Reciprocal polar varieties

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Israel 70
Belo Horizonte
November 7, 2018

UiO : University of Oslo

Arcata 1974

Bull. Soc. math. France,
86, 1958, p. 137 à 154.

LA THÉORIE DES CLASSES DE CHERN

[Appendice au Mémoire de A. Borel et J.-P. Serre]

PAR

ALEXANDER GROTHENDIECK.

Introduction. — Dans cet appendice, nous développons une théorie axiomatique des classes de Chern, qui permet en particulier de définir les classes de Chern d'un fibré vectoriel algébrique E sur une variété algébrique non singulière quasi projective X comme des éléments de l'anneau de Chow $A(X)$ de X, i. e. comme des classes de cycles pour l'équivalence rationnelle. Cet exposé est inspiré du livre de HIRZEBRUCH d'une part (où les *propriétés formelles* essentielles caractérisant une théorie des classes de Chern étaient bien mises en évidence), et d'une idée de CHERN [2], qui consiste à utiliser

Polar varieties

Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a projective variety of dimension m, and $L_k \subset \mathbb{P}^n$ a linear subspace of codimension $m-k+2$.

The *kth polar variety* of of X (with respect to L_k) is

$$M_k := \overline{\{x \in X_{\mathrm{sm}} \mid \dim(T_{X,x} \cap L_k) \geq k-1\}}.$$

The classes $[M_k]$ are *projective invariants* of X: the kth class of a (general) projection of X is the projection of the kth class of X, and the kth class of a (general) linear section is the linear section of the kth class.

Mather Chern classes

Define

$$c_k^M(X) := \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap [M_i],$$

where $h := c_1(\mathcal{O}_X(1))$ denotes the class of a hyperplane section.

Let $\nu : \overline{X} \to X$ be the Nash transform of X and Ω the Nash cotangent bundle on $\overline{X}$. Then

$$c_k^M(X) = \nu_*(c_k(\Omega^\vee) \cap [\overline{X}]).$$

The Todd–Eger formula

We have

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap c_i^M(X).$$

It follows that

$$\sum_{k=0}^{m} h^{m-k} \cap [M_k] = \sum_{i=0}^{m} (-1)^i (2^{m-i+1} - 1) h^{m-i} \cap c_i^M(X),$$

hence

$$\sum_{k=0}^{m} \deg M_k = \sum_{i=0}^{m} (-1)^i (2^{m-i+1} - 1) \deg c_i^M(X).$$

Talbot House, Vermont, 1975 or 1976

Reciprocal polar varieties

Define polar varieties using conditions on the normal spaces instead of the tangent spaces?[1]

Problem: We have no normal spaces.

Solution: Pretend we are in Euclidean space.

Need to define a linear space orthogonal to a given linear space at a point, of complementary dimension.

[1] Bank–Giusti–Heintz–Pardo, Mork–P.

Affine space as Euclidean space

Coxeter:

"Kepler's invention of points at infinity made it possible to regard the projective plane as the affine plane plus the line at infinity. A converse relationship was suggested by Poncelet (1822) and von Staudt (1847): regard the affine plane as the projective plane minus an arbitrary line ℓ, and then regard the Euclidean plane as the affine plane with a special rule for associating pairs of points on ℓ (in "perpendicular directions")."

This way we can consider affine space with an added notion of orthogonality (or perpendicularity) as "Euclidean space" (no distance function).

Polarity w.r.t. a quadratic form

Let V and V' be a vector spaces of dimensions $n+1$ and n, and $V \to V'$ a surjection.

Let $H_\infty := \mathbb{P}(V') \subset \mathbb{P}(V)$ be the hyperplane at infinity, so $\mathbb{P}(V) \setminus H_\infty \cong V'$ is affine n-space.

A non-degenerate quadratic form on V' gives an isomorphism $V' \cong (V')^\vee$ and a non-singular quadric $Q_\infty \subset H_\infty$.

Let $L' = \mathbb{P}(W) \subset \mathbb{P}(V')$ be a linear space, and set

$$K := \operatorname{Ker}((V')^\vee \cong V' \to W).$$

Then $L'^\perp := \mathbb{P}(K^\vee) \subset \mathbb{P}(V')$ is the polar of L'.

Orthogonality

Given a linear space $L \subset \mathbb{P}(V)$, $L \nsubseteq H_\infty$, and $P \in L$. The *orthogonal space to L at P* is

$$L_P^\perp := \langle P, (L \cap H_\infty)^\perp \rangle.$$

Example

In $\mathbb{P}^2$, take $H_\infty : z = 0$, $Q_\infty : x^2 + y^2 = 0$, $L : x - 2y - 4z = 0$, $P = (2 : -1 : 1)$.

Then $L \cap H_\infty : (2 : 1 : 0)$, $(L \cap H_\infty)^\perp = (1 : -2 : 0)$, and $L_P^\perp : 2x + y - 3z = 0$.

Porto de Galinho, Easter 1981

Euclidean normal spaces

Let $X \subset \mathbb{P}(V)$ be a variety, with $X \not\subseteq H_\infty$.

For $P \in X_{\text{sm}} \setminus H_\infty$, let $T_P X$ denote the tangent space to X at P, and define[2] the *normal space to X at P*:

$$N_P X := (T_P X)_P^\perp.$$

The exact sequence

$$0 \to \mathcal{N}_{X_{\text{sm}}}(1) \to V_{X_{\text{sm}}} \to \mathcal{P}^1_{X_{\text{sm}}}(1) \to 0.$$

extends on the Nash transform $\nu : \overline{X} \to X$ to

$$0 \to \mathcal{N} \to V_{\overline{X}} \to \mathcal{P} \to 0.$$

[2] Catanese–Trifogli

Euclidean normal bundle

Consider $0 \to V'' \to V \to V' \to 0$ $(\dim V'' = 1)$.

Assuming $H_\infty = \mathbb{P}(V')$ is general with respect to X, we get

$$0 \to \mathcal{N} \to V'_{\overline{X}} \to \overline{\mathcal{P}} \to 0.$$

The polarity in H_∞ w.r.t. Q_∞ gives $V' \cong (V')^\vee$, so we have

$$V'_{\overline{X}} \cong V'^\vee_{\overline{X}} \to \mathcal{N}^\vee$$

whose fibers give the spaces polar to the spaces $T_P X \cap H_\infty$, and combining $V_{\overline{X}} \to V'_{\overline{X}}$ and $V_{\overline{X}} \to \mathcal{O}_{\overline{X}}(1)$, we get

$$V_{\overline{X}} \to \mathcal{E} := \mathcal{N}^\vee \oplus \mathcal{O}_{\overline{X}}(1)$$

whose fibers correspond to the Euclidean normal spaces $N_P X$.

We call $\mathcal{E}$ the *Euclidean normal bundle* of X.

Bernina Pass, June 1981

Reciprocal polar varieties

Instead of imposing conditions on the tangent spaces of a variety, one can similarly impose conditions on the Euclidean normal spaces.

Let $L \subset \mathbb{P}(V)$, $L \nsubseteq H_\infty$, have codimension w, $n - m \leq w \leq n$. Set $k = w - (n - m)$ and define *reciprocal polar varieties*

$$M_k(L)^\perp := \overline{\{P \in X_{\mathrm{sm}} \setminus H_\infty | N_P X \cap L \neq \emptyset\}}.$$

Then (Porteous' formula) $M_k^\perp$ have classes

$$[M_k^\perp] = \nu_*(s_k(\mathcal{E}) \cap [\overline{X}]) = \nu_*\big([s(\mathcal{N}^\vee)s(\mathcal{O}_{\overline{X}}(1))]_k \cap [\overline{X}]\big),$$

hence, since $s(\mathcal{N}^\vee) = c(\mathcal{P})$,

$$[M_k^\perp] = \sum_{i=0}^{k} h^{k-i} \cap [M_i].$$

Aftermath in Rome: June 21, 1986

Toric varieties

The Schwartz–MacPherson Chern class of a toric variety X with orbits $\{X_\alpha\}_\alpha$ is equal to (Ehler's formula[3])

$$c^{\mathrm{SM}}(X) = \textstyle\sum_\alpha [\overline{X}_\alpha].$$

This implies (proof by the definition of $c^{SM}(X)$ and induction on $\dim X$) that the Mather Chern class of a toric variety X is equal to

$$c^{\mathrm{M}}(X) = \textstyle\sum_\alpha \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

where $\mathrm{Eu}_X(X_\alpha)$ denotes the value of the local Euler obstruction of X at a point in the orbit X_α.

[3] see also Barthel–Brasselet–Fieseler, Maxim–Schürmann, Aluffi

Reciprocal polar classes of toric varieties

The polar classes of a toric variety X of dimension m are

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap \sum_{\alpha} \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

and

$$[M_m^{\perp}] = \sum_{k=0}^{n} h^{m-k} \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap \sum_{\alpha} \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

$$= \sum_{i=0}^{n} (-1)^i (2^{m-i+1} - 1) h^{m-i} \cap \sum_{\alpha} \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

where the second sum in each expression is over α such that $\mathrm{codim} X_\alpha = i$.

If X is a projective toric variety corresponding to a convex lattice polytope P,

$$\deg M_k = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} \mathrm{Vol}^i(P),$$

(cf. Matsui–Takeuchi) and

$$\deg M_m^{\perp} = \sum_{i=0}^{n} (-1)^i (2^{m-i+1} - 1) \mathrm{Vol}^i(P),$$

where $\mathrm{Vol}^i(P)$ denotes the sum of the normalized volume of the faces of P of codimension i (cf. Helmer–Sturmfels).

Happiness 2011

Weighted projective spaces (Nødland)

$\mathbb{P}(1, a, b, c)$ is a threefold with isolated singularities.

$P = \mathrm{Conv}\{(0,0,0),\ (bc,0,0),\ (0,ac,0),\ (0,0,ab)\}$.

Volume of P: $\mathrm{Vol}^0(P) = a^2b^2c^2$
Area of facets of P: $\mathrm{Vol}^1(P) = abc(1+a+b+c)$
Length of edges of P: $\mathrm{Vol}^2(P) = a+b+c+bc+ac+ab$
Number of vertices of P: $\mathrm{Vol}^3(P) = 4$

Example

$\mathrm{Eu}_X(v) = 1$ for all vertices of $\mathbb{P}(1,2,3,5)$ (counterexample to a conjecture of Matsui and Takeuchi!).
So the degree of $M_3^\perp$ is

$$a^2b^2c^2 + abc(1+a+b+c) + a+b+c+bc+ac+ab+4 = 1275.$$

The Euclidean endpoint map

Consider $\mathbb{P}(\mathcal{E}) \subset \overline{X} \times \mathbb{P}(V)$.

Let $p\colon \mathbb{P}(\mathcal{E}) \to \overline{X}$ and $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}(V)$ denote the projections on the first and second factor. The map q is called the *endpoint map*.

Let $A \in \mathbb{P}(V) \setminus H_\infty$. Then $p(q^{-1}(A))$ is a reciprocal polar variety:

$$p(q^{-1}(A)) = \{P \in X \mid A \in N_P X\} = M_m(A)^\perp.$$

Hence:

$$\deg q = \deg M_m^\perp = \sum_{i=0}^{m} \deg M_i.$$

Euclidean distance degree

The degree of the endpoint map $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}(V)$ is also called the (general) *Euclidean distance degree*[4]:

$$\mathrm{E}\deg X = \deg q = \deg M_m^{\perp} = \sum_{i=0}^{m} \deg M_i.$$

The points in $M_m(A)^{\perp}$ are the points $P \in X$ where the line $\langle P, A\rangle$ is orthogonal to the tangent space $T_P X$. Hence they are max/min points for the "distance function" induced by the Euclidean orthogonality defined by the quadric $Q_\infty \subset H_\infty$.

[4] Draisma–Horobet–Ottaviani–Sturmfels–Thomas

Rio 2012

Hypersurfaces with isolated singularities

If $X \subset \mathbb{P}(V)$ is a smooth hypersurface of degree d, then $\deg M_k = d(d-1)^k$.

If X has only isolated singularities, then only $\deg M_{n-1}$ is affected, and we get (Teissier, Laumon)

$$\mathrm{E}\deg X = \frac{d((d-1)^n - 1)}{d-2} - \sum_{P \in \mathrm{Sing}(X)} (\mu_P^{(n)} + \mu_P^{(n-1)}),$$

where $\mu_P^{(n)}$ is the Milnor number and $\mu_P^{(n-1)}$ is the sectional Milnor number of X at P.

Surface with ordinary singularities

Assume $X \subset \mathbb{P}(V) \cong \mathbb{P}^3$ is a generic projection of a smooth surface of degree d, so that X has *ordinary* singularities: a double curve of degree ϵ, t triple points, and ν_2 pinch points. Then (using known formulas for $\deg M_1$ and $\deg M_2$)

$$\mathrm{E}\deg X = \deg X + \deg M_1 + \deg M_2 = d^3 - d^2 + d - (3d-2)\epsilon - 3t - 2\nu_2$$

Example

The Roman Steiner surface: $d = 4$, $\epsilon = 3$, $t = 1$, $\nu_2 = 6$

$$\mathrm{E}\deg X = 7$$

The focal locus

The focal locus Σ_X is the branch locus of the map $q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(V)$:

$$\Sigma_X = \{Q \in \mathbb{P}(V) \,|\, \# q^{-1}(Q) < \deg q\}$$

It is the image of the subscheme R_X given by the Fitting ideal $F^0(\Omega^1_{\mathbb{P}(\mathcal{E})/\mathbb{P}(V)})$, so (if $\overline{X}$ is smooth) its class is

$$[\Sigma_X] = q_*\big((c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^* c_1(\Omega^1_{\mathbb{P}(V)})\big) \cap [\mathbb{P}(\mathcal{E})])$$

The focal locus of a hypersurface

Assume $X \subset \mathbb{P}(V)$ and $X^\vee \subset \mathbb{P}(V^\vee)$ are hypersurfaces, and Z a common desingularization. Let h_1 and h_2 denote the pullbacks of the hyperplane section line bundles to Z. Then $\mathcal{E} = h_2 \oplus h_1$, and

$$\deg \Sigma_X = (c_1(\Omega^1_Z)+c_1(h_2)+c_1(h_1))s_{n-2}(h_1\oplus h_2)+(n-1)s_{n-1}(h_1\oplus h_2),$$

with $s_j(h_1 \oplus h_2) = \sum_{i=0}^{j} c_1(h_1)^{j-i} c_1(h_2)^i$.

This expression is symmetric in h_1 and h_2, hence

$$\deg \Sigma_X = \deg \Sigma_{X^\vee}.$$

Example

$X \subset \mathbb{P}(V) \cong \mathbb{P}^2$ is a plane curve of degree d. Then the focal locus is the *evolute* of X, namely the *envelope* of its normals. Its degree is

$$\deg\big(c_1(\Omega^1_Z) + 2c_1(h_1) + 2c_1(h_2)\big) = 2g - 2 + 2\deg X + 2\deg X^\vee$$

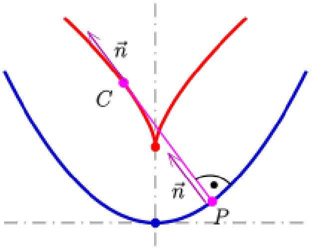
If $X = Z$ is smooth, we get $\deg \Sigma_X = 3d(d-1)$.

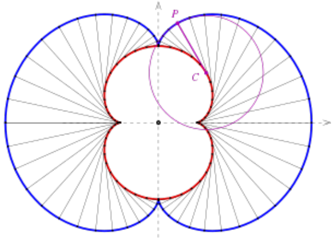
In the case that X is a "Plücker curve" of degree d having only δ nodes and κ ordinary cusps as singularities, then we obtain the classical formula due to Salmon

$$\deg \Sigma_X = 3d(d-1) - 6\delta - 8\kappa = 3\deg X^\vee + \kappa.$$

In fact: $3\deg X^\vee + \kappa = 3\deg X + \iota$, so $\deg \Sigma_X = \deg \Sigma_{X^\vee}$.

Evolutes of parabola and nephroide





The focal locus of a smooth hypersurface

Let $X \subset \mathbb{P}(V)$ be a general (smooth) hypersurface ($m = n - 1$) of degree d. It is known that in this case $R_X \to \Sigma_X$ is birational. We compute

$$\deg \Sigma_X = (n+1) \deg M_{n-1} + 2 \sum_{i=1}^{n-2} \deg M_i$$

$$= (n+1)d(d-1)^{n-1} + 2d\big((d-1)^{n-1} - 1\big)(d-2)^{-1}$$

which checks with the formula found by Trifogli.

Happy birthday, Israel!