

Generating functions in enumerative geometry

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ICREM 5
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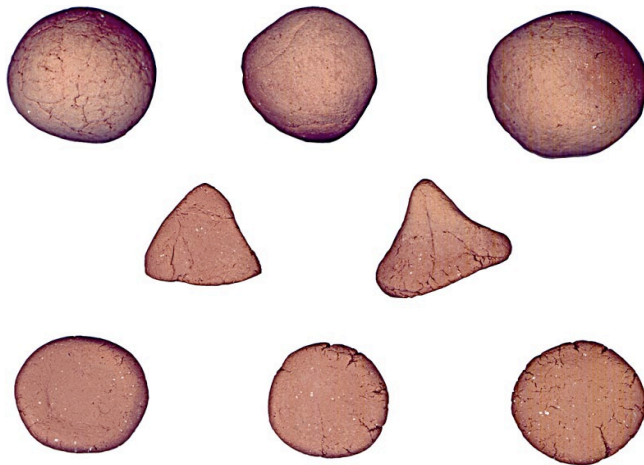


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Counting 1



MS 5067/1-8

Neolithic plain counting tokens. Near East, ca. 8000--3500 BC

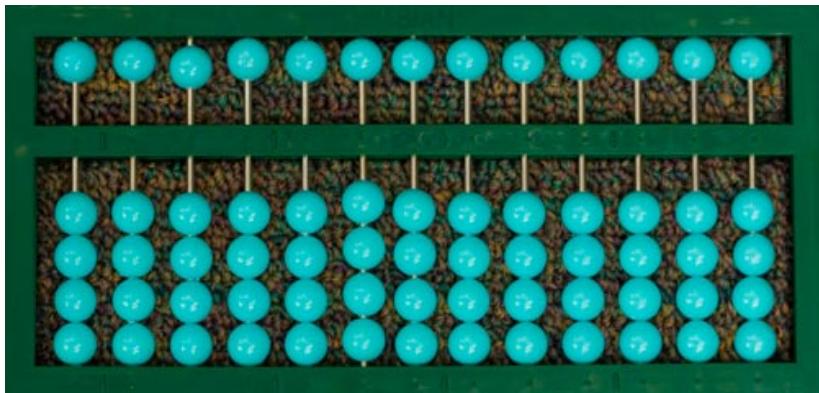


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Counting 2



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Counting 3

Nomor	
0 nol	
1 satu	11 sebelas
2 dua	12 dua belas
3 tiga	13 tiga belas
4 empat	14 empat belas
5 lima	15 lima belas
enam	16 enam belas



Partitions

Let n be a positive integer.

In how many ways $p(n)$ can we write n as a sum of positive integers?

$$1 = 1$$

$$2 = 2 = 1 + 1$$

$$3 = 3 = 2 + 1 = 1 + 1 + 1$$

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$



A partition of 7



MS 4647

Numbers 3+4. Sumer, ca. 3500-3200 BC



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Partitions

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In how many ways $p(n)$ can we write n as a sum of positive integers?

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$$2 = 2 = 1 + 1$$

$$3 = 3 = 2 + 1 = 1 + 1 + 1$$

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

$$p(1) = 1$$

$$p(2) = 2$$

$$p(3) = 3$$

$$p(4) = 5$$

A closed formula for $p(n)$?



The partition function

Consider the infinite product

$$(1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots) \cdots (1 + q^m + q^{2m} + \cdots) \cdots$$

Expanding and taking the coefficient of q^n gives us $p(n)$.

Conclusion: Let $p(n)$ denote the number of ways we can write $n = n_1 + n_2 + \dots$, where $n_1 \geq n_2 \geq \dots > 0$.

Let $f(q) := \sum p(n)q^n$ denote the *generating function* of the $p(n)$'s. Then

$$f(q) = \prod_{m \geq 1} (1 + q^m + q^{2m} + q^{3m} + \cdots) = \prod_{m \geq 1} (1 - q^m)^{-1}.$$

The function $f(q)$ displays the integer sequence

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, 77, 101, \dots$$

Generating function



“A generating function is a clothesline on which we hang up a sequence of numbers for display.”

H.S.Wilf
generatingfunctionology
Academic Press Inc., 1990



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Plane partitions

Consider next the number of *plane partitions* $\pi(n)$:

$$\pi(1) = 1$$

$$\pi(2) = 3 \leftrightarrow 2, \quad 1 \mid 1, \quad \frac{1}{1}$$

$$\pi(3) = 6 \leftrightarrow 3, \quad 2 \mid 1, \quad \frac{2}{1}, \quad \frac{1}{1}, \quad 1 \mid 1 \mid 1, \quad \frac{1}{1} \mid \frac{1}{1}$$

The generating function for the $\pi(n)$ is the MacMahon function:

$$M(q) = \prod_{m \geq 1} (1 - q^m)^{-m}$$



Modulo q^4 :

$$\begin{aligned} M(q) &\equiv (1 + q + q^2 + q^3)(1 + q^2)^2(1 + q^3)^3 \\ &\equiv 1 + q + q^2 + 2q^2 + q^3 + q \cdot 2q^2 + 3q^3 \end{aligned}$$

$$q \leftrightarrow 1 \leftrightarrow \langle 1 \rangle$$

$$q^2 \leftrightarrow 2 \leftrightarrow \langle 1, z \rangle$$

$$2q^2 \leftrightarrow 1 \mid 1 \quad \text{and} \quad \frac{1}{1} \leftrightarrow \langle 1, y \rangle \quad \text{and} \quad \langle 1, x \rangle$$

$$q^3 \leftrightarrow 3 \leftrightarrow \langle 1, z, z^2 \rangle$$

$$q \cdot 2q^2 \leftrightarrow 2 \mid 1 \quad \text{and} \quad \frac{2}{1} \leftrightarrow \langle 1, z, y \rangle \quad \text{and} \quad \langle 1, z, x \rangle$$

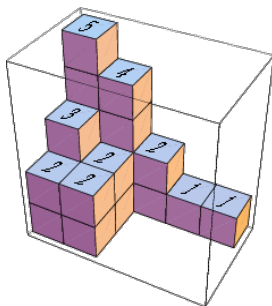
$$3q^3 \leftrightarrow 1 \mid 1 \mid 1, \frac{1}{1} \mid \frac{1}{1}, \frac{1}{1} \leftrightarrow \langle 1, y, y^2 \rangle, \langle 1, x, y \rangle, \langle 1, x, x^2 \rangle$$



A plane partition of 22

5	4	2	1	1
3	2			
2	2			

 $\longleftrightarrow \langle 1, z, z^2, z^3, z^4, y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2z \rangle$



Catalan numbers

The n th Catalan number is

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

They satisfy the convolution formula

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

From this one deduces the equation $q C(q)^2 - C(q) + 1 = 0$ for the generating function:

$$C(q) := \sum_{n=0}^{\infty} C_n q^n = \frac{1 - \sqrt{1 - 4q}}{2q}$$



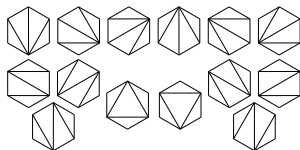
The Catalan numbers count:

- ▶ The number of ways to insert n pairs of parentheses in a word of $n + 1$ letters.

For $n = 3$ there are 5 ways:

$((ab)(cd))$, $((ab)c)d$, $((a(bc))d)$, $(a((bc)d))$, $(a(b(cd)))$.

- ▶ The ways of joining $2n$ points on a circle to form n nonintersecting chords.
- ▶ The number of ways to triangulate a regular $(n + 2)$ -gon.
For $n = 4$ there are 14 ways:



Schubert calculus

- ▶ How many lines intersect four given lines in $\mathbb{P}^3$?

Degenerate the four lines to two pairs of intersecting lines. Then the line through the two points of intersection, and the line which is the intersection of the two planes are the lines meeting all four lines – the answer is 2.

- ▶ How many lines meet six given planes in $\mathbb{P}^4$?

The answer is 5.

- ▶ How many lines meet $2n$ given $(n-1)$ -planes in $\mathbb{P}^{n+1}$?

The answer is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

The function $C(q) = \frac{1-\sqrt{1-4q}}{2q}$ displays the sequence

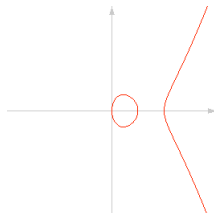
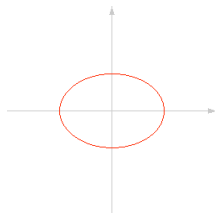
$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \dots$$



Plane curves

A plane curve of degree d is given as the set of zeros of a (homogeneous) polynomial $F(y_0, y_1, y_2)$ in three variables of degree d .

The curve on the left is a conic ($d = 2$). It is a *rational* curve (it has genus 0). The one to the right is a cubic ($d = 3$). It is *elliptic* (it has genus 1).



Counting plane curves

The space of all curves of degree d in $\mathbb{P}^2$ is thus a projective space of dimension $\binom{d+2}{2} - 1$.

The *rational* curves are those which have $\binom{d-1}{2}$ singular points. They form a sub-“space” of dimension $3d - 1$.

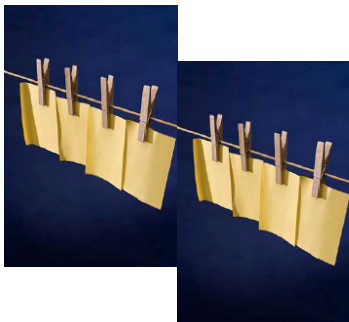
Problem: Find the number N_d of plane rational curves of degree d passing through $3d - 1$ points.

Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

What is the generating function $\sum N_d q^d = ??$

To display on the clothesline:



...this sequence of numbers:

$$N_1 = 1$$

$$N_2 = 1$$

$$N_3 = 12$$

$$N_4 = 620$$

$$N_5 = 87304$$

$$N_6 = 26312976$$

$$N_7 = 14616808192$$

$$N_8 = 13525751027392$$

$$N_9 = 19385778269260800$$

$$N_{10} = 40739017561997799680$$

$$N_{11} = 120278021410937387514880$$

$$N_{12} = 482113680618029292368686080$$



K3 and the partition function

Let S be a K3 surface (e.g., $S \subset \mathbb{P}^3$ is a quartic surface) and $\mathcal{L}$ a line bundle. The *rational* (genus 0) curves in the linear system $|\mathcal{L}|$ are the curves with $r := (\mathcal{L}^2 + 2)/2$ singularities. Their number, N_r , is independent of S and $\mathcal{L}$.

Yau–Zaslow: The generating function is

$$\sum_{r=0}^{\infty} N_r q^r = \prod_{m \geq 1} (1 - q)^{-24} = (\sum p(n) q^n)^{24} = f(n)^{24}.$$

Geometric reason (Bryan–Leung): The Euler number $c_2(S)$ is 24, and S can be degenerated (in symplectic geometry) to an elliptic fibration over $\mathbb{P}^1$ with 24 singular (hence rational) fibers. There are $p(j_i)$ degree j_i maps from a stable rational curve (a bunch of $\mathbb{P}^1$'s) to a singular fiber, hence

$$n_r = \sum_{j_1 + \dots + j_{24} = r} p(j_1) \cdots p(j_{24}).$$

Bell numbers

The *Stirling* numbers $S_{n,k}$ count the number of partitions of a set with n elements into k blocks. The *Bell* numbers count *all* partitions: $B_n = \sum_{k=1}^n S_{n,k}$. They can be defined recursively:

$$B_0 = 1 \text{ and } B_{n+1} = \sum_{i=0}^n \binom{n}{i} B_i$$

Their *exponential* generating function

$$B(q) := \sum_{n=0}^{\infty} B_n q^n / n!$$

satisfies the differential equation

$$\frac{B'(q)}{B(q)} = e^q,$$

and hence

$$B(q) = e^{e^q - 1}.$$



Polydiagonals

Let X be a space, and consider

$$X^n = X \times \cdots \times X = \{(x_1, \dots, x_n) \mid x_i \in X\}.$$

For each $k \leq n$ and each partition $\mathbf{i}_k$ of $\{1, 2, \dots, n\}$ into k disjoint subsets $\{i_1^j, i_2^j, \dots, i_{l(j)}^j\}$, $j = 1, \dots, k$, $\sum l(j) = n$, the corresponding *polydiagonal* is defined as

$$\Delta(\mathbf{i}_k) = \{(x_1, \dots, x_n) \mid x_{i_r^j} = x_{i_s^j} \text{ for all } r, s, j\}.$$

Then $\#\{\Delta(\mathbf{i}_k)\} = S_{n,k}$, and $B_n = \#\cup_{k=1}^n \{\Delta(\mathbf{i}_k)\}$ is the number of all polydiagonals in X^n , and there are

$$\frac{n!}{j_1! \cdots j_n!} \left(\frac{1}{1!}\right)^{j_1} \cdots \left(\frac{1}{n!}\right)^{j_n}$$

polydiagonals of each type ($j_i = \#$ blocks of i elements).

Example

$n = 4$, $k = 2$, $\mathbf{i}_2$:

$\{1\}, \{2, 3, 4\}; \{2\}, \{1, 3, 4\}; \{3\}, \{1, 2, 4\}; \{4\}, \{1, 2, 3\};$

$\{1, 2\}, \{3, 4\}; \{1, 3\}, \{2, 4\}; \{1, 4\}, \{2, 3\}.$

There are $\frac{4!}{1!1!}(\frac{1}{1!})^1(\frac{1}{3!})^1 = 4$ of the first type,

and $\frac{4!}{2!}(\frac{1}{2!})^2 = 3$ of the second type.



Bell polynomials

The *partial Bell polynomials* are

$$B_{n,k}(a_1, \dots, a_{n-k+1}) = \sum \binom{n}{j_1 \dots j_{n-k+1}} \left(\frac{a_1}{1!}\right)^{j_1} \dots \left(\frac{a_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

summing over $j \vdash n, \sum j_i = k$.

The *complete (exponential) Bell polynomials* are

$$B_n(a_1, \dots, a_n) = \sum_{k=1}^n B_{n,k}(a_1, \dots, a_{n-k+1}).$$

$$B_1(a_1) = a_1,$$

$$B_2(a_1, a_2) = a_1^2 + a_2,$$

$$B_3(a_1, a_2, a_3) = a_1^3 + 3a_1a_2 + a_3$$

$$B_4(a_1, a_2, a_3, a_4) = a_1^4 + 6a_1^2a_2 + 4a_1a_3 + 3a_2^2 + a_4$$



Bell polynomials

Recursively defined by $B_0 = 1$ and

$$B_{n+1}(a_1, \dots, a_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(a_1, \dots, a_{n-i}) a_{i+1}$$

or by the formal identity

$$\sum_{i \geq 0} B_i(a_1, \dots, a_i) q^i / i! = \exp(\sum_{j \geq 1} a_j q^j / j!).$$

Note that $S_{n,k} = B_{n,k}(1, \dots, 1)$ and $B_n = B_n(1, \dots, 1)$.



Faà di Bruno's formula

Let $h(t) = f(g(t))$ be a composed function.

Differentiate once: $h'(t) = f'(g(t))g'(t)$

and twice: $h''(t) = f''(g(t))g'(t)^2 + f'(t)g''(t)$.

Set $h_i = h^{(i)}(t)$, $f_i(t) = f^{(i)}(g(t))$, $g_i = g^{(i)}(t)$. Then

$$h_1 = f_1 g_1$$

$$h_2 = f_2 g_1^2 + f_1 g_2$$

$$h_3 = f_3 g_1^3 + 3f_2 g_1 g_2 + f_1 g_3$$

and indeed

$$h_n = \sum_{k=1}^n f_k B_{n,k}(g_1, \dots, g_{n-k+1})$$



Göttsche's conjectures

Let S be a smooth, projective surface, $\mathcal{L}$ a line bundle, $m = \mathcal{L}^2$, $k = K_S \cdot \mathcal{L}$, $s = K_S^2$, $x = c_2(S)$ the four Chern numbers.

An r -nodal curve is a curve with precisely r ordinary double points.

Let N_r denote the number of r -nodal curves in the linear system $|\mathcal{L}|$ passing through the appropriate number of points on S .

Goettsche conjectured:

$$\sum N_r q^r = A_1^m A_2^k A_3^s A_4^x,$$

where the $A_i = A_i(q)$ are (known and unknown) universal power series.

Proved in 2010 by Tzeng and by Kool–Shende–Thomas.

Bell polynomials and nodal curves

From Goettsche's conjecture it follows (Qviller) that there exist universal linear polynomials $a_1, a_2, \dots$ in four variables, such that

$$\sum N_r q^r = \sum B_r(a_1(m, k, s, x), \dots, a_r(m, k, s, x))/r! q^r$$

where B_r is the r th Bell polynomial.

Proved by Kleiman–P. for $r \leq 8$, together with an algorithm to compute the a_i .

Note that there are three aspects:

- ▶ *shape* of generating function
- ▶ *explicit computation*
- ▶ *validity* of the formulas



Why Bell polynomials?

- ▶ A recursive formula for N_r fits with the Bell polynomials (reminiscent of “derivations” like in the Faà di Bruno formula).
- ▶ An intersection theoretic approach on the configuration space S^r , shows that each a_i comes from an intersection class supported on a diagonal, and each product of a_i ’s to a class on a corresponding polydiagonal (Qviller).

The advantage of knowing the Bell form of the node polynomials N_r is that in order to compute N_{r+1} from the previous N_i ’s one only needs to compute one new term, a_{r+1} :

$$N_{r+1} = \frac{1}{(r+1)!} \sum_{i=0}^{r-1} \binom{r}{i} (r-i)! N_{r-i} a_{i+1} + \frac{1}{(r+1)!} a_{r+1}$$



The recursion

Express the number of r -nodal curves in terms of $(r - 1)$ -nodal and lower.

Blow up the surface to get rid of one node in each curve, then use the formula for $(r - 1)$ -nodal curves on the blown up surface and push it down. This creates a “derivation formula” of the form

$$ru_r = u_{r-1}u_1 + \partial(u_{r-1})$$

or

$$r!u_r = (r - 1)!u_{r-1}u_1 + \partial((r - 1)!u_{r-1})$$

Set $a_1 = u_1$ and $a_2 = \partial(a_1)$. Then

$$2!u_2 = a_1^2 + a_2$$

and, pretending ∂ is a derivation:

$$3!u_3 = (a_1^2 + a_2)a_1 + \partial(a_1^2 + a_2) = a_1^3 + a_1a_2 + 2a_1a_2 + a_3.$$

Donaldson–Thomas and MacMahon

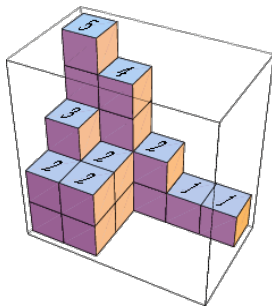
A plane partition of n can be viewed as a set of n monomials in three variables such that all the other monomials generate an ideal in $\mathbb{C}[x, y, z]$.



Recall the plane partition of 22:

5	4	2	1	1
3	2			
2	2			

 $\longleftrightarrow \langle 1, z, z^2, z^3, z^4, y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2z \rangle$



Donaldson–Thomas and MacMahon

A plane partition of n can be viewed as a set of n monomials in three variables such that all the other monomials generate an ideal in $\mathbb{C}[x, y, z]$.

Hence the integer $\pi(n)$ counts the number of monomial ideals in $\mathbb{C}[x, y, z]$ of colength n .

These correspond to the colength n ideals that are invariant under the torus action,

or to the length n fixed points of $\mathbb{C}^3$ under the torus action.

This is the virtual count of the **degree zero Donaldson–Thomas invariants**:

$$DT_0(\mathbb{C}^3)(q) = M(-q) = \prod_{m \geq 1} (1 - (-q)^m)^{-m}$$

THANK YOU FOR YOUR ATTENTION!



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