Polytopes, discriminants and toric geometry

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Resultants and discriminants

Il faut éliminer la théorie de l'élimination.

J. Dieudonné (1969)

Eliminate, eliminate, eliminate
Eliminate the eliminators of elimination theory.

S. S. Abhyankar (1970)

Résultant, discriminant

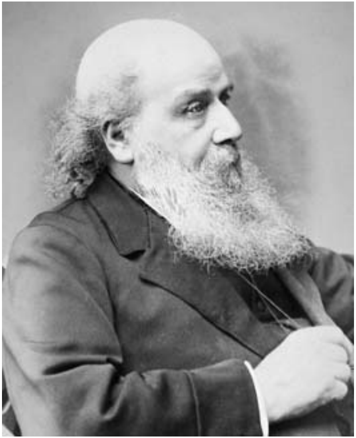
M. Demazure (2011) – à J.-P. Serre pour son 85-ième anniversaire

Question: For which $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ do

$$f(x) = a_m x^m + \cdots + a_0 \text{ and } g(x) = b_n x^n + \cdots + b_0$$

have a common root?

James Joseph Sylvester (1814–1897)



The *Sylvester matrix* is the $(m+n) \times (m+n)$-matrix

$$\begin{pmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & \cdots \\ 0 & a_m & a_{m-1} & a_{m-2} & \cdots \\ \vdots & & & \vdots & \\ b_n & b_{n-1} & b_{n-2} & \cdots & \cdots \\ 0 & b_n & b_{n-1} & b_{n-2} & \cdots \\ \vdots & & & \vdots & \end{pmatrix}$$

The *resultant* $\mathrm{Res}(f,g)$ is the determinant of this matrix.

A student of Sylvester: Florence Nightingale (1820-1910)

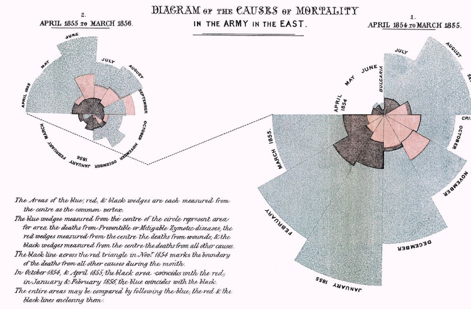
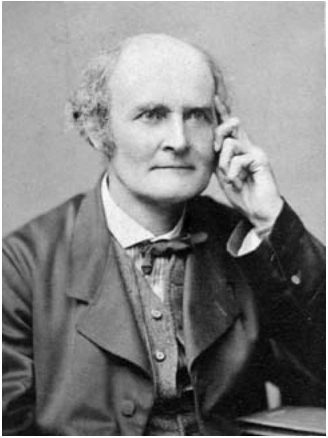


Figure: Diagram of the Causes of Mortality in the Army in the East

Arthur Cayley (1821–1895)



Set

$$h(x, y) := f(x) + yg(x).$$

If α is a common root of f and g, then

$$(\alpha, -\frac{f_x(\alpha)}{g_x(\alpha)})$$

is a common zero of h, h_x, h_y.

The Cayley trick

Consider

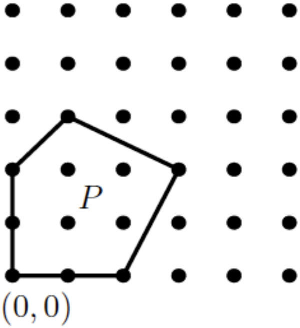
$$h(x_1, \dots, x_k, y_1, \dots, y_k) := \\ f_0(x_1, \dots, x_k) + y_1 f_1(x_1, \dots, x_k) + y_k f_k(x_1, \dots, x_k).$$

The *discriminant* $\Delta(h)$ of h is obtained by eliminating the x_i's and y_i's from the $2k+1$ equations

$$h = 0, \partial h / \partial x_i = 0, \partial h / \partial y_j = f_j = 0.$$

Hence $\Delta(h) \sim \mathrm{Res}(f_0, \dots, f_k)$.

Convex lattice polytopes



Cayley polytopes

Let $P_0, \ldots, P_k \subset \mathbb{R}^{n-k}$ be convex lattice polytopes, and $e_0, \ldots, e_k$ are the vertices of $\Delta_k \subset \mathbb{R}^k$.

The polytope

$$P = \text{Conv}\{e_0 \times P_0, \ldots, e_k \times P_k\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n,$$

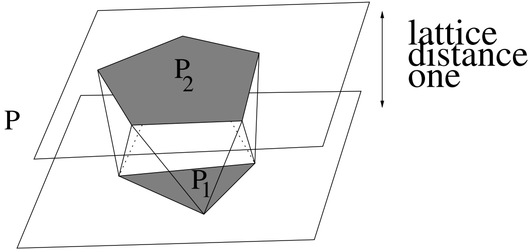
is called a *Cayley polytope*.

We write

$$P = P_0 \star \cdots \star P_k$$

A Cayley polytope is “hollow”: it has no interior lattice points.

An example



The codegree and degree of a polytope

$$\operatorname{codeg}(P) = \min\{m \mid mP \text{ has interior lattice points}\}.$$

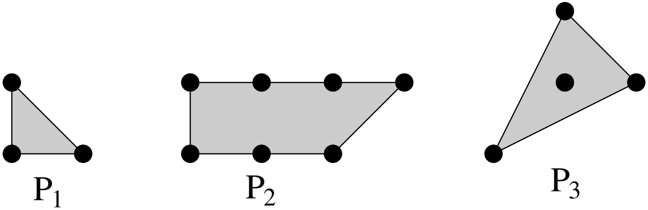
$$\deg(P) = n + 1 - \operatorname{codeg}(P)$$

Example (1)

$$\operatorname{codeg}(\Delta_n) = n + 1 \text{ and } \operatorname{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil.$$

Example (2)

$$P = P_0 \star \cdots \star P_k \text{ implies } \operatorname{codeg}(P) \geq k + 1.$$



$\mathrm{codeg}(P_1) = 3$ $\mathrm{codeg}(P_2) = 2$ $\mathrm{codeg}(P_3) = 1$

The Cayley polytope conjecture

Question (Batyrev–Nill): Is there an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Answer (Haase–Nill–Payne): Yes, and $N(d) \leq (d^2 + 19d - 4)/2$

Question: Is $N(d)$ linear in d?

Answer (Dickenstein–Di Rocco–P.): Yes, $N(d) = 2d + 1$ (if P is smooth and $\mathbb{Q}$-normal).

Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.

Theorem (Dickenstein, Di Rocco, P., Nill)

Let P be a smooth lattice polytope of dimension n. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = P_0 \star \cdots \star P_k$ *is a smooth Cayley polytope with* $k + 1 = \operatorname{codeg}(P)$ *and* $k > \frac{n}{2}$.

(3) P *is defective, with defect* $2k - n > 0$.

The proof is algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

Lattice polytopes and toric embeddings

The polytope P_0:

• —— • —— •

corresponds to the toric embedding $\mathbb{C}^* \to \mathbb{P}^2$ given by $x \mapsto (1 : x : x^2)$; its closure X_{P_0} is a conic.

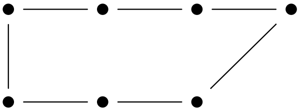
The polytope P_1:

• —— • —— • —— •

corresponds to the toric embedding $\mathbb{C}^* \to \mathbb{P}^3$ given by $x \mapsto (1 : x : x^2 : x^3)$; its closure X_{P_1} is a twisted cubic curve.

The Cayley sum

The polytope $P = P_0 \star P_1$:



corresponds to the embedding

$$(\mathbb{C}^*)^2 \to \mathbb{P}^6$$

given by

$$(x, y) \mapsto (1 : x : x^2 : y : xy : x^2y : x^3y);$$

its closure X_P is a rational normal scroll of type $(2, 3)$.

Hyperplane sections and discriminants

$P = P_0 \star \cdots \star P_k$ gives $X_P \subseteq \mathbb{P}^N$.

A hyperplane section of X_P:

$$h(x_1, \ldots, x_{n-k}, y_1, \ldots, y_k) := f_0 + y_1 f_1 + \cdots + y_k f_k = 0,$$

($f_i = 0$ is a hyperplane section of X_{P_i}) is singular if
$h = \partial h / \partial x_i = \partial h / \partial y_j = 0$.

Generalize the Cayley trick:

$$\mathrm{Res}(f_0(x_1, \ldots, x_{n-k}), \ldots, f_k(x_1, \ldots, x_{n-k})) \sim \Delta(h).$$

A Cayley poytope with $k = 2 > n - k = 3 - 2 = 1$

$P = P_0 \star P_1 \star P_2$

$P_j \subset \mathbb{R}$

$h(x, y, z) = f_0(x) + y_1 f_1(x) + y_2 f_2(x)$

$\Delta(h)$: eliminate x from $f_0 = f_1 = f_2 = 0$

The ideal $\Delta(h)$ has three generators:

$\mathrm{Res}(f_0, f_1)$, $\mathrm{Res}(f_0, f_2)$, $\mathrm{Res}(f_1, f_2)$

X_P is a 3-dimensional rational normal scroll. The set of hyperplanes tangent to X_P is not a hypersurface.

Discriminants and dual varieties

If $k \leq n-k$, then $\Delta(h)$ is a polynomial in the coefficients of h, and defines a hypersurface: the *dual variety* $X_P^\vee \subseteq (\mathbb{P}^N)^\vee$ of X_P.

If $k > n-k$, the system $f_0 = \cdots = f_k = 0$ has too many equations. Hence the discriminant ideal of h is not principal, and the dual variety is not a hypersurface.

A variety X is called *defective* if its dual variety $X^\vee$ is not a hypersurface. A polytope P is defective if X_P is defective.

The *defect* of a defective variety X is the positive integer $\operatorname{codim} X^\vee - 1$.

Hence: *The Cayley polytope* $P = P_0 \star \cdots \star P_k$ *is defective if* $k > n-k$.

The degree of the dual variety

Theorem (Gelfand–Kapranov–Zelevinski)

If X_P is smooth,

$$\deg X_P^\vee = \sum_{F \subseteq P} (-1)^{\operatorname{codim} F} (\dim F + 1) \mathrm{Vol}_{\mathbb{Z}}(F).$$

Proof. $\deg X_P^\vee = c_n(\mathcal{P}^1(L_P))$ is a polynomial in the Chern classes of X_P and the hyperplane bundle L_P.

$c_1(L_P)^n = \mathrm{Vol}_{\mathbb{Z}}(P) = \deg X_P$

$c_i(T_{X_P}) c_1(L_P)^{n-i} = \sum_{\operatorname{codim} F_i = i} \mathrm{Vol}_{\mathbb{Z}}(F_i)$.

$c_n(T_{X_P}) = \#$ vertices of P

kth order dual varieties

$$X^{(k)} = \overline{\{H \in \mathbb{P}^{m\vee} \mid H \text{ is tangent to } X \text{ to the order } k\}}$$
$$= \overline{\{H \in \mathbb{P}^{m\vee} \mid H \supseteq \mathbb{T}^k_{X,x} \text{ for some } x \in X_{\text{smooth}}\}},$$

$\mathbb{T}^k_{X,x} = k$th osculating space to X at x.

$\dim \mathbb{T}^k_{X,x} \leq \binom{n+k}{k} - 1$, $n = \dim X$.

$X^{(1)} = X^\vee$ and $X^{(k-1)} \supseteq X^{(k)}$

Expected dimension of $X^{(k)} = n + m - \binom{n+k}{k}$.

X is *k-defective* if $\dim X^{(k)} < n + m - \binom{n+k}{k}$.

Toric threefolds

Theorem (Dickenstein–Di Rocco–P.)

$(X,P)=(X_P,L_P)$ *smooth, 2-regular toric threefold embedding is 2-defective if and only if* $(X,L)=(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$. *Moreover:*

(1) $\deg X^{(2)}=120$ *if* $(X,L)=(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(3))$

(2) $\deg X^{(2)}=6(8(a+b+c)-7)$ *if* $(X,L)=(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(b)\oplus\mathcal{O}_{\mathbb{P}^1}(c)),2\xi)$, *where* ξ *denotes the tautological line bundle,*

(3) *In all other cases,*
$\deg X^{(2)}=62V-57F+28E-8v+58V_1+51F_1+20E_1$,
where V, F, E *(resp.* V_1, F_1, E_1*) denote the (lattice) volume, area of facets, length of edges of* P *(resp. the adjoint polytope* $\mathrm{Conv}(\mathrm{int}P)$*), and* $v=\#\{$*vertices of* $P\}$.

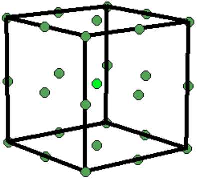
Example

If P is a cube with edge lengths 2, then $(X_P, L_P) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2,2))$.

$V = 3!8 = 48$, $F = 6 \cdot 2 \cdot 4 = 48$, $E = 12 \cdot 2 = 24$, $v = 8$.

$V_1 = F_1 = E_1 = 0$ ($\mathrm{int}(P) = \{(1,1,1)\}$ is a point)

$$\deg X^{(2)} = 62V - 57F + 28E - 8v = 848.$$



k-selfdual toric varieties (joint with A. Dickenstein)

$\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ a lattice point configuration, and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric embedding.

Form the matrix A by adding a row of 1's to the matrix $(a_0|\cdots|a_N)$. Denote by $\mathbf{v}_0 = (1, \ldots, 1)$, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^{N+1}$ the row vectors of A.

For any $\alpha \in \mathbb{N}^{n+1}$, denote by $\mathbf{v}_\alpha \in \mathbb{Z}^{N+1}$ the vector obtained as the coordinatewise product of α_0 times the row vector $\mathbf{v}_0$ times $\ldots$ times α_n times the row vector $\mathbf{v}_n$.

Order the vectors $\{\mathbf{v}_\alpha : |\alpha| \leq k\}$. Let $A^{(k)}$ be the $\binom{n+k}{k} \times (N+1)$ integer matrix with these rows.

Rational normal curve

Take $\mathcal{A} = \{0, \ldots, d\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \end{pmatrix},$$

and

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 1 & 4 & 9 & \cdots & d^2 \\ 0 & 1 & 8 & 27 & \cdots & d^3 \end{pmatrix}.$$

Note that

$$A^{(3)} \cong \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 3 & \cdots & \binom{d}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{d}{3} \end{pmatrix}.$$

The case $k = 2$

Denote by $\mathbf{v}_i * \mathbf{v}_j \in \mathbb{Z}^{m+1}$ the vector given by the coordinatewise product of these vectors. Define the $\binom{n+2}{2} \times (m+1)$-matrix

$$A^{(2)} = \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{v}_1 * \mathbf{v}_1 \\ \mathbf{v}_1 * \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} * \mathbf{v}_n \\ \mathbf{v}_n * \mathbf{v}_n \end{pmatrix},$$

$\mathbf{v}_i * \mathbf{v}_j$, $1 \leq i \leq j \leq n$. Then, $\mathbb{P}(\text{Rowspan}(A^{(2)})) = \mathbb{T}^2_{X_{\mathcal{A}},\mathbf{1}}$ describes the second osculating space of $X_{\mathcal{A}}$ at the point $\mathbf{1}$.

Non-pyramidal configurations

The configuration $\mathcal{A}$ is *non-pyramidal* (nap) if the configuration of columns in A is not a pyramid (i.e., no basis vector e_i of $\mathbb{R}^{N+1}$ lies in the rowspan of the matrix).

The configuration $\mathcal{A}$ is $k$nap if the configuration of columns in $A^{(k)}$ is not a pyramid.

Example

A is a pyramid:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$

Characterization of k-self dual configurations

$X_{\mathcal{A}}$ is *k-selfdual* if $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$ for some $\phi\colon \mathbb{P}^N \cong (\mathbb{P}^N)^\vee$.

Theorem (Dickenstein–P.)

(1) *$X_{\mathcal{A}}$ is k-selfdual if and only if $\dim X_{\mathcal{A}} = \dim X_{\mathcal{A}}^{(k)}$ and $\mathcal{A}$ is knap.*

(2) *If $\mathcal{A}$ is knap and $\dim \operatorname{Ker} A^{(k)} = 1$, then $X_{\mathcal{A}}$ is k-selfdual.*

(3) *If $\mathcal{A}$ is knap and k-selfdual, and $\dim \operatorname{Ker} A^{(k)} = r > 1$, then $\mathcal{A} = e_0 \times \mathcal{A}_0 \cup \ldots \cup e_{r-1} \times \mathcal{A}_{r-1}$ is r-Cayley.*

The proof generalizes [Bourel–Dickenstein–Rittatore] ($k = 1$).

A surface in $\mathbb{P}^3$

$$\mathcal{A} = \{(0,0),(1,0),(1,1),(0,2)\}$$

gives

$$X_{\mathcal{A}} : (x,y) \mapsto (1 : x : xy : y^2)$$

and

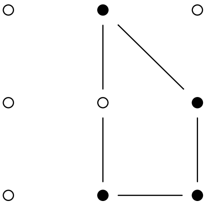
$$X_{\mathcal{A}}^{\vee} \cong X_{\mathcal{A}^{\vee}} : (x,y) \mapsto (-y^2 : 2x^{-1}y^2 : -2x^{-1}y : 1),$$

with

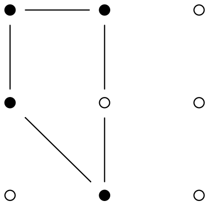
$$\mathcal{A}^{\vee} = \{(0,2),(-1,2),(-1,1),(0,0)\}.$$

This surface is self dual.

The corresponding polytopes



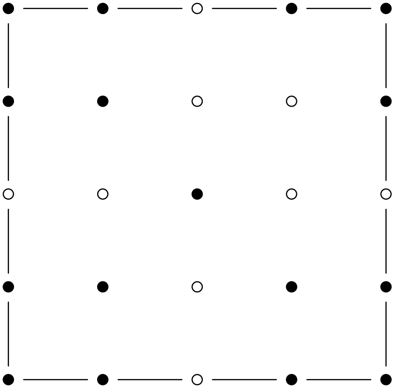
$\mathcal{A}$



$\mathcal{A}^\vee$

Example

This square is an example of a 4-selfdual smooth surface which is not centrally symmetric.



Connections with number theory

Non-trivial linear relations between the rows of $A^{(k)}$ correspond to polynomials of degree $\leq k$ vanishing on $\mathcal{A}$ (D. Perkinson).

Example

Three quadrics $Q_1, Q_2, Q_3 \in \mathbb{Z}[x_1, x_2, x_3]$ with

$$Q_1 \cap Q_2 \cap Q_3 = \{a_0, \ldots, a_7\} = \mathcal{A} \subset \mathbb{Z}^3 \subset \mathbb{R}^3.$$

Then $X_{\mathcal{A}}$ is a 2-selfdual threefold:

The rank of $A^{(2)}$ is $10 - 3 = 7$, which is one less than the maximal rank.

Such constructions give an interesting connection to diophantine theory: polynomials with many integer solutions.

Togliatti's surface

Togliatti's surface: $X_{\mathcal{A}} \subset \mathbb{P}^5$, with

$$\mathcal{A} = \{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\},$$

(omitting the interior lattice point $(1,1)$ of the hexagon).

All 2nd order osculating spaces have dimension 4 (instead of 5).

Then $\mathcal{A}$ is 2nap and $\dim \operatorname{Ker} A^{(2)} = 1$, so $X_{\mathcal{A}}$ is 2-selfdual.

The unique quadric through $\mathcal{A}' := \mathcal{A} \setminus \{(2,2)\}$ also go through the points $(4,3)$ and $(4,2)$. Thus,

$$\mathcal{A}' \cup \{(4,3)\} \text{ and } \mathcal{A}' \cup \{(4,2)\}$$

give (non-smooth, non centrally symmetric) 2-selfdual surfaces.

THANK YOU FOR YOUR ATTENTION!