

A Characterization of Balanced Rational Normal Scrolls in Terms of their Osculating Spaces

by
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1. Introduction

Consider a smooth, complex projective surface $X \subset \mathbf{P}^N$, $N \geq 5$, and assume X is not contained in a hyperplane. Recall ([2], [4]) that the m -th order osculating space to X at a point x is the linear subspace $\text{Osc}_X^m(x)$ of $\mathbf{P}^N$ determined by the partial derivatives of order $\leq m$ of the coordinate functions, with respect to a system of local parameters for X at x , and evaluated at x . At a general point $x \in X$, one expects the m -th order osculating space to X to have dimension $\binom{m+2}{2} - 1$, if m is such that this number is not greater than N . Points where the dimension of the osculating space is smaller than expected, are called points of hyperosculation — these are "flat" points of X . Certain surfaces are such that all points are points of hyperosculation, in this sense. For example, if X is a ruled surface, then

$$(L) \quad \dim \text{Osc}_X^2(x) \leq 4 \quad \text{for all points } x \in X$$

and hence $\dim \text{Osc}_X^m(x) \leq 2m$, for all m . Not all surfaces satisfying the condition (L) are ruled: Togliatti ([10]) gives as an example a special projection to $\mathbf{P}^5$ of the Del Pezzo surface of degree 6 in $\mathbf{P}^6$. (This seems to be the only known example of a non-ruled smooth surface satisfying (L).)

If a surface satisfies (L), one can show that, locally around a point, the coordinate functions of the surface satisfy a linear partial differential equation of order 2, or a Laplace equation — a term used classically by projective differential geometers (see [6], [10], [3], [7]).

Here we study those surfaces that satisfy condition (L) and do not exhibit further hyperosculating behavior. Our aim is to give a characterization of rational normal scrolls (of dimension 2) similar to the one for the Veronese embeddings of projective space given in ([2]).

Definition. A surface $X \subset \mathbf{P}^N$, $N \geq 5$, satisfies condition $(L^\sharp)$ if X is smooth, X is not contained in a hyperplane, X satisfies condition (L), and in addition $\dim \text{Osc}_X^n(x) = 2n$, for all $x \in X$, where $n = \lfloor \frac{N-1}{2} \rfloor$.

Then we have the following conjecture.

Conjecture. Let $X \subset \mathbf{P}^N$ be a smooth, projective surface that satisfies $(L^\sharp)$.

- (i) If $N = 2n+1$ is odd, then X is a balanced rational normal scroll, of degree $2n$ (i.e., of type (n, n)).
- (i) If $N = 2n+2$ is even, then X is a semi-balanced rational normal scroll, of degree $2n+1$ (i.e., of type $(n, n+1)$).

Recall ([5]) that a rational normal scroll X of type (d_1, d_2) is defined as the image of a $\mathbf{P}^1$ -bundle:

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2)) \rightarrow \mathbf{P}(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_1)) \oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_2))) \cong \mathbf{P}^{d_1+d_2+1}.$$

We call X *balanced* if $d_1 = d_2$, and *semi-balanced* if $|d_1 - d_2| = 1$. A balanced scroll is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and a semi-balanced scroll is isomorphic to the surface $\mathbf{F}_1$ obtained from $\mathbf{P}^2$ by blowing up a point. Note that both $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_1$ have Chern numbers $c_2 = 4$ and $c_1^2 = 8$.

The only rational normal scrolls satisfying $(L^\sharp)$ are the balanced and semi-balanced ones; moreover, these are the only ones with the property that the strict dual variety $(X^*)^*$ of the strict dual variety X^* of X is equal to X ; the balanced ones are the only ones such that X and X^* are isomorphic ([5]).

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Case (i) of the above Conjecture was formulated by the second named author and proved by him in the case $n = 2$, under some additional assumptions ([9]). The purpose of the present paper is to give a proof in case (i) that works for $n \leq 4$. In particular, by taking $n = 2$, this proves the conjecture in ([7]).

We begin the next section by recalling the definition of osculating spaces and strict dual variety. Then we prove several lemmas and a proposition, that are valid for any surface $X \subset \mathbf{P}^{2n+1}$ satisfying $(L^\sharp)$. We then prove our main result by showing how to deduce case (i) of the Conjecture from the Proposition, under the additional assumption $n \leq 4$.

In the last section we discuss briefly how one could approach case (ii) of the Conjecture. We also speculate on the possibility of proving directly that a surface in $\mathbf{P}^N$, satisfying (L) and with $N \geq 6$, must be linearly normal and ruled. Needless to say, we hope to return to a complete proof of the Conjecture in a future work.

2. Surfaces in $\mathbf{P}^{2n+1}$

Let V be a complex vector space of dimension $2n+2$, $n \geq 2$, and $X \subset \mathbf{P}(V) \cong \mathbf{P}^{2n+1}$ a smooth surface. Let $\mathcal{P}_X^m(1)$ denote the sheaf of principal parts of order m of the line bundle $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbf{P}(V)}(1)|_X$, for $m \geq 0$. Recall ([4]) that there are homomorphisms

$$a^m : V_X \rightarrow \mathcal{P}_X^m(1),$$

such that $\text{Im}(a^m(x))$ defines the m -th order osculating space to X at x , i.e.,

$$\text{Osc}_X^m(x) = \mathbf{P}(\text{Im}(a^m(x))) \subset \mathbf{P}(V).$$

The sheaf $\mathcal{P}_X^m(1)$ is locally free, with rank $\binom{m+2}{2}$, and there are natural exact sequences (compatible with the maps a^m)

$$0 \rightarrow S^m \Omega_X^1 \otimes \mathcal{O}_X(1) \rightarrow \mathcal{P}_X^m(1) \rightarrow \mathcal{P}_X^{m-1}(1) \rightarrow 0.$$

Set $K_m = \text{Ker}(a^m)$, $P_m = \text{Im}(a^m)$, and $Q_m = \text{Coker}(a^m)$. Then Q_m is locally free at x if and only if $\text{Im}(a^m(x)) = P_m(x)$. If P_m is locally free, we call P_m the m -th order osculating bundle of X (note that P_m may be locally free even if Q_m is not). Since X is a smooth surface, K_m is always locally free.

Denote by $s(m)$ the integer such that a^m has generic rank $s(m) + 1$. Then $\text{Osc}_X^m(x)$ has dimension $s(m)$ for almost all points $x \in X$. If $x \in X$ is such that $\dim \text{Osc}_X^m(x) \leq s(m) - 1$, we call x a point of hyperosculatation (of order $\leq m$).

Let $\bar{m}$ denote the largest integer such that $s(\bar{m}) < 2n + 1$, $s(m) < s(\bar{m})$ for all $m < \bar{m}$, and such that $P_{\bar{m}}$ is not the trivial bundle. Define the strict dual variety of X , as in ([5]), $X^* \subset \mathbf{P}(V^\vee)$, to be the closure of the set $\{H \subset \mathbf{P}(V); H \text{ hyperplane, } H \supseteq \text{Osc}_X^{\bar{m}}(x), \text{ for some } x \text{ s.t. } \dim \text{Osc}_X^{\bar{m}}(x) = s(\bar{m})\}$.

Assume now that the surface X satisfies condition (L) . The maps a^m are, locally at each point x , Taylor series expansions of the coordinate functions of X , of order $\leq m$. The relation that makes a^2 drop rank can be interpreted, around each point of X , as a second order linear partial differential equation of the coordinate functions of X (in terms of local parameters of X at the point). This is the Laplace equation referred to in the introduction. When we differentiate further, this relation gives more relations — so the generic rank of a^m is $\leq 2m + 1$, for all m . Moreover, if $\text{rank}(a^m(x)) < 2m + 1$ for some x , then $\text{rank}(a^{m'}(x)) < 2m' + 1$ for all $m' \geq m$. Hence, if X satisfies $(L^\sharp)$, then $\text{rank}(a^m(x)) = 2m + 1$ for all $x \in X$ and all m , $0 \leq m \leq n$. In particular, the sheaves Q_m are locally free, with rank $\binom{m}{2}$, and the osculating bundles P_m , with rank $2m + 1$, represent the osculating spaces at all points of X , for $m \leq n$.

Suppose X satisfies $(L^\sharp)$. Then we have $\bar{m} = n$, $K_n = \text{Ker}(a^n)$ is locally free with rank 1, and the dual homomorphism $V_X^\vee \rightarrow K_n^\vee$ is surjective. Denote by

$$\pi : X \rightarrow \mathbf{P}(V^\vee) \cong (\mathbf{P}^{2n+1})^\vee$$

the morphism defined by this 1-quotient. Then we have $\pi(X) = X^*$, and we call π the strict dual morphism.

Lemma 1. Assume X satisfies $(L^\sharp)$. Then the strict dual morphism $\pi : X \rightarrow X^*$ is finite; in particular, the strict dual variety X^* is a surface.

Proof: Since $X \not\subset$ hyperplane, X^* cannot be a point. Assume $C \subset X$ is an integral curve such that $\pi(C) = y$ is a point. If H_y denotes the hyperplane corresponding to y , then $H_y \cap X$ contains $(n+1)C$. Since X has no points of hyperosculation, no points of the curve $H_y \cap X$ can have multiplicity greater than $n+1$, hence C must be smooth, and $H_y \cap X$ can have no other components (since it is connected). But this gives a contradiction: the curve C has self-intersection ≤ 0 since it is contracted, whereas $H_y^2 \cap X$ has positive degree (the degree of X). $\square$

Lemma 2. Assume X satisfies $(L^\sharp)$. Then, for $m \geq 2$, there exist exact sequences

$$0 \longrightarrow S^{m-2}\Omega_X^1 \otimes Q \longrightarrow Q_m \xrightarrow{\beta} Q_{m-1} \longrightarrow 0.$$

Proof: The surjection β is induced by the natural surjection $\mathcal{P}_X^m(1) \rightarrow \mathcal{P}_X^{m-1}(1)$, because of the commutative diagram

$$\begin{array}{ccccccc} V_X & \xrightarrow{a^m} & \mathcal{P}_X^m(1) & \longrightarrow & Q_m & \longrightarrow & 0 \\ \parallel & & \downarrow & & \beta \downarrow & & \\ V_X & \xrightarrow{a^{m-1}} & \mathcal{P}_X^{m-1}(1) & \longrightarrow & Q_{m-1} & \longrightarrow & 0 \end{array}$$

Now set $A_m = \text{Ker}(P_m \rightarrow P_{m-1})$ and $B_m = \text{Ker}(\beta)$ — then A_m is locally free with rank 2, and B_m is locally free with rank $m-1$ — and consider the diagrams of locally free sheaves

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_m & \longrightarrow & P_m & \longrightarrow & P_{m-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S^m\Omega_X^1 \otimes \mathcal{O}_X(1) & \longrightarrow & \mathcal{P}_X^m(1) & \longrightarrow & \mathcal{P}_X^{m-1}(1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_m & \longrightarrow & Q_m & \xrightarrow{\beta} & Q_{m-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Note that $Q_1 = 0$, so $B_2 \cong Q_2$. Set $Q = Q_2$, and observe that Q is invertible. Then we may consider $Q^\vee \rightarrow S^2T_X \otimes \mathcal{O}_X(-1)$ as a sub-bundle, where $T_X = (\Omega_X^1)^\vee$ denotes the tangent bundle of X . Consider the map $Q_m^\vee \rightarrow S^mT_X \otimes \mathcal{O}_X(-1)$, whose image is $B_m^\vee$, and the composed map

$$\gamma : S^{m-2}T_X \otimes Q^\vee \rightarrow S^{m-2}T_X \otimes S^2T_X \otimes \mathcal{O}_X(-1) \rightarrow S^mT_X \otimes \mathcal{O}_X(-1).$$

It is enough to show $\gamma(S^{m-2}T_X \otimes Q^\vee) = B_m^\vee$, since this gives a surjection $S^{m-2}T_X \otimes Q^\vee \rightarrow B_m^\vee$, which must be an isomorphism, since the bundles have the same rank. This verification can now be done locally: Suppose (u, v) are local parameters of X at x . Then, around x , $Q^\vee \rightarrow S^2T_X \otimes \mathcal{O}_X(-1)$ is determined by an element

$$q = a \frac{\partial^2}{\partial u^2} + b \frac{\partial^2}{\partial u \partial v} + c \frac{\partial^2}{\partial v^2}.$$

Then $B_m^\vee \subset S^mT_X \otimes \mathcal{O}_X(-1)$ is spanned, locally around x , by the elements

$$q_i = a \frac{\partial^m}{\partial u^{i+1} \partial v^{m-i-1}} + b \frac{\partial^m}{\partial u^i \partial v^{m-i}} + c \frac{\partial^m}{\partial u^{i-1} \partial v^{m-i+1}},$$

for $i = 1, \dots, m-1$. But the image of γ is spanned by

$$\gamma\left(\frac{\partial^{m-2}}{\partial u^{i-1} \partial v^{m-i-1}} \otimes q\right) = q_i. \quad \square$$

This lemma implies that Q_m resembles $\mathcal{P}_X^{m-2}(Q)$ — in particular, the Chern classes of these two bundles are equal.

Lemma 3. *Assume X satisfies $(L^\sharp)$. Then the strict dual surface X^* is smooth, and there exists an exact sequence*

$$0 \longrightarrow \pi^*(\Omega_{X^*}^1 \otimes \mathcal{O}_{X^*}(1))^\vee \longrightarrow S^n \Omega_X^1 \otimes \mathcal{O}_X(1) \longrightarrow S^{n-2} \Omega_X^1 \otimes Q \longrightarrow 0.$$

Proof: Let $\mathcal{P}_{X^*}^m(1)$ denote the sheaf of principal parts of order m of $\mathcal{O}_{X^*}(1) = \mathcal{O}_{\mathbf{P}(V^\vee)}(1)|_{X^*}$, and let

$$a_*^m : V_{X^*}^\vee \rightarrow \mathcal{P}_{X^*}^m(1)$$

denote the natural homomorphisms. As in ([4]) one observes that the composed map

$$\pi^*(a_*^1) \circ (a^{n-1})^\vee : (\mathcal{P}_X^{n-1}(1))^\vee \rightarrow V_X^\vee \rightarrow \pi^*(\mathcal{P}_{X^*}^1(1))$$

is 0. Hence there is an induced surjection

$$K_{n-1}^\vee \rightarrow \pi^*(\mathcal{P}_{X^*}^1(1)).$$

Since $\mathcal{P}_{X^*}^1(1)$ has generic rank 3, and K_{n-1} is locally free with rank 3, this surjection must be an isomorphism. Therefore $\pi^*(\mathcal{P}_{X^*}^1(1))$ is locally free with rank 3, hence $\pi^*\Omega_{X^*}^1$ is locally free with rank 2. Since the strict dual map π is a surjective morphism between integral schemes, and $\Omega_{X^*}^1$ has generic rank 2, it follows — from the functoriality of Fitting ideals — that $\Omega_{X^*}^1$ is locally free, with rank 2. Hence X^* is smooth, and $K_{n-1}^\vee \cong \pi^*(\mathcal{P}_{X^*}^1(1))$.

In order too establish the exact sequence, it suffices to observe that A_n is isomorphic to $\text{Coker}(K_n \rightarrow K_{n-1})$, which again is isomorphic to $\text{Ker}(\pi^*(\mathcal{P}_{X^*}^1(1)) \rightarrow \pi^*\mathcal{O}_{X^*}(1))^\vee = (\pi^*\Omega_{X^*}^1 \otimes \mathcal{O}_X(1))^\vee$, and then apply the diagram of Lemma 2 with $m = n$. $\square$

Lemma 4. *Assume X satisfies $(L^\sharp)$. Then the strict dual morphism $\pi : X \rightarrow X^*$ is finite and étale, and the first Chern classes of the line bundles $\mathcal{O}_X(1)$ and $\pi^*(\mathcal{O}_{X^*}(1))$ are equal in $A^1 X \otimes \mathbb{Q}$.*

Proof: Since we know already that X and X^* are smooth and π is finite, π is flat. In order to prove the first part, it therefore suffices to prove that π is unramified.

Let $y \in X^*$, let H_y denote the corresponding hyperplane, and consider the flat family of curves on X (over X^*)

$$\mathcal{H} = \{H_y \cap X | y \in X^*\}.$$

Since X satisfies $(L^\sharp)$, each point $x \in \pi^{-1}(y)$ is a point of multiplicity exactly $n+1$ on the curve $H_y \cap X$ — and all other points on this curve have multiplicity $\leq n$. Set $e = \deg(\pi)$. For most y , $\pi^{-1}(y)$ consists of e distinct points — since no member of $\mathcal{H}$ has points of multiplicity $> n+1$, no member can have fewer than e points of multiplicity $n+1$. Hence $\#\pi^{-1}(y) = e$ for all $y \in X^*$, and so π has no points of ramification.

From this it follows that $\pi^*\Omega_{X^*}^1 \cong \Omega_X^1$. Set $K = c_1(\Omega_X^1)$, $H = c_1(\mathcal{O}_X(1))$, and $H^* = c_1(\pi^*\mathcal{O}_{X^*}(1))$.

To prove the second part, we shall compute two expressions for $c_1(Q)$ in $A^1 X$. First we note that

$$\pi^*\mathcal{O}_{X^*}(1) \cong K_n^{-1} \cong \bigwedge^{max} P_n,$$

hence

$$H^* = -c_1(K_n) = c_1(P_n).$$

From the various exact sequences established earlier, we obtain the following equalities in $A^1 X$:

$$c_1(Q_n) = c_1(\mathcal{P}_X^n(1)) - c_1(P_n) = \binom{n+2}{3} K + \binom{n+2}{2} H - H^*.$$

From Lemma 2 we obtain

$$c_1(Q_n) = c_1(\mathcal{P}_X^{n-2}(Q)) = \binom{n}{3} K + \binom{n}{2} c_1(Q).$$

This gives

$$\binom{n}{2} c_1(Q) = n^2 K + \binom{n+2}{2} H - H^*.$$

The sequence of Lemma 3 yields

$$(n-1)c_1(Q) = c_1(S^n \Omega_X^1 \otimes \mathcal{O}_X(1)) - c_1(S^{n-2} \Omega_X^1) + K + 2H^*.$$

Carrying out the computations and eliminating $c_1(Q)$, we obtain the equality in $A^1 X$,

$$2(n+1)H = 2(n+1)H^*. \quad \square$$

Now let c_2 denote the degree of the second Chern class $c_2(T_X)$ of X . For $A, B \in A^1 X$, we let $A \cdot B \in \mathbb{Z}$ denote the intersection number. Note that $H^2 = H \cdot H$ is the degree of X .

Proposition. *Assume the surface X satisfies condition $(L^\sharp)$. Then the following formulas hold:*

- (1) $(n-1)^2 c_2 + (nK + 2H) \cdot (K + 2H) = 0$
- (2) $(n^2 + 2n + 3)c_2 + nK^2 + 2(n+3)K \cdot H = 0.$

Or, equivalently,

- (1') $(2n+1)c_2 + 2K \cdot H - 2H^2 = 0$
- (2') $n(2n+1)K^2 + 2n(n+5)K \cdot H + 2(n^2 + 2n + 3)H^2 = 0.$

Proof: Since H^* and H are numerically equivalent, we may identify them in the computations. By Lemma 4, we may replace $\pi^* \Omega_X^1$ by Ω_X^1 in the exact sequence of Lemma 3. This sequence gives the equality in $A^1 X \otimes \mathbb{Q}$,

$$(n-1)c_1(Q) = 2nK + (n+3)H.$$

Taking this equality into account, we then obtain (1) by computing $c_2(\Omega_X^1)$ from the same exact sequence.

The formula (2) is obtained by setting equal the two expressions for $c_2(P_n)$ obtained from the two exact sequences

$$0 \longrightarrow \pi^* \mathcal{O}_{X^*}(-1) \longrightarrow V_X \longrightarrow P_n \longrightarrow 0$$

and

$$0 \longrightarrow P_n \longrightarrow \mathcal{P}_X^n(1) \longrightarrow Q_n \longrightarrow 0,$$

using the fact

$$c(Q_n) = c(\mathcal{P}_X^{n-2}(Q))$$

proved in Lemma 2. $\square$

Corollary. *If the surface X satisfies $(L^\sharp)$, then X is birationally equivalent to a ruled surface, and $X \not\cong \mathbb{P}^2$.*

Proof: We use only formula (1). Suppose X is not birationally ruled, and let $X \rightarrow S$ be a minimal model of X . Then $c_2 = c_2(S) + b$ and $K^2 = c_1(S)^2 - b$, for some integer $b \geq 0$. Since S is minimal and not ruled, we have $c_2(S), c_1(S)^2 \geq 0$. Since X is not birationally ruled and H is ample, we have $K \cdot H \geq 0$. From (1) we obtain

$$(n-1)^2 c_2(S) + n c_1(S)^2 + 2(n+1)K \cdot H + 4H^2 = -(n^2 - 3n + 1)b.$$

For $n \geq 3$ this gives a contradiction, since $n^2 - 3n + 1 > 0$ in that case.

For $n = 2$, (1) becomes

$$c_2 + 2(K + H)^2 + 2K \cdot H + 2H^2 = 0.$$

Hence we must have $(K + H)^2 < 0$, since $c_2 \geq 0$, $K \cdot H \geq 0$, and $H^2 > 0$. Now we apply a theorem of Sommese and Van de Ven ([8], [11]), which says that $K + H$ is generated by global sections (and hence

that $(K + H)^2 \geq 0$) unless X is a plane, the Veronese surface, or is ruled (by lines). This gives the desired contradiction.

The last assertion holds because (1) implies that c_2 and K^2 cannot both be odd. $\square$

We now state our main result and show how to deduce it from the formulas of the Proposition.

Theorem. *Assume the surface $X \subset \mathbf{P}^{2n+1}$ satisfies condition $(L^\sharp)$. If $n \leq 4$, then X is a balanced rational normal scroll, of degree $2n$ and type (n, n) .*

Proof: Let

$$\tau = \frac{1}{3}(K^2 - 2c_2)$$

denote the Hirzebruch index of X . Since X is birationally ruled and $X \not\cong \mathbf{P}^2$, we have $\tau \leq 0$, and $\tau = 0$ holds if and only if X is ruled (not necessarily by lines in $\mathbf{P}^{2n+1}$).

We claim that in order to prove the Theorem for any n (and hence case (i) of the Conjecture), it suffices to prove $\tau = 0$. For suppose this holds. Then (1') and (2') imply

$$nK \cdot H + (n+1)H^2 = 0.$$

Hence

$$n(2g - 2) + H^2 = 0,$$

where g denotes the sectional genus of X , i.e., $2g - 2 = K \cdot H + H^2$. Since X is not contained in a hyperplane, $H^2 \geq 2n$ holds; hence we get $g = 0$ and $H^2 = 2n$. But then it is well known (see e.g. [1]) that X is a rational normal scroll (since $X \not\cong \mathbf{P}^2$, the exceptional case of the Veronese surface is excluded). It follows from ([5]) that the only such scrolls that satisfy condition $(L^\sharp)$, are the balanced ones.

To finish the proof of the Theorem, it remains to show that if $n \leq 4$, then $\tau = 0$ holds.

From the formulas (1') and (2') we obtain the following expression for the index τ :

$$3n(2n+1)\tau = -2(n+3)(nK \cdot H + (n+1)H^2).$$

This gives

$$(K + H)^2 = -\frac{3(n-4)}{n+3}\tau - \frac{n-2}{n}H^2.$$

If $n = 3$ or $n = 4$, then this implies, since $\tau \leq 0$,

$$(K + H)^2 \leq -\frac{n-2}{n}H^2 < 0,$$

and we conclude by the theorem of Sommese–Van de Ven [loc.cit.] that X is ruled (even by lines), hence that $\tau = 0$, and we are done.

If $n = 2$, we obtain

$$(K + H)^2 = \frac{6}{5}\tau.$$

Now if $\tau < 0$, then X is not ruled, so it cannot be a scroll — this gives a contradiction, again by the theorem of Sommese–Van de Ven. Hence we must have $\tau = 0$ also in this case. $\square$

Clearly, one would like to prove directly that $\tau = 0$ holds (for all n), i.e., that X is ruled. Geometrically, it seems likely that the presence of reducible rulings would give points of hyperosculation on X , namely the points of intersection of the irreducible components of these rulings, and thus contradict the fact that X satisfies $(L^\sharp)$. Unfortunately, we have not been able to make this idea work.

Remark. Instead of using the numerical invariants c_2 , K^2 , and $K \cdot H$, one could use τ , g , and q , where τ is the Hirzebruch index of X , g is the sectional genus, and $q = h^1(X, \mathcal{O}_X(1))$ is the irregularity. Since X

is birationally ruled, we have $K^2 = 8(1 - q) + \tau$ and $c_2 = 4(1 - q) - \tau$. One has $g \geq q \geq 0$, and one easily sees that the formulas (1) and (2) imply that c_2 and K^2 , hence also τ , are divisible by 4, and that H^2 is divisible by $2n$. Set $e = \frac{H^2}{2n}$. We get

$$(1'') \quad e - 1 + q = -\frac{n+6}{4(n+3)}\tau$$

and

$$(2'') \quad g - q = -\frac{5n-3}{4(n+3)}\tau.$$

This shows (again)

$$\tau = 0 \iff e = 1 \quad \text{and} \quad g = q = 0.$$

In fact, any one of the conditions $e = 1$, $g = q$, or $g = 0$ implies $\tau = 0$. We have already observed this for $e = 1$. If $g = q$, then $\tau = 0$. If $g = 0$, then $q = 0$, and hence $\tau = 0$. Moreover, it is also true that in order to prove the Theorem for $n \geq 3$, it suffices to show that $q = 0$ holds, i.e., that X is a rational surface. For suppose $q = 0$ holds. Compute $(K + H)^2$ using this and (1'') to get

$$(K + H)^2 = -\frac{2}{n+6}((n^2 - 2n + 12)e + 6(n - 4)).$$

This shows that $(K + H)^2 < 0$ if $n \geq 3$. Hence $K + H$ cannot be generated by its global sections, and we conclude once again by the theorem of Sommese – Van de Ven.

3. Some remarks

Suppose the surface $X \subset \mathbf{P}(V) \cong \mathbf{P}^{2n+2}$ satisfies the condition $(L^\sharp)$. Then the strict dual variety $X^* \subset \mathbf{P}(V^\vee)$ is of dimension ≤ 3 , since it is the image of a $\mathbf{P}^1$ -bundle on X ; the strict dual morphism is in this case

$$\pi : \mathbf{P}(K_n^\vee) \longrightarrow \mathbf{P}(V^\vee),$$

defined by the 2-quotient $V_X^\vee \rightarrow K_n^\vee$, where $K_n = \text{Ker}(a^n)$, and $X^* = \pi(\mathbf{P}(K_n^\vee))$.

If X is a semi-balanced rational normal scroll, then $\dim X^* = 3$ holds ([5]), so one wants to show this holds whenever X satisfies $(L^\sharp)$. Granted this, one should proceed by trying to obtain formulas similar to those of the Proposition, hoping that the fact that π is no longer defined on X , but on $\mathbf{P}(K_n^\vee)$, does not complicate matters too much.

One could also ask whether there are any surfaces $X \subset \mathbf{P}^{2n+2}$ satisfying $(L^\sharp)$ and such that $\text{Osc}_X^{n+1}(x) = \mathbf{P}^{2n+2}$ for all points $x \in X$. But if the Conjecture is true, no such surface exist, since the only possibility — a semi-balanced scroll — satisfies $\dim \text{Osc}_X^{n+1}(x) = 2n + 1$ for all points x on a rational normal curve of degree n on X ([5]).

Another approach to the Conjecture (in both cases) would be to establish more directly that a surface satisfying $(L^\sharp)$ must be linearly normal and ruled. Togliatti ([10]) gives an example of a surface in $X \subset \mathbf{P}^5$ satisfying (L) and such that the Laplace equation satisfied by X , at a general point, is of hyperbolic type, hence X is not ruled. This surface is obtained by projecting the Del Pezzo surface of degree 6 in $\mathbf{P}^6$ from a point common to all the 2nd order osculating spaces to X (such a point exists!), so it is not linearly normal. It would be interesting to know whether this is in fact the only example of a smooth surface satisfying (L) which is neither ruled nor linearly normal — this would give a proof of our Conjecture.

References

- [1] D. Eisenbud and J. Harris, "On varieties of minimal degree (a centennial account)," *Proc. Symp. Pure Math.* **46**(1987), 3–13.
- [2] W. Fulton, S. Kleiman, R. Piene, and H. Tai, "Some intrinsic and extrinsic characterizations of the projective space," *Bull. Soc. math. France* **113**(1985), 205–210.
- [3] P. Griffiths and J. Harris, "Algebraic Geometry and Local Differential Geometry," *Ann. scient. Ec. Norm. Sup.* **12**(1979), 355–452.
- [4] R. Piene, "A note on higher order dual varieties, with an application to scrolls," *Proc. Symp. Pure Math.* **40**(1983), Part 2, 335–342.
- [5] R. Piene and G. Sacchiero, "Duality for rational normal scrolls," *Comm. in Algebra* **12**(9)(1984), 1041–1066.
- [6] C. Segre, "Su una classe di superficie degl'iperspazi, legata colle equazioni lineari alle derivate parziali di 2° ordine," *Atti Acc. Torino* **42**(1906–07), 1047–1079.
- [7] T. Shifrin, "The osculatory behavior of surfaces in $\mathbf{P}^5$," *Pac. J. Math.* **123**(1986), 227–256.
- [8] A.J. Sommese, "Hyperplane sections of Projective surfaces I — the adjunction mapping," *Duke Math. J.* **46**(1979), 377–401.
- [9] Hsin-sheng Tai, "On Second Order Osculating Spaces of Surfaces in $\mathbf{P}^5$," Preprint 1984.
- [10] E. Togliatti, "Alcuni esempi di superficie algebriche degli iperspazi che rappresentano un'equazione di Laplace," *Comm. Math. Helv.* **1**(1929), 255–272.
- [11] A. Van de Ven, "On the 2-connectedness of very ample divisors on a surface," *Duke Math. J.* **46**(1979), 403–407.

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