

Hilbert schemes and singular curves on surfaces

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In a series of papers we develop methods to enumerate singular curves on families of smooth surfaces. The goal is to express the enumerating cycles as universal polynomials in classes derived from the Chern classes of the family. Here I shall report on some aspects of this work that concerns Hilbert schemes.

Let Y be an equidimensional Cohen–Macaulay scheme of finite type over the complex numbers. Let $\pi : F \rightarrow Y$ be a smooth projective family of surfaces, and D a relative effective divisor on F/Y . Fix $r \geq 0$, and consider the set $Y(r)$ of points $y \in Y$ parameterizing the curves D_y with precisely r -nodes.

Theorem. Assume the fibers of D are reduced and that for each $\mathbf{D}$ such that $Y(\mathbf{D}) \neq \emptyset$, we have

$$\mathrm{cod}Y(\mathbf{D}) \geq \min(r + 1, \mathrm{cod}\mathbf{D}).$$

Then either $Y(r)$ is empty, or it has pure codimension r ; in either case, its closure is the support of a natural nonnegative cycle $U(r)$. Furthermore, if $r \leq 8$, then the rational equivalence class $[U(r)]$ is given by the formula

$$[U(r)] = P_r(a_1, \dots, a_r)/r!$$

where $a_q := \pi_* b_q$ and b_q is a polynomial, output by a certain algorithm, in the Chern classes $v = [D]$, $w_i = c_i(\Omega_{F/Y}^1)$, and the P_j are Bell polynomials.

The *Bell* polynomials $P_i(a_1, \dots, a_i)$ are defined via the formal identity in t :

$$\sum_{i \geq 0} P_i t^i / i! := \exp(\sum_{j \geq 1} a_j t^j / j!).$$

For example,

$$P_0 = 1,$$

$$P_1 = a_1,$$

$$P_2 = a_1^2 + a_2,$$

$$P_3 = a_1^3 + 3a_1a_2 + a_3,$$

$$P_4 = a_1^4 + 6a_1^2a_2 + 4a_1a_3 + 3a_2^2 + a_4.$$

If we assign a_j weight j , then $P_i(a_1, \dots, a_i)$ is weighted homogeneous of degree i .

Assume $x \in C \subset S$ is a singular point on a reduced curve on a smooth surface. Let $\beta : S^* \rightarrow S$ be a good embedded resolution, and let $\mathcal{C}$ denote the corresponding weighted configuration of infinitely near points of x . The configuration determines a weighted diagram with a proximity relation, $\mathbf{D} = \{(V, m_V), \succ\}$, the *Enriques diagram* of the singularity. The Enriques diagram determines the topological type of the singularity.

The *Zariski cluster* of the singularity is the subscheme defined by the complete ideal

$$\mathcal{I} := \beta_* \mathcal{O}_{S^*}(-\sum_V m_V E_V),$$

where E_V is the full pullback to S^* of the exceptional divisor coming from blowing up the point V . Note that a general element of $\mathcal{I}$ determines a curve singularity with Enriques diagram $\mathbf{D}$.

Set

$$\deg \mathbf{D} := \sum_{V \in \mathbf{D}} \binom{m_V + 1}{2}.$$

A classical formula asserts

$$\dim \mathcal{O}_S/\mathcal{I} = \deg \mathbf{D}.$$

Define also

$$\dim \mathbf{D} := \text{rts}\mathbf{D} + \text{frs}\mathbf{D},$$

where $\text{rts}\mathbf{D}$ is the number of roots of $\mathbf{D}$, i.e., the number of singular points, and $\text{frs}\mathbf{D}$ is the number of free (non-satellite) vertices of $\mathbf{D}$. We define the codimension of the diagram as

$$\text{cod}\mathbf{D} := \deg \mathbf{D} - \dim \mathbf{D}.$$

Let $\mathbf{D}$ be an Enriques diagram, and set $d := \deg \mathbf{D}$. The set of Zariski clusters in the fibers of π defined by complete ideals $\mathcal{I}$ with diagram $\mathbf{D}$, determines a locally closed subset $H(\mathbf{D}) \subset \text{Hilb}_{F/Y}^d$.

Proposition. $H(\mathbf{D})$ is smooth, irreducible, and of dimension $\dim \mathbf{D} := \text{rts} \mathbf{D} + \text{frs} \mathbf{D}$.

Sketch of proof: We construct by induction on the number of vertices of $\mathbf{D}$ a smooth, irreducible variety P parameterizing ordered configurations with (unweighted) diagram $\mathbf{D}$. Forgetting the ordering gives a map to $H(\mathbf{D})$, which is therefore irreducible and of the right dimension. Smoothness comes from Wahl's result, and we use Zariski's Main Theorem to show that $H(\mathbf{D})$ is the quotient of P by the automorphisms of the orderings.

In the case of a very ample linear system on a surface, we can give a different proof. We show that $Y(\mathbf{D})$ is nonempty (using Gotzmann's regularity theorem), smooth, locally closed, and of codimension $\text{cod} \mathbf{D}$, and we then deduce the result from this.

Let $Y(\mathbf{D})$ denote the set of closed points $y \in Y$ such that the fibre D_y has singularities precisely of type $\mathbf{D}$. We want to determine a *cycle* on Y , supported on the closure of $Y(\mathbf{D})$, that *enumerates* the curves in the family with this type of singularities.

The *candidate* for such an enumerating cycle is the pushdown $U(\mathbf{D})$ of the fundamental cycle of the closure of $Z(\mathbf{D})$, where

$$Z(\mathbf{D}) := H(\mathbf{D}) \cap \mathrm{Hilb}_{D/Y}^d \subset \mathrm{Hilb}_{F/Y}^d.$$

Unfortunately, it seems too hard to compute $U(\mathbf{D})$ directly — not enough is known about the cohomology ring of the Hilbert schemes.

In the case the family is a linear system of plane curves, Harris and Pandharipande were able to use this approach to compute formulas, but only for $\mathbf{D}$ equal to one, two, or three nodes. They used a resolution of the closure of $H(\mathbf{D})$, together with Bott's residue formula (in the manner of E-S). In the same situation, but for arbitrary many nodes, Göttsche gave the class as a Chern class of the pushdown of a certain vector bundle on the Hilbert scheme, but was unable to compute it in terms of Chern classes of the family.

In order to get a formula, we try instead to use a recursive procedure, which also involves properties of the Hilbert schemes.

Let $\pi : F \rightarrow Y$ be a family of smooth, projective surfaces, and $D \subset F$ a relative, effective divisor. Assume Y is equidimensional and Cohen–Macaulay.

Let X_i denote the scheme of points $x \in F$ at which $D_{\pi(x)}$ has multiplicity at least i ; more precisely, X_i is the scheme of zeros of a section of the bundle of twisted principal parts $\mathcal{P}_{F/Y}^{i-1}(D)$, namely, the section induced by the inclusion $D \subset F$. Then $X_0 = F$, $X_1 = D$, and $X_i \subseteq X_{i-1}$ for $i \geq 0$, because of the surjections

$$\mathcal{P}_{F/Y}^{i-1}(D) \rightarrow \mathcal{P}_{F/Y}^{i-2}(D).$$

Let $b : F' \rightarrow F \times_Y F$ be the blowup along the diagonal Δ , so $E := b^{-1}\Delta$ is the exceptional divisor. Let p_j be the projection on the j th factor, and set $\tilde{p}_j := p_j \circ b$. Then $\tilde{p}_2 : F' \rightarrow F$ is again a smooth, projective family of surfaces. Set $F_i := \tilde{p}_2^{-1}(X_i)$, and let $\pi_i : F_i \rightarrow X_i$ be the restriction of $\tilde{p}_2$. Consider the relative divisor

$$D_i := \tilde{p}_1^{-1}D - i(b^{-1}\Delta)$$

on F'/F . Note that $D_i|_{F_i}$ is effective; in fact, X_i is the largest subscheme of F over which this holds.

This setup is summarized in following diagram:

$$\begin{array}{ccccccc} F & \xleftarrow{p_1} & F \times_Y F & \xleftarrow{b} & F' & \hookleftarrow & F_i \hookleftarrow D_i \\ \downarrow \pi & & \downarrow p_2 & & \downarrow \tilde{p}_2 & & \downarrow \pi_i \\ Y & \xleftarrow{\pi} & F & \xlongequal{\quad} & F & \hookleftarrow & X_i \end{array}$$

Write $U(r) := U(r\mathbf{A}_1)$. The key to computing a formula for $U(r)$ is the following proposition.

Proposition. Assume the family of curves D on $\pi : F \rightarrow Y$ is reduced in codimension $\leq r$ and that $\text{cod}Y(\mathbf{D}) \geq \min(r + 1, \text{cod}\mathbf{D})$. Set $r_i := r - \binom{i+1}{2} + 2$. Then the following formula holds:

$$ru(D, r) = \pi_* \sum_{i \geq 2} (-1)^i u(D_i, r_i).$$

Proof. The proof is based on the following result:

There exists an open dense subset $U \subseteq F$ such that for all i , $X_i \cap U$ is dense in X_i , and a canonical isomorphism

$$Z(D_i, r_i)|_U \rightarrow Z(D_{i-1}, r_{i-1})|_U.$$

Take U to be the complement of $\pi^{-1}Y(\infty)$, the closure of $\pi^{-1}(Y(r))$, and all $\pi^{-1}(Y(\mathbf{D}))$ with $\text{cod}\mathbf{D} \geq r + 1$. There is a natural map $Z(D_i, r_i)|_U \rightarrow Z(D_{i-1}, r_{i-1})|_U$ given by “adding” the exceptional divisor E , and from the assumptions it follows that this map is bijective on the underlying topological spaces.

We shall show that it is surjective on tangent spaces.

Let A be a local ring with maximal ideal $\mathfrak{m}$, and assume $\mathfrak{m}^2 = 0$. A map

$$\mathrm{Spec} A \rightarrow Z(D_{i-1}, r_{i-1})|_U$$

corresponds to an infinitesimal equisingular r_{i-1} -nodal family of curves. By assumption, i of the r_{i-1} equimultiple sections (of multiplicity 2) of this family lie on the exceptional divisor E modulo $\mathfrak{m}$. We want to show that they also do so over A .

The situation is as follows: we have $f \in A[[x, y]]$, $f = f_{i-1} + f_i + \dots$, with $f_{i-1} \in \mathfrak{m}[[x, y]]$. Blowing up the ring gives $A[[x, y]][z]/(x - yz)$. Our family is given by the strict transform of f , $\tilde{f}(y, z)$. We need to show that $\tilde{f}$ has y as a factor, which is the same as showing that $f_{i-1} = 0$.

We can write $f_{i-1} = a_{i-1}x^{i-1} + a_{i-2}x^{i-2}y + \dots + a_0y^{i-1}$, with $a_j \in \mathfrak{m}$, and we may assume that modulo $\mathfrak{m}$, we have $\overline{f_i} = x(x-y)(x-2y)\dots(x-(i-1)y)$ and hence $f_i = \overline{f_i} + \sum_k b_k f'_k$, where $b_k \in \mathfrak{m}$ and the f'_k are monomials in x and y of degree i . We get $f(yz, y) = y^{i-1}\tilde{f}$, with

$$\begin{aligned}\tilde{f}(y, z) &= a_{i-1}z^{i-1} + a_{i-2}z^{i-2} + \dots + a_0 \\ &+ y(z(z-1)(z-2)\dots(z-(i-1))) + \sum_k b_k f''_k \\ &+ y^2(\dots),\end{aligned}$$

where the f''_k are polynomials in z of degree $\leq i$.

Assume i sections $s_j : A[[x, y]][z]/(x - yz, \tilde{f}) \rightarrow A$ are given by $s_j(y) = c_j$ and $s_j(z) = j + d_j$, with $c_j, d_j \in \mathfrak{m}$, $j = 0, \dots, i - 1$. We get i relations $\tilde{f}(c_j, j + d_j) = 0$. Taking $j = 0$, we see that $a_0 \in \mathfrak{m}^2$, hence $a_0 = 0$, and the other $i - 1$ relations give $i - 1$ independent linear relations (the coefficient matrix is a Vandermonde matrix) between $a_1, \dots, a_i$, showing that these must also be 0.

Therefore D_{i-1} splits as $D_i + E$ in the infinitesimal family, and so the family factors via X_i , since $X_i \subseteq X_{i-1}$ is the largest subscheme where this happens, according to the previous lemma. Hence the given tangent to $Z(D_{i-1}, r_{i-1})|_U$ comes from a tangent to $Z(D_i, r_i)|_U$.