Chapter 9
Algebraic Spline Geometry: Some Remarks

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9.1 Algebraic Geometry and Geometric Modeling

Applications of algebraic geometry to Computer Aided Geometric Design often rely on mathematical foundations based on the use of complex numbers and projective geometry. However, all the considered practical questions are presented over the real numbers and in an affine setting. This obvious remark implies that one needs to focus on the study of *real algebraic geometry* and consider curves, surfaces and solids as semi-algebraic sets (i.e., sets defined by means of polynomial equalities and inequalities).

Certain aspects of the foundations of algebraic geometry, with special emphasis on classical projective geometry of curves and surfaces, need to be extended and developed for the real, affine, bounded – and in particular polyhedral – cases, having in mind applications to CAGD. Particular issues that could be considered are singularity theory: existence and description of real surface singularities; the theory of polar and dual varieties in order to relate the description of real polar varieties to Sturm–Habicht methods for determining the topology of real surfaces, and to find efficient ways of computing points on the components of real varieties [5, 6]; the theory of moduli spaces of varieties and of parameterized varieties and the study of the semi-algebraic stratification of these moduli spaces; the design of parametric catalogues of surfaces (parametric or not) that can be used in CAGD.

When modeling curves and surfaces algebraically, using just one interpolating polynomial does not necessarily give a good approximation, due to Runge's phenomenon, even if the polynomial has high degree. It is preferable to use

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piecewise polynomials to approximate larger regions of a CAGD-model. By using more polynomials, one can keep the polynomial degree low and still get better approximations. An *algebraic spline function* is simply a piecewise polynomial function on a polyhedral subdivision of a region in $\mathbb{R}^d$.

By replacing affine, real algebraic varieties in $\mathbb{R}^d$ by simplicial complexes Δ embedded in $\mathbb{R}^d$, and polynomial functions by algebraic spline functions, one gets a theory that we can call *algebraic spline geometry*. The purpose of such a theory is to establish a better basis for the application of multivariate algebraic splines to problems in geometric modeling. To bring real algebraic geometry and spline surface representation in CAGD closer together, we should also develop and understand certain aspects of the "classical" semi-algebraic real geometry so that it can extend to the theory of multivariate algebraic splines.

A simplicial complex in $\mathbb{R}^d$ can be thought of as a union of semi-algebraic sets, each determined by linear equations and linear inequalities, fitting together in a prescribed way. One can generalize this approach by using polynomials of arbitrary degree, as is done in [9], and to a certain extent in [12]. Still another point of view is the following (cf. [11]): replace ordinary algebraic sets in $\mathbb{R}^d$ (defined as zero sets of polynomials) by "piecewise" algebraic sets defined as zero sets of algebraic splines on some given domain in $\mathbb{R}^d$.

9.2 Algebraic Spline Spaces

Consider a (pure) d-dimensional simplicial complex Δ in $\mathbb{R}^d$. We let $C^r(\Delta)$ denote the set of piecewise polynomials (algebraic splines) on Δ of smoothness r. This set is a ring under the usual pointwise addition and multiplication. The (global) polynomial functions $\mathbb{R}[x_1, \ldots, x_d]$ are of course r-smooth for any r, and hence can be considered as a subring of every $C^r(\Delta)$. These functions will be called the *trivial splines*.

The subsets $C_k^r(\Delta) \subset C^r(\Delta)$ consisting of splines of degree $\leq k$ are $\mathbb{R}$-vector spaces, called *algebraic spline spaces*, and a major problem is to determine their dimension. In addition, one would like to find explicit bases for these spaces.

In the context of the SAGA project, Mourrain and Villamizar addressed the first problem in the case of $d = 2$ [7] and $d = 3$ [10, Ch. 3]. Using the homological approach introduced by Billera [1] and further developed by Billera–Rose [2, 3], they gave upper and lower bounds for the dimension of the vector spaces $C_k^r(\Delta)$.

Yuzvinsky [13] considered $C^r(\Delta)$ as the global sections of a sheaf $\mathcal{F}^r$ on the poset associated to Δ. He was, like Billera and Rose, interested in the structure of $C^r(\Delta)$ as a module over the ring of trivial splines, for example finding conditions for it to be a free module or a Cohen–Macaulay module, and conditions for the sheaf $\mathcal{F}^r$ to be flasque.

9.3 Generalized Stanley–Reisner Rings

Let Δ be a simplicial complex in $\mathbb{R}^d$, with vertices $\{v_1, \ldots, v_n\}$. The face ring, or Stanley–Reisner ring of Δ, is the ring $A_\Delta := \mathbb{R}[Y_1, \ldots, Y_n]/I_\Delta$, where I_Δ is the monomial ideal generated by the products $Y_{i_1} \cdots Y_{i_j}$ such that $\{v_{i_1}, \ldots, v_{i_j}\}$ is not a face of Δ. It is known [4] that if two simplicial complexes have isomorphic Stanley–Reisner rings, then they are themselves isomorphic.

By identifying Y_i with the Courant function $Y_i(v_j) = \delta_{ij}$ extended by linearity, we would have $\sum_i Y_i = 1$, and the ring $A_\Delta/(\sum Y_i - 1)$ can be viewed as the ring $C^0(\Delta)$ of (continuous) splines on Δ. This ring is sometimes referred to as the *affine* Stanley–Reisner ring of Δ. The points of $\mathrm{Spec}(C^0(\Delta))$ are the points on $\mathrm{Spec}(A_\Delta) \subset \mathbb{R}^n$ that lie on the hyperplane $\sum_i Y_i = 1$, and those points that have non-negative coordinates, give a model of Δ.

As observed by Billera and Rose [3], one can "homogenize" the Stanley–Reisner ring by replacing Δ by the $(d+1)$-dimensional simplicial complex $\hat{\Delta}$ obtained by embedding Δ in the hyperplane $x_0 = 1$ in $\mathbb{R}^{d+1}$ and taking its cone with vertex at the origin $\hat{v}_0 = (0, \ldots, 0)$. Then

$$C^0(\hat{\Delta}) = \mathbb{R}[\hat{Y}_0, \hat{Y}_1, \ldots, \hat{Y}_n]/(I_{\hat{\Delta}} + (\sum_{i=0}^{n} \hat{Y}_i - 1)) = A_{\hat{\Delta}}/(\sum_{i=0}^{n} \hat{Y}_i - 1) \cong A_\Delta,$$

where the variables $\hat{Y}_i$ represent the Courant functions of $\hat{\Delta}$ (see [8]). Since I_Δ is a homogeneous ideal, this ring is graded, and Proj $C^0(\hat{\Delta})$ also "describes" Δ.

For $r \geq 0$, consider the subalgebras $C^r(\Delta) \subseteq C^0(\Delta) = A_\Delta/(\sum Y_i - 1)$ and $C^r(\hat{\Delta}) \subseteq C^0(\hat{\Delta}) = A_\Delta$ consisting of splines of smoothness r. We call these rings *generalized Stanley–Reisner rings*. As observed by Billera [1], the space of splines on Δ of smoothness r and degree $\leq k$ correspond to the space of splines on $\hat{\Delta}$ of smoothness r and degree exactly k, in particular the dimensions of these two vector spaces are equal.

Here we concentrate on the "affine" generalized Stanley–Reisner rings $C^r(\Delta)$. In the following, we compute these rings and explain their geometric realization $\mathrm{Spec}(C^r(\Delta))$ only in two particularly simple cases. Nevertheless, we think that even these two examples support our conjecture, namely that the part of $\mathrm{Spec}(C^r(\Delta))$ with non-negative coordinates (with respect to a certain embedding into an affine space) models a twisted version of the simplicial complex Δ.

Example 1: $d = 1$**.** Let Δ be a one-dimensional simplicial complex with three vertices $v_1, v_2, v_3 \in \mathbb{R}$, and assume $v_1 < v_2 < v_3$ (Fig. 9.1).

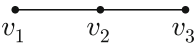

Fig. 9.1 The one-dimensional simplicial complex of Example 1

Since only $\{V_1, V_3\}$ is not a face, we have

$$C^0(\Delta) = A_\Delta/(\sum_{i=1}^{3} Y_i - 1) = \mathbb{R}[Y_1, Y_2, Y_3]/(Y_1Y_3, Y_1 + Y_2 + Y_3 - 1).$$

Then $\mathrm{Spec}(C^0(\Delta)) \subset \mathbb{R}^3$ is the union of the lines $Y_1 = Y_2 + Y_3 - 1 = 0$ and $Y_3 = Y_1 + Y_2 - 1 = 0$. The segments of these two lines contained in the positive octant mimic the two 1-faces of Δ, and they intersect transversally. Set

$$H := v_1Y_1 + v_2Y_2 + v_3Y_3.$$

Then $H(x) = x$ for any point $x \in |\Delta|$. So $\mathbb{R}[H]$ is the subring of trivial splines, and $\mathbb{R}[H] \subseteq C^r(\Delta)$, for any $r \geq 0$.

By performing the (affine) computations similar to what is done in [8], one sees that H, Y_1^{r+1}, and Y_3^{r+1} generate the subring $C^r(\Delta)$ of $C^0(\Delta)$. Note that, since

$$Y_1H = (v_1 - v_2)Y_1^2 + v_2Y_1 \text{ and } Y_3H = (v_3 - v_2)Y_3^2 + v_2Y_3$$

in $C^0(\Delta)$, all powers of Y_1 and Y_3 higher than $r + 1$ are in the subring generated by (the images of) H, Y_1^{r+1}, and Y_3^{r+1}. Now consider the homomorphism

$$\mathbb{R}[x, y, z] \to \mathbb{R}[Y_1, Y_2, Y_3]/(Y_1Y_3, Y_1 + Y_2 + Y_3 - 1)$$

that sends x to $H - v_2$, y to $(v_1 - v_2)^{r+1}Y_1^{r+1}$, and z to $(v_3 - v_2)^{r+1}Y_3^{r+1}$. The image of this map is $C^r(\Delta)$ and the kernel is the ideal $(yz, y + z - x^{r+1})$. Hence

$$C^r(\Delta) \cong \mathbb{R}[x, y, z]/(yz, y + z - x^{r+1}),$$

and $\mathrm{Spec}(C^r(\Delta))$ is the union of the plane curves $y = z - x^{r+1} = 0$ and $z = y - x^{r+1} = 0$. For $r \geq 1$, these two curves both have the x-axis as tangent at their point of intersection (the origin), and the tangent intersects each curve with multiplicity $r + 1$ (Fig. 9.2).

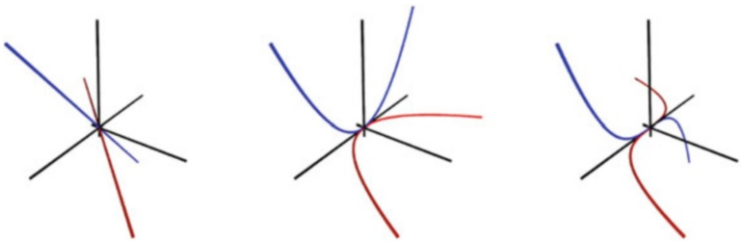

Fig. 9.2 $\mathrm{Spec}(C^r(\Delta))$ for $r = 0, 1, 2$

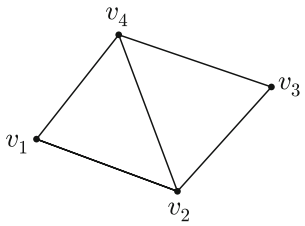

Fig. 9.3 The two-dimensional simplicial complex of Example 2

Example 2: $d = 2$**.** Consider a two-dimensional simplicial complex Δ, with four vertices $v_1, \ldots, v_4 \in \mathbb{R}^2$, no three on a line, and $\{v_1, v_3\}$ as the only non-face. Write $v_i = (v_{i1}, v_{i2})$ (Fig. 9.3).

Then

$$C^0(\Delta) = \mathbb{R}[Y_1, Y_2, Y_3, Y_4]/(Y_1 Y_3, Y_1 + Y_2 + Y_3 + Y_4 - 1),$$

where the variables Y_i represent the Courant functions. The trivial splines in this case are

$$H_j := v_{1j} Y_1 + v_{2j} Y_2 + v_{3j} Y_3 + v_{4j} Y_4, \text{ for } j = 1, 2.$$

As in the previous example, we observe that Y_1^{r+1} and Y_3^{r+1} are in $C^r(\Delta)$. Moreover, we deduce the following linear relation (assuming for simplicity that none of the denominators are zero):

$$\frac{H_1 - v_{21}}{v_{41} - v_{21}} - \frac{H_2 - v_{22}}{v_{42} - v_{22}} = \Big(\frac{v_{11} - v_{21}}{v_{41} - v_{21}} - \frac{v_{12} - v_{22}}{v_{42} - v_{22}}\Big) Y_1 + \Big(\frac{v_{31} - v_{21}}{v_{41} - v_{21}} - \frac{v_{32} - v_{22}}{v_{42} - v_{22}}\Big) Y_3.$$

Multiplying this relation by powers of Y_1 and of Y_3, we see that the subring of $C^r(\Delta)$ generated by the images of $H_1, H_2, Y_1^{r+1}, Y_3^{r+1}$ contain all powers of Y_1 and Y_3 higher than $r + 1$. Hence we get that $C^r(\Delta)$ is the image of the homomorphism

$$\mathbb{R}[x_1, x_2, y, z] \to \mathbb{R}[Y_1, Y_2, Y_3, Y_4]/(Y_1 Y_3, Y_1 + Y_2 + Y_3 + Y_4 - 1),$$

sending x_1 to $\frac{H_1 - v_{21}}{v_{41} - v_{21}}$, x_2 to $\frac{H_2 - v_{22}}{v_{42} - v_{22}}$, y to $\big(\frac{v_{11} - v_{21}}{v_{41} - v_{21}} - \frac{v_{12} - v_{22}}{v_{42} - v_{22}}\big)^{r+1} Y_1^{r+1}$, and z to $\big(\frac{v_{31} - v_{21}}{v_{41} - v_{21}} - \frac{v_{32} - v_{22}}{v_{42} - v_{22}}\big)^{r+1} Y_3^{r+1}$. The kernel of this homomorphism is generated by yz and $y + z - (x_1 - x_2)^{r+1}$, so that

$$C^r(\Delta) \cong \mathbb{R}[x_1, x_2, y, z]/(yz, y + z - (x_1 - x_2)^{r+1}).$$

Hence $\mathrm{Spec}(C^r(\Delta))$ is the union of the surfaces $y = z - (x_1 - x_2)^{r+1} = 0$ and $z = y - (x_1 - x_2)^{r+1} = 0$. The intersection of these surfaces is the line $y = z = x_1 - x_2 = 0$. The plane $y = z = 0$ is the tangent plane to both surfaces at all points

of their line of intersection. The intersection of this tangent plane and each surface is the line, with multiplicity $r+1$.

If we instead consider the "homogeneous" Stanley–Reisner ring $C^0(\hat{\Delta}) \cong A_\Delta$, we set $H_0 := Y_1 + Y_2 + Y_3 + Y_4$ and define the homomorphism

$$\mathbb{R}[x_0, x_1, x_2, y, z] \to \mathbb{R}[Y_1, Y_2, Y_3, Y_4]/(Y_1 Y_3),$$

by sending x_0 to H_0, X_1 to $\frac{H_1 - v_{21} H_0}{v_{41} - v_{21}}$, x_2 to $\frac{H_2 - v_{22} H_0}{v_{42} - v_{22}}$, and y and z as before. We then get

$$C^r(\hat{\Delta}) \cong \mathbb{R}[x_0, x_1, x_2, y, z]/(yz, y + z - (x_1 - x_2)^{r+1}).$$

This is a graded ring, where the x_i have weight 1 and y and z have weight $r+1$. Hence Conjecture 3.1 of [10, Ch. 3], stated only for $r = 1$ and with a different choice of variables, holds for all r. Note that $\mathrm{Spec}(C^r(\hat{\Delta}))$ is an affine cone, and that we can view $\mathrm{Spec}(C^r(\Delta))$ as the intersection of this cone with the hyperplane $x_0 = 1$.

We see that, as a module over the ring of trivial splines $\mathbb{R}[x_1, x_2]$, $C^r(\Delta)$ is isomorphic to $\mathbb{R}[x_1, x_2] \oplus y\mathbb{R}[x_1, x_2]$ (note that the isomorphism depends on r). Moreover, the r-splines of degree $\leq k$ are of the form $f(x_1, x_2) + yg(x_1, x_2)$, where f is a polynomial of degree $\leq k$ and g is of degree $\leq k - (r+1)$. Hence

$$\dim C_k^r(\Delta) = \binom{k+2}{2} + \binom{k+2-(r+1)}{2},$$

in accordance with [7, Thm. 13, p. 572], since there are no interior vertices in Δ.

It is natural to conjecture that the situation of these examples is the general one, namely that $\mathrm{Spec}(C^r(\Delta))$ represents a twisted version of Δ, in which the varieties corresponding to the d-faces are curved and intersect along varieties corresponding to the $(d-1)$-faces "with multiplicity $r+1$" as described above. More precisely, based on the above two examples, we conjecture the following "local" description of $\mathrm{Spec}(C^r(\Delta))$.

Local Spline Ring Conjecture *Let Δ be a d-dimensional simplicial complex consisting of two d-simplices intersecting in a $(d-1)$-simplex. Then we can realize $\mathrm{Spec}(C^r(\Delta)) \subset \mathbb{R}^{d+2}$ as the union of two smooth d-dimensional varieties V_1 and V_2 intersecting along a linear $(d-1)$-dimensional space L, such that V_1 and V_2 have the same d-dimensional linear space T as tangent space at each point of L and such that V_i and T have order of contact $r+1$ at each point of L.*

In [10, Ch. 4], Villamizar considered several examples of simplicial complexes Δ in the special case $d = 2$, $r = 1$, and gave conjectural generators for $C^1(\hat{\Delta})$ as a $\mathbb{R}$-subalgebra of $C^0(\hat{\Delta})$. The computations performed there for splines of low degree, using again the method of [8], support the above conjecture.

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