_X be the cone above it. Let C_X° denote the cone minus its vertex. It is possible to view a regular n -form on X as a k^* -equivalent n -form on C_X° .

Due to the smoothness of X , there is a defined and nowhere zero n -form on C_X° that is given by

$$\Sigma_X := \frac{1}{\partial H / \partial x_{n+1}} dx_0 \wedge \dots \wedge x_n.$$

In particular, we have that

$$m_\lambda^* \Sigma_X = \lambda^{n+1-(d-1)} \Sigma_X.$$

Now if G is a homogeneous polynomial, then

$$m_\lambda^*(G \Sigma_X) = \lambda^{\deg G + n - d + 2} G \Sigma_X.$$

Hence G is k^* invariant if and only if $\deg G = d - n - 2$. If $d \geq n + 2$ such a G exists and then $m_\lambda^*(G \Sigma_X)$ is a defined, nonzero, regular n -form on X . ■

Contrary to the theorem above, it is known that a hypersurface can be rational if it is very singular.

Example 6.4. Following the proof of Theorem 3.1 we find that if $X \subseteq \mathbb{P}^n$ is given by the zero locus of

$$H_{d-1}(x_0, \dots, x_n)x_{n+1} + H_d(x_0, \dots, x_n),$$

where H_i is a homogeneous polynomial of degree d , then projecting from $(0 : \dots : 0 : 1)$ to the hyperplane $H = Z(x_{n+1})$ gives a birational map. Its inverse is

$$(x_0 : \dots : x_n) \mapsto \left(x_0 : \dots : x_n : \frac{H_d(x_0 : \dots : x_n)}{H_{d-1}(x_0 : \dots : x_n)} \right).$$

If $d = 2$, then this is the stereographic projection. If $d > 2$, then $(0 : \dots : 0 : 1)$ is a singular point.

Example 6.5 ([Kol19, Example 21.3]). Take $n + 1$ homogenous polynomials $p_0(x), \dots, p_n(x)$ of degree d . They define a rational map

$$\varphi: \begin{cases} \mathbb{P}^n & \dashrightarrow \mathbb{P}^{n+1} \\ x & \mapsto (p_0(x), \dots, p_n(x)) \end{cases}.$$

If the $p_i(x)$ are general, then φ is a morphism and the image of φ is a rational hypersurface of degree d^n . The inverse function theorem allows us to find a rational local inverse for every point. Patching together these local inverses we obtain a global inverse. However, this hypersurface often has very complicated self-intersections.

7 Recent progress and open problems

As we saw in the previous chapter, much is known about hypersurfaces of high degree in low dimension. For lower degree hypersurfaces much is still unknown, but we present some of what is known.

In section 5 we saw that the cubic threefold was irrational due to a famous paper by Clemens and Griffiths [CG72] from 1972. In 1971, Iskovskih and Manin proved the long standing conjecture that the smooth quartic threefold V_4 is irrational [IM71] by showing that the group of birational automorphisms of V_4 is finite, while the corresponding group for $\mathbb{P}^n$ is enormous for all n (see [Har77, Example II.7.1.1]). Unfortunately, the calculation of the group of birational automorphisms on general hypersurfaces is not easy to work out and so far applies only to very specific hypersurfaces.

Example 7.1 (The group of birational automorphisms on a cubic). Let X be a cubic in $\mathbb{P}^{n+1}$ and fix a point $p \in X$. To any other point $q \in X$, consider the unique line $\langle p, q \rangle$ that goes through p and q . It generically intersects X in a third point q' . We thus get a birational automorphism ϕ_p that swaps the points q and q' for each point p in X . Thus $\text{Bir}(X)$ is huge!

Recently, Nicaise and Ottem showed that a very general quintic fivefold is irrational [NO19, Corollary 4.1.2], thus settling the case for $d = 4, n = 5$. Moreover, they showed that if a very general hypersurface of degree d and dimension n is stably irrational, then a very general hypersurface of degree $d + 1$ and dimension $n + 1$ is stably irrational as well [NO19, Theorem 4.3.1].

These three cases are the only settled problems for low degree hypersurfaces in high dimension. Because of this there are a lot of unsolved problems and conjectures.

We have already seen that it is unknown whether or not the cubic fourfold is rational. Some of the other famous conjectures are:

Conjecture 7.2. *The smooth generic cubic of dimension greater than 3 is irrational.*

More generally, we have been unable to find any rational hypersurface of degree greater than 3 that is rational.

Conjecture 7.3. *Every smooth generic hypersurface of dimension and degree greater than 3 is irrational.*

We know that a very general fourfold of degree 4 is irrational, but it is still unsolved if there exists any rational ones. The same goes for the five- and sixfolds in degree 5. We also note that it is not known of any rational smooth cubics in odd dimension.

As we have seen, it turns out very difficult to determine if a general hypersurface is rational. We therefore conclude our examination of rational hypersurfaces by introducing some weaker notions that hopefully is easier to calculate and thus maybe shedding some light on the rationality problem.

Definition 7.4. Let X be a variety of dimension n . We say that X is *ruled* if there exists a variety Y of dimension $n-1$ and a birational map $\phi: Y \times \mathbb{P}^1 \dashrightarrow X$. We say that X is *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$.

It is clear from the definition that stably rational varieties are ruled and that rational varieties are stably rational.

The most general theorem for hypersurfaces over $\mathbb{C}$ is provided by Kollár in [Kol95]. Using differential forms modulo p , he finds a variety that is not ruled and which can be obtained as a limit in a family of hypersurfaces.

Theorem 7.5 ([Kol95, Theorem 2]). *Let $X_d \subseteq \mathbb{P}^{n+1}$ be a generic hypersurface over $\mathbb{C}$ of degree*

$$d \geq \lceil \frac{2}{3}(n+3) \rceil.$$

Then X_d is not ruled.

In 2016, Totaro further showed that if $d \geq 2\lceil \frac{n+2}{3} \rceil$, then X_d is not stably rational [Tot16]. This bound was improved to $d \geq \lceil \log_2(n+2) \rceil$ by Schreieder in 2019 [Sch19].

It is a more subtle question whether stably rationality implies rationality. This is however false due to a construction of A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer from 1985 [Bea+85].

Definition 7.6. A variety X is *unirational* if there is a rational map $\mathbb{P}^n \dashrightarrow X$. That is, if X can be covered by a rational variety.

This is equivalent to asking if $K(X)$ can be embedded in $k(x_1, \dots, x_n)$. Clearly any (stably) rational variety is unirational, but it is not clear if the converse is true. This is the Lüroth problem that was asked by Jacob Lüroth in 1876 [Lür76]. It holds true for curves [Har77, Example 2.5.5] and surfaces (due to Castelnuovo's criterion Theorem 4.2), but turned out false with the constructions of Iskovskih and Manin [IM71] and Clemens and Griffiths [CG72] since they were known to be unirational. Artin and Mumford observed in 1971 that the torsion subgroup of $H^3(X, \mathbb{Z})$ is a stable birational invariant and used it to construct a irrational, but unirational, hypersurface in any dimension [AM72].

Definition 7.7 ([Bea16]). A variety is *rationally connected* if every two points can be connected by a rational curve.

It is clear that unirational varieties are rationally connected, but it is not known if the converse is true.

For curves and surfaces we are able to give a condition that completely characterised whether or not they were rational. Similarly, it is known that if

$H^0(K_X^m) = 0$ for all m , then X is ruled, but it is unknown whether the converse is true. Equally, if $H^0(\Omega_X^m) = 0$ for all m , then X is rationally connected, but again it is unknown whether the converse is true.

8 Conclusion

The question of whether a variety is rational is very interesting because it uses many different methods. To show that something is rational tends to use a lot of geometric arguments: Generalizing the stereographic projection, we proved that the quadric is rational (3.1) and projecting from two lines we showed that the cubic surface is rational (4.1). A similar argument proves that a cubic fourfold that contains two planes are rational. On the other hand, showing that something is irrational often revolves around finding, and calculating, birational invariants that takes on a specific value on $\mathbb{P}^n$. They are the genus, plurigenera, $\text{Bir}(X)$ ([IM71]), and global differential forms, as was presented here, and the fundamental group, $H^3(X, \mathbb{Z})_{\text{tors}}$ and the intermediate jacobian ([CG72]), to name a few. We also surveyed some of recent progress, which provided a significant progress in the question of rational hypersurfaces and surveyed some weaker conditions than rationality. Still, there are a lot of unsolved problems, especially for low degree hypersurfaces in high dimension.

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