

Enriques diagrams and equisingular strata of families of curves

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Motivation

- Count the number of curves in a given family which have certain types of singularities, or that have a given genus (e.g., are rational), and which satisfy some other conditions (like going through given points, or are tangent to a given curve).
- Even better: determine the generating function of the problem, if this makes sense. For example, if $N_r = \#$ curves with r nodes, then determine

$$\sum N_r t^r.$$

- More generally, the number N_r can be replaced with a cycle class on the parameter variety of the family.

Enriques diagrams

In a directed graph with no loops, if W is a predecessor or successor to a vertex V , then W is said to be *remote* if there is a distinct third vertex lying between V and W ; otherwise, W is said to be *immediate*.

A *tree* is a directed graph without loops. It has a single initial vertex, or *root*, and every other vertex has a unique immediate predecessor. A final vertex is called a *leaf*. A disjoint union of trees is called a *forest*.

An *Enriques diagram* is a finite forest $\mathbb{T}$ where each vertex V is assigned a weight $m_V \geq 1$. The forest is equipped with a binary relation, denoted ' $\succ$ ' and called *proximity*. A vertex V is called a *satellite* if it is proximate to a remote predecessor; if not, V is said to be *free*. A root is free, and a leaf can be either free or a satellite.

Law of Proximity

A root is proximate to no vertex. If a vertex is not a root, then it is proximate to its immediate predecessor and to at most one other vertex; the latter must be a remote predecessor. If one vertex is proximate to a second, and if a distinct third lies between the two, then it too is proximate to the second.

Proximity Inequality

For each vertex V ,

$$m_V \geq \sum_{W \succ V} m_W.$$

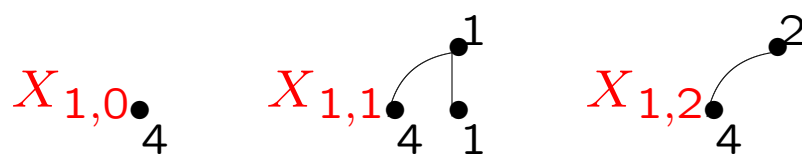
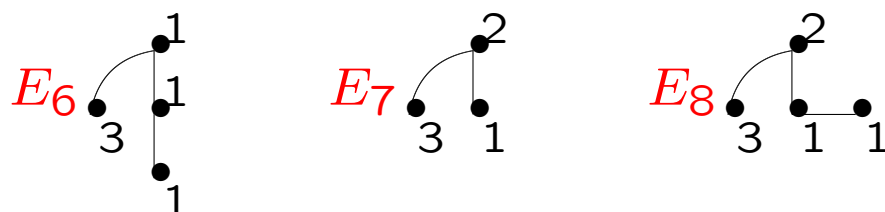
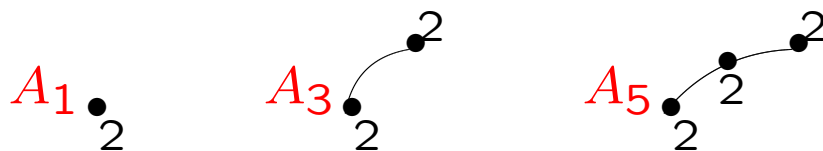
Law of Succession

A vertex V may have any number of free immediate successors, but at most two may be satellites, and they must be satellites of different vertices.

Law of Minimality

There are only finitely many vertices, and every leaf of weight 1 is a satellite.

Examples of diagrams



Numerical characters

Fix an Enriques diagram $\mathbf{T}$ and introduce some numerical characters:

$\text{rts}(\mathbf{T}) :=$ the number of roots

$\text{frs}(\mathbf{T}) :=$ the number of free vertices

$\text{sts}(\mathbf{T}) :=$ the number of satellites

In terms of the preceding characters, define the following ones:

$\text{dim}(\mathbf{T}) := \text{rts}(\mathbf{T}) + \text{frs}(\mathbf{T})$

$\text{deg}(\mathbf{T}) := \sum_{V \in \mathbf{T}} \binom{m_V + 1}{2}$

$\text{cod}(\mathbf{T}) := \text{deg}(\mathbf{T}) - \text{dim}(\mathbf{T})$

$\delta(\mathbf{T}) := \sum_{V \in \mathbf{T}} \binom{m_V}{2}$

$r(\mathbf{T}) := \sum_{V \in \mathbf{T}} (m_V - \sum_{W \succ V} m_W)$

$\mu(\mathbf{T}) := 2\delta(\mathbf{T}) - r(\mathbf{T}) + \text{rts}(\mathbf{T})$

$e(\mathbf{T}) := \mu(\mathbf{T}) + \sum_{R \text{ root of } \mathbf{T}} (m_R - 1)$

$\text{mod}(\mathbf{T}) := \mu(\mathbf{T}) - \text{cod}(\mathbf{T})$

$\bar{M}(\mathbf{T}) := \text{cod}(\mathbf{T}) - \delta(\mathbf{T})$

Lemma. Let T be an Enriques diagram with one root R , and let S be the set of satellites. Then:

$$\begin{aligned}
m_R &= r(T) + \sum_{V \in S} m_V \\
\bar{M}(T) &\geq 0 \text{ with } = \text{ iff } T = A_1 \\
\text{mod}(T) &\geq 0 \text{ with } = \text{ iff } T = A_k, D_k, E_6, E_7, E_8 \\
\mu(T) &\leq 2\delta(T) \text{ with } = \text{ iff } r(T) = 1 \\
\text{cod}(T) &\leq 2\delta(T) \text{ with } = \text{ iff } T = A_{2i}, E_6, E_8 \\
2\delta(T) &\leq e(T) \text{ with } = \text{ iff } m_R = r(T) \\
e(T) &\leq \text{cod}(T) + \delta(T) \text{ with } = \text{ iff } T = A_1, A_2 \\
e(T) &\leq 2 \text{cod}(T) \text{ with } = \text{ iff } T = A_1
\end{aligned}$$

Proof of (ii): We see that

$$\begin{aligned}
\bar{M}(T) &= \text{cod}(T) - \delta(T) \\
&= (m_R - 2) + \sum_{V \notin S, V \neq R} (m_V - 1) + \sum_{V \in S} m_V,
\end{aligned}$$

is a sum of three non-negative terms, and all are zero only for $T = A_1$.

Example: Fibonacci diagrams

Let φ_k denote the k th Fibonacci number:

$$\varphi_0 = 0, \varphi_1 = 1, \varphi_2 = 1, \varphi_3 = 2, \varphi_4 = 3, \dots$$

The k th Fibonacci diagram, for $k \geq 3$, is the weighted tree with one leaf, starting with a root of weight φ_k , followed by a free vertex of weight φ_{k-1} , followed by a succession of satellites with weights $\varphi_{k-2}, \dots, 1, 1$. It has the following numerical characters:

$$\text{rts} = 1, \text{frs} = 2, \text{sts} = k - 2$$

$$\text{dim} = 3$$

$$\text{deg} = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 1$$

$$\text{cod} = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 4$$

$$\delta = (\varphi_k - 1)(\varphi_{k+1} - 1)/2$$

$$r = 1$$

$$\mu = (\varphi_k - 1)(\varphi_{k+1} - 1)$$

$$e = (\varphi_k - 1)\varphi_{k+1}$$

$$\text{mod} = (\varphi_k - 3)(\varphi_{k+1} - 3)/2$$

$$\bar{M} = \varphi_{k+2} - 4$$

Zariski clusters

Assume $x \in C \subset S$ is a singular point on a reduced curve on a smooth surface. Let $\beta : S' \rightarrow S$ be a good embedded resolution — it determines a weighted diagram with a proximity relation, $\mathbf{T} = \{(V, m_V), \succ\}$, the *Enriques diagram* of the singularity. The Enriques diagram determines the topological type of the singularity.

The *Zariski cluster* of the singularity is the subscheme defined by the complete ideal

$$\mathcal{I} := \beta_* \mathcal{O}_{S'}(-\sum_V m_V E_V),$$

where E_V is the full pullback to S' of the exceptional divisor coming from blowing up the point V .

Geometric interpretation

Let $x \in C \subset S$ be a singular point of a reduced curve on a smooth projective surface and T its Enriques diagram. There are the following interpretations:

$$\text{rts}(T) = 1$$

$$\text{frs}(T) = \text{the number of sprouting blowups}$$

$$\text{sts}(T) = \text{the number of subdivisinal blowups}$$

$$\text{deg}(T) = \dim \mathcal{O}_C / \mathcal{I}$$

$$\dim(T) = \text{dimension of the } T\text{-stratum} \\ \text{of the Hilbert scheme}$$

$$\text{cod}(T) = \text{codimension of equisingular stratum} \\ \text{in versal deformation space}$$

$$\delta(T) = \text{the usual delta invariant (genus defect)}$$

$$r(T) = \text{the number of branches}$$

$$\mu(T) = \text{the Milnor number}$$

$$e(T) = \text{the multiplicity of the Jacobian ideal}$$

$$\text{mod}(T) = \text{the modality}$$

$$\bar{M}(T) = \text{the "rough" } M\text{-number}$$

Example: Fibonacci curves

The rational curve of degree φ_{k+1} ,

$$C_k: y^{\varphi_k} z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0$$

has precisely two cusps as singularities: the Fibonacci cusp, and a “semi-Fibonacci” cusp

$$z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0.$$

The strict transform of this cusp is a $(k-1)$ th Fibonacci cusp.

- The curves C_k are self-dual:

$$\begin{aligned} \deg C_k^* &= \\ 2\varphi_{k+1} - 2 - (\varphi_{k-1} - 1 + \varphi_k - 1) &= \varphi_{k+1}, \\ \text{and the cusps are “interchanged”.} \end{aligned}$$

- The curves C_k are “maximally inflected” (Kharlamov–Sottile).

- The sum of the $\bar{M}$ -numbers is

$$\varphi_{k+2} - 4 + \varphi_{k+1} - 1 + \varphi_{k-1} - 4 = 3\varphi_{k+1} - 9$$

(cf. a conjecture by Orevkov).

Orevkov's conjecture

If $C \subset \mathbb{P}^2$ is a rational curve of degree m , then

$$\bar{M} \leq 3m - 9.$$

Since C is rational,

$$\bar{M} = \text{cod} - \delta = \text{cod} - \frac{1}{2}(m-1)(m-2),$$

hence the conjecture is equivalent to

$$\text{cod} \leq \frac{1}{2}m(m+3) - 8.$$

For any reduced plane curve C of degree m , cod = the expected codimension of the set of curves with the same singularity type as C . This set contains the orbit of C under $\text{PGL}(3)$, which has dimension 8 if $g(C) \geq 2$. We have cod = the actual codimension if $\text{cod} < 4m - 4$ (using Wahl, Greuel-Lossen).

For the Fibonacci curves, the orbit has dimension 7, but the inequality is an equality, so cod is one less than expected.

For rational curves of degree $m \geq 10$, $\text{cod} \geq \delta = \frac{1}{2}(m-1)(m-2) \geq 4m - 4$.

If C is a Plücker curve of degree $m \geq 14$ and $g \geq 2$, with κ cusps, Lefschetz gave the inequality

$$\kappa \leq 3m + g - 9.$$

Note that $\bar{M}(A_1) = 0$ and $\bar{M}(A_2) = 1$, so $\bar{M} = \kappa$ in this case, and taking $g = 0$, we get a “special case” of Orevkov’s conjecture.

But for a rational Plücker curve, we have

$$\bar{M} = \kappa \leq \frac{3}{2}(m - 2).$$

This follows from the fact that for any rational curve of degree m we have

$$2k_0 + k_1 = 3(m - 2),$$

where k_i is the i th stationary index (the numbers of cusps and flexes, counted properly). But there is no way in general to compute $\bar{M}$ just from the stationary indices.

(If the curve is defined over $\mathbb{R}$, it is called *maximally inflected* if all the cusps and flexes are real.)

Families of curves

Let Y be an equidimensional Cohen–Macaulay scheme of finite type over the complex numbers. Let $\pi : F \rightarrow Y$ be a smooth projective family of surfaces, and D a relative effective divisor on F/Y .

For a given Enriques diagram $\mathbf{T}$, we want to define an “enumerating cycle” on Y , supported on the closure of the locus $Y(\mathbf{T})$ consisting of those points y such that D_y has Enriques diagram $\mathbf{T}$.

There are only finitely many diagrams arising from the fibers of D/Y .

(One shows that δ is bounded in the family, and this implies that the number of vertices V with weight ≥ 2 is bounded, as are those weights. One can then bound the number of satellites with weight 1.)

Hilbert schemes

Let $\mathbf{T}$ be an Enriques diagram, and set $d := \deg \mathbf{T}$. The set of Zariski clusters in the fibers of π defined by complete ideals $\mathcal{I}$ with diagram $\mathbf{T}$, determines a locally closed subset $H(\mathbf{T}) \subset \mathrm{Hilb}_{F/Y}^d$ (Nobile–Villamayor, Lossen).

Proposition. $H(\mathbf{T})$ is irreducible and smooth over Y , with fibers of dimension $\dim \mathbf{T}$.

Sketch of proof (cf. Ro  ): Construct, by induction on the number of vertices of $\mathbf{T}$, a Y -smooth, irreducible variety $P(\mathbf{T})$, with fiber dimension $\dim \mathbf{T}$, representing the functor of ordered configurations of (infinitely near) points with (unweighted) diagram $\mathbf{T}$. Forgetting the ordering gives a finite map to $H(\mathbf{T})$, hence $H(\mathbf{T})$ is irreducible and of the right dimension.

The enumerating cycle

The candidate for the enumerating cycle is the pushdown $U(\mathbf{T})$ of the fundamental cycle of the closure of $Z(\mathbf{T})$, where

$$Z(\mathbf{T}) := H(\mathbf{T}) \cap \mathrm{Hilb}_{D/Y}^d \subset \mathrm{Hilb}_{F/Y}^d.$$

Theorem 1. Assume the fibers of D are reduced and that for each $\mathbf{T}$ such that $Y(\mathbf{T}) \neq \emptyset$, we have

$$\mathrm{cod} Y(\mathbf{T}) \geq \min(r + 1, \mathrm{cod} \mathbf{T}).$$

Then either $Y(r\mathbf{A}_1)$ is empty, or it has pure codimension r ; in either case, its closure is the support of the natural nonnegative cycle $U(r\mathbf{A}_1)$.

Sketch of proof: $\dim Z(r) \geq \dim Y(r)$, since $\dim H(r) = \dim Y + 2r$ and $\mathrm{Hilb}_{D/Y}^d \subset \mathrm{Hilb}_{F/Y}^d$ is the zero scheme of a regular section of a rank $3r$ bundle. And $Z(r)/Y$ is finite, and 1-1 over $Y(r)$, and there are only finitely many $Y(\mathbf{T})$ with $\mathrm{cod} Y(\mathbf{T}) > r$.

The recursion

Write $U(r) := U(r\mathbf{A}_1)$. The key to computing a formula for $U(r)$ is the following proposition.

Proposition. Assume the family of curves D on $\pi : F \rightarrow Y$ is reduced in codimension $\leq r$ and that $\text{cod } Y(\mathbf{T}) \geq \min(r + 1, \text{cod } \mathbf{T})$. Set $r_i := r - \binom{i+1}{2} + 2$. Then the following formula holds:

$$ru(D, r) = \pi_* \sum_{i \geq 2} (-1)^i u(D_i, r_i).$$

Here D_i , $\pi_i : F_i \rightarrow Y_i$ are families derived by blowing up i -multiple points of the fibers of D . The new base Y_i is a modification of the scheme of zeros of the section of $\mathcal{P}_{F/Y}^{i-1}(D)$ induced by D .

Bell polynomials

The Bell polynomials $P_i(a_1, \dots, a_i)$ are defined via the formal identity in t :

$$\sum_{i \geq 0} P_i t^i / i! := \exp\left(\sum_{j \geq 1} a_j t^j / j!\right)$$

We can write

$$\begin{aligned} P_{n+1}(a_1, \dots, a_{n+1}) \\ = \sum_{k=0}^n \binom{n}{k} P_{n-k}(a_1, \dots, a_{n-k}) a_{k+1} \end{aligned}$$

For example,

$$P_0 = 1$$

$$P_1 = a_1$$

$$P_2 = a_1^2 + a_2$$

$$P_3 = a_1^3 + 3a_1a_2 + a_3$$

$$P_4 = a_1^4 + 6a_1^2a_2 + 4a_1a_3 + 3a_2^2 + a_4$$

If we assign a_j weight j , then $P_i(a_1, \dots, a_i)$ is weighted homogeneous of degree i .

The generating function

Ideally we want to determine the generating function

$$u(t) := \sum_{r \geq 1} u(D, r) t^r.$$

We note that

$$\begin{aligned} u'(t) &= \sum_{r \geq 1} r u(D, r) t^{r-1} \\ &= \sum_r \pi_* \sum_{i \geq 2} (-1)^i u(D_i, r_i) t^{r-1} \\ &= \sum_{i \geq 2} (-1)^i \pi_* u_i(t) t^{r-1-r_i}. \end{aligned}$$

Define

$$a(t) = \sum a_j / j! t^j := \ln u(t),$$

so that $u(t) = \exp a(t)$. This will give us

$$u(D, r) = P_r(a_1, \dots, a_r) / r!$$

and all we need now is to find an algorithm to determine the a_j .

Node polynomials

Theorem 2. With the above assumptions, if $r \leq 8$, then the rational equivalence class $[U(r)]$ is given by the formula

$$[U(r)] = P_r(a_1, \dots, a_r)/r!$$

where a_q is π_* of a polynomial, output by a certain algorithm, in the Chern classes $c_1(\mathcal{O}_F(D))$, $c_1(\Omega_{F/Y}^1)$, $c_2(\Omega_{F/Y}^1)$, and the P_j are Bell polynomials.

Sketch of proof: Rewrite

$$\exp a(t)a'(t) = \sum_{i \geq 2} (-1)^i \pi_* \exp a_i(t) t^{r-1-r_i}$$

We proceed by induction on r , and assume, for $j < r$, $a_j = \pi_* b_j - \pi_* b_j^\infty$, where the b_j are universal polynomials in the Chern classes of the family and b_j^∞ is a correction term which only enters in the final result for $r \geq 8$. We apply the induction hypothesis to the coefficients of $a_i(t)$; what makes it work is that we know how to compute the Chern classes, and their pushdowns, of the derived families, and because the generating function was differentiated.

Other enumerative problems

Another type of enumerative question is to ask how many non-isomorphic curves exist in a family with given types of singularities, leading to the question of which plane curves are rigid in the sense that the only isomorphic ones are the projectively equivalent ones. And one can ask how many translates of a given plane curve go through an appropriate number of points.

Back to the Fibonacci curves: the orbit O_{C_k} has dimension 7. Applying a result of Aluffi-Faber, we can compute the degree of the orbit:

$$\deg O_{C_k} = 6\varphi_{k+1}^2 \varphi_k^2 \varphi_{k-1}^2 - 84\varphi_{k+1}^2 + 252\varphi_{k+1} - 192$$

Checking for C_3 , the cuspidal cubic, we get as we should

$$\deg O_{C_3} = 24.$$