Euclidean projective geometry: reciprocal polar varieties and focal loci

Ragni Piene

Boris 60
Stockholm University
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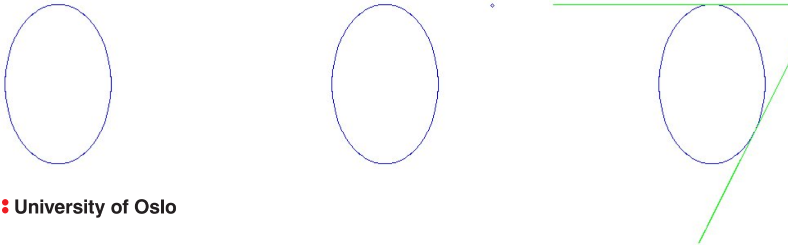
UiO : University of Oslo

Pole and polars in the plane

Let Q be a conic section. Let P be a point in the plane. There are two tangents to Q passing through P. The polar of P is the line joining the two points of tangency. Conversely, if L is a line, it intersects the conic in two points. The pole of L is the intersection of the tangents to Q at these two points.



Polarity (reciprocation)

A quadric hypersurface Q in $\mathbb{P}^n$ given by a quadratic form q, sets up a polarity between points and hyperplanes:

$$P = (b_0 : \cdots : b_n) \mapsto P^\perp = H\colon \sum b_i \frac{\partial q}{\partial X_i} = 0$$

The polar hyperplane $P^\perp$ of P is the linear span of the points on Q such that the tangent hyperplane at that point contains P.

If $P \in Q$, then $P^\perp = T_P Q$.

If $L \subset \mathbb{P}^m$ is a linear space, then $L^\perp = \cap_{P \in Q \cap L} T_P Q$.

If H is a hyperplane, its pole $H^{\perp}$ is the intersection of the tangent hyperplanes to Q at the points of intersection with H.

Example

If the quadric is $q = \sum X_i^2$, then the polar of $P = (b_0 : \cdots : b_m)$ is the hyperplane $P^{\perp} : b_0 X_0 + \ldots + b_n X_n = 0$.

The polar of the hyperplane $H : b_0 X_0 + \ldots + b_n X_n = 0$ is the point $H^{\perp} = (b_0 : \cdots : b_m)$.

Grassmann and Schubert varieties

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{G}(m, n)$ denote the Grassmann variety of $(m+1)$-spaces in k^{n+1}, or equivalently, of m-dimensional linear subspaces of $\mathbb{P}^n_k$.

Let

$$L_\bullet \colon L_0 \subset L_1 \subset \cdots \subset L_m \subset \mathbb{P}^n_k$$

be a flag of linear subspaces, with $\dim L_i = a_i$.

The Schubert variety $\Omega(L_\bullet)$ is defined by

$$\Omega(L_\bullet) := \{W \in \mathbb{G}(m, n) \mid \dim W \cap L_i \geq i, 0 \leq i \leq m\}.$$

The class of $\Omega(L_\bullet)$ depends only on the a_i. Write

$$\Omega(L_\bullet) = \Omega(a_0, \ldots, a_m).$$

Example

- $m = 1$, $n = 3$: $\mathbb{G}(1,3) =$ lines in $\mathbb{P}^3$.

$$\Omega(1,3) = \text{lines meeting a given line}$$

$$\Omega(0,3) = \text{lines through a given point}$$

$$\Omega(1,2) = \text{lines in a given plane}$$

$$\Omega(0,2) = \text{lines in a plane through a point in the plane}$$

- $m = 2$, $n = 5$: $\mathbb{G}(2,5) =$ planes in $\mathbb{P}^5$.

$$\Omega(1,4,5) = \text{planes meeting a given line}$$

$$\Omega(2,4,5) = \text{planes meeting a given plane}$$

The Gauss map

- Projective variety $X \subset \mathbb{P}^n$, $\dim X = m$. The Gauss map is

$$\gamma\colon X \dashrightarrow \mathbb{G}(m,n)\,;P \mapsto T_P X$$

$T_P X$ =the projective tangent space to X at P.

- Affine variety $X \subset \mathbb{A}^n$, $\dim X = m$. The Gauss map is

$$\gamma\colon X \dashrightarrow \mathbb{G}(m-1,n-1)\,;P \mapsto t_P X$$

$t_P X$ = the affine tangent space to X at P (considered as a subspace of k^n).

Polar varieties

The polar varieties of $X \subset \mathbb{P}^n$ are the inverse images

$$P(L_\bullet) := \gamma^{-1}\Omega(L_\bullet)$$

of the Schubert varieties via the Gauss map.

Example

- $X \subset \mathbb{P}^2$ (resp. $\mathbb{P}^3$) is a curve (m=1). Then $P(0,2)$ (resp. $P(1,3)$) is the set of points $P \in X$ such that T_P meets a given point (resp. line), i.e., the ramification points of the projection map $X \to \mathbb{P}^1$.
- $X \subset \mathbb{P}^5$ is a surface ($m = 2$). Then $P(1,4,5)$ is the ramification locus of the projection map $X \to \mathbb{P}^3$ with center a line L_0.

Polar varieties and Chern classes

Let $L_k \subset \mathbb{P}^n$ be a linear subspace of codimension $m - k + 2$.

The *kth polar variety* of of $X \subset \mathbb{P}^n$ with respect to L_k is

$$M_k := \{x \in X \mid \dim(T_{X,x} \cap L_k) \geq k - 1\}.$$

Its class is $[M_k] = c_k(\mathcal{P}^1_X(1)) \cap [X]$, hence we get the Todd–Eger relation

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} c_i(T_X) \cap [X], \qquad (1)$$

where $h = c_1(\mathcal{O}_X(1))$ is the class of a hyperplane.

Singular varieties and Nash transform

If X is singular, we take its Nash transform $\pi : \overline{X} \to X$ and replace Ω^1_X by the Nash bundle Ω on $\overline{X}$.

The Mather–Chern classes of X are

$$c_i^M(X) = \pi_*(c_i(\Omega^\vee) \cap [\overline{X}]),$$

and we get for the polar varieties:

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap c_i^M(X). \tag{2}$$

Affine space as Euclidean space

Coxeter:

"Kepler's invention of points at infinity made it possible to regard the projective plane as the affine plane plus the line at infinity. A converse relationship was suggested by Poncelet (1822) and von Staudt (1847): regard the affine plane as the projective plane minus an arbitrary line ℓ, and then regard the Euclidean plane as the affine plane with a special rule for associating pairs of points on ℓ (in "perpendicular directions")."

This way we can consider affine space with an added notion of orthogonality (or perpendicularity) as "Euclidean space" (no distance function).

Euclidean normal spaces

Consider $X \subset \mathbb{P}^n = \mathbb{P}(V)$, fix a hyperplane $H_\infty \subset \mathbb{P}^n$ at infinity and a smooth quadric Q in H_∞.

Use the polarity in $H_\infty \cong \mathbb{P}^{n-1} = \mathbb{P}(V')$ induced by Q to define *Euclidean normal spaces* at each smooth point $P \in X \setminus H_\infty$:

$$N_P X = \langle P, (T_P X \cap H_\infty)^\perp \rangle$$

Consider $0 \to V'' \to V \to V' \to 0$ and

$$0 \to \mathcal{N}_X(1) \to V_X \to \mathcal{P}^1_X(1) \to 0.$$

Assume (transversality of tangent spaces and H_∞) this induces

$$0 \to V'' \to \mathcal{P}^1_X(1) \to \mathcal{P} \to 0.$$

Then we get

$$0 \to \mathcal{N}_X(1) \to V'_X \to \mathcal{P} \to 0.$$

The polarity in H_∞ given by Q, gives $V'^\vee \cong V'$, so we have

$$V'_X \to \mathcal{N}_X(1)^\vee$$

corresponding to the spaces orthogonal to the spaces $T_P X \cap H_\infty$, and combining with $V_X \to V'_X$ and $V_X \to \mathcal{O}_X(1)$, we get (assuming transversality of X and Q)

$$V_X \to \mathcal{N}_X^\vee(-1) \oplus \mathcal{O}_X(1)$$

whose fibers correspond to the Euclidean normal spaces $N_P X$.

Euclidean normal bundle (Catanese–Trifogli)

We call $\mathcal{E} := \mathcal{N}_X^\vee(-1) \oplus \mathcal{O}_X(1)$ the *Euclidean normal bundle* of X.

The normal spaces are the fibers of the projective bundle

$$\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n \to X.$$

Reciprocal polar varieties

There are several (essentially equivalent) ways of defining *reciprocal* polar varieties of projective and affine varieties.

We can mimic the definition of classical polar varieties, by exchanging the tangent spaces with the normal spaces.

Consider the normal map $\nu : X \dashrightarrow \mathbb{G}(n-m, n)$ given by $P \mapsto N_P X$. The *reciprocal polar varieties* are

$$R(L_\bullet) := \nu^{-1}(\Omega(L_\bullet)),$$

where $\Omega(L_\bullet) \subset \mathbb{G}(n-m, n)$ is the Schubert variety corresponding to the flag $L_\bullet$.

Applications

Polar and reciprocal polar varieties have been applied to study

- singularities
- the topology of real affine varieties
- real solutions of polynomial equations
- complexity questions
- foliations
- finding nonsingular points on every component of a real affine plane curve (Banks et al. for smooth curves, Mork–P for compact curves with ordinary multiple points, counterexamples for curves with worse singularities).

Figure: A sextic and its polar.

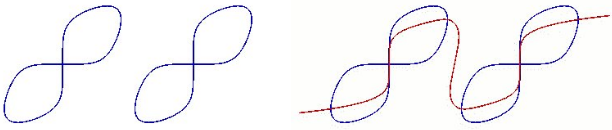
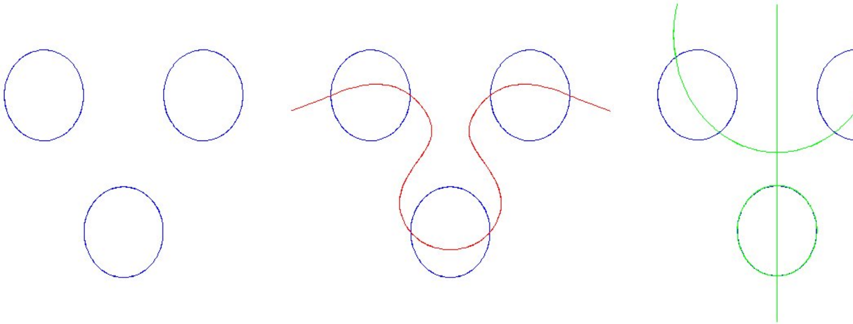


Figure: A sextic curve with its polar with its reciprocal polar.



There exist compact singular real affine plane curves such that no polar variety contains a point from each connected component, e.g. this sextic with eight cusps:

The Euclidean endpoint map

Consider $\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n$.

Let $p\colon \mathbb{P}(\mathcal{E}) \to X$ and $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ denote the projections on the first and second factor. The map q is called the *endpoint map*.

Let $A \in \mathbb{P}^n \setminus H_\infty$. Then $p(q^{-1}(A))$ is a reciprocal polar variety:

$$p(q^{-1}(A)) = \{P \in X \mid A \in \langle P, (T_P X \cap H_\infty)^\perp \rangle$$

so the degree of q is the degree of the reciprocal polar variety.

Euclidean distance degree[1]

The (general) *Euclidean distance degree* is the degree of the map $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Hence

$$\operatorname{E}\deg X = \deg p_* c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^n \cap [X] = \deg s_m(\mathcal{E}),$$

where $m = \dim X$. We can compute:

$$s(\mathcal{E}) = s(\mathcal{N}^\vee_{X/\mathbb{P}^n}(-1))s(\mathcal{O}_X(1)) = c(\mathcal{P}^1_X(1))c(\mathcal{O}_X(-1))^{-1}$$

We conclude:

$$\operatorname{E}\deg X = \sum_{k=0}^m \mu_k,$$

where μ_k is the degree of the kth polar variety $[M_k]$ of X.

[1] Draisma–Horobet–Ottaviani–Sturmfels–Thomas.

Hypersurfaces with isolated singularities

If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $\mu_0 = d$, then $\mu_k = d(d-1)^k$.

If X has only isolated singularities, then only μ_{n-1} is affected, and we get (from Teissier's formula and the Plücker formula for hypersurfaces with isolated singularities (Teissier, Laumon))

$$\operatorname{ED}\deg X = \frac{d((d-1)^n - 1)}{d-2} - \sum_{P \in \operatorname{Sing}(X)} (\mu_P^{(n)} + \mu_P^{(n-1)}),$$

where $\mu_P^{(n)}$ is the Milnor number and $\mu_P^{(n-1)}$ is the sectional Milnor number of X at P.

Surface with ordinary singularities

Assume $X \subset \mathbb{P}^3$ is a generic projection of a smooth surface of degree $\mu_0 = d$, so that X has *ordinary* singularities: a double curve of degree ϵ, t triple points, and ν_2 pinch points. Then (using known formulas for μ_1 and μ_2)

$$\mathrm{ED}\deg X = \mu_0 + \mu_1 + \mu_2 = d^3 - d^2 + d - (3d-2)\epsilon - 3t - 2\nu_2.$$

The focal locus

The focal locus Σ_X is the branch locus of the map q.

It is the image of the subscheme R_X given by the ideal $F^0(\Omega^1_{\mathbb{P}(\mathcal{E})/\mathbb{P}^n})$, so its class is

$$[\Sigma_X] = q_*((c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^*c_1(\Omega^1_{\mathbb{P}^n})\big) \cap [\mathbb{P}(\mathcal{E})]).$$

Example

$X \subset \mathbb{P}^2$ is a (general) plane curve of degree d. Then the focal locus is the *evolute* (or caustic) of X. Its degree is the degree of the class

$$q_*\big(\big(c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^*c_1(\Omega^1_{\mathbb{P}^2})\big)c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E}]\big)$$

hence is given by $\deg \Sigma_X = 3d(d-1)$.

In the case that X is a "Plücker curve" of degree $d = \mu_0$ and having only δ nodes and κ ordinary cusps as singularities, and ι ordinary inflection points, then we obtain the classical formula due to Salmon

$$\deg \Sigma_X = 3d(d-1) - 6\delta - 8\kappa.$$

The focal locus of a hypersurface

Let $X \subset \mathbb{P}^n$ be a general hypersurface $(m = n - 1)$ of degree μ_0. It is known that in this case $R_X \to \Sigma_X$ is birational. We compute

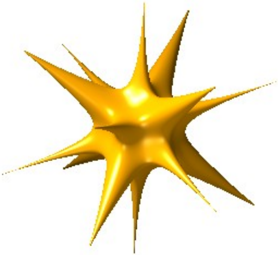
$$\deg \Sigma_X = (n-1)\mu_{n-1} + 2(\mu_0 - 1) \sum_{i=0}^{n-2} \mu_i.$$

For a smooth hypersurface of degree d in $\mathbb{P}^n$, we have $\mu_i = d(d-1)^i$. Hence

$$\deg \Sigma_X = (n-1)d(d-1)^{n-1} + 2d(d-1)((d-1)^{n-1} - 1)(d-2)^{-1},$$

which checks with the formula found by Trifogli.

Happy birthday, Boris!



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