Euclidean projective geometry: reciprocal polar varieties and focal loci

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Pole and polars in the plane

Let $Q$ be a conic section. Let $P$ be a point in the plane. There are two tangents to $Q$ passing through $P$. The polar of $P$ is the line joining the two points of tangency. Conversely, if $L$ is a line, it intersects the conic in two points. The pole of $L$ is the intersection of the tangents to $Q$ at these two points.
Polarity (reciprocation)

A quadric hypersurface $Q$ in $\mathbb{P}^n$ given by a quadratic form $q$, sets up a polarity between points and hyperplanes:

$$P = (b_0 : \cdots : b_n) \mapsto P^\perp = H : \sum b_i \frac{\partial q}{\partial X_i} = 0$$

The polar hyperplane $P^\perp$ of $P$ is the linear span of the points on $Q$ such that the tangent hyperplane at that point contains $P$.

If $P \in Q$, then $P^\perp = T_PQ$.

If $L \subset \mathbb{P}^m$ is a linear space, then $L^\perp = \cap_{P \in Q \cap L} T_PQ$. 
If $H$ is a hyperplane, its pole $H^\perp$ is the intersection of the tangent hyperplanes to $Q$ at the points of intersection with $H$.

**Example**

If the quadric is $q = \sum X_i^2$, then the polar of $P = (b_0 : \cdots : b_m)$ is the hyperplane $P^\perp : b_0X_0 + \ldots + b_nX_n = 0$.

The polar of the hyperplane $H : b_0X_0 + \ldots + b_nX_n = 0$ is the point $H^\perp = (b_0 : \cdots : b_m)$. 
Grassmann and Schubert varieties

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{G}(m, n)$ denote the Grassmann variety of $(m + 1)$-spaces in $k^{n+1}$, or equivalently, of $m$-dimensional linear subspaces of $\mathbb{P}^n_k$.

Let

$$L_\bullet : L_0 \subset L_1 \subset \cdots \subset L_m \subset \mathbb{P}^n_k$$

be a flag of linear subspaces, with $\dim L_i = a_i$.

The Schubert variety $\Omega(L_\bullet)$ is defined by

$$\Omega(L_\bullet) := \{ W \in \mathbb{G}(m, n) \mid \dim W \cap L_i \geq i, 0 \leq i \leq m \}.$$

The class of $\Omega(L_\bullet)$ depends only on the $a_i$. Write

$$\Omega(L_\bullet) = \Omega(a_0, \ldots, a_m).$$
Example

- \( m = 1, \ n = 3 \): \( G(1, 3) = \) lines in \( \mathbb{P}^3 \).

\( \Omega(1, 3) = \) lines meeting a given line

\( \Omega(0, 3) = \) lines through a given point

\( \Omega(1, 2) = \) lines in a given plane

\( \Omega(0, 2) = \) lines in a plane through a point in the plane

- \( m = 2, \ n = 5 \): \( G(2, 5) = \) planes in \( \mathbb{P}^5 \).

\( \Omega(1, 4, 5) = \) planes meeting a given line

\( \Omega(2, 4, 5) = \) planes meeting a given plane
The Gauss map

- Projective variety $X \subset \mathbb{P}^n$, $\dim X = m$. The Gauss map is
  \[ \gamma: X \rightarrow \mathbb{G}(m, n); P \mapsto T_P X \]
  \[ T_P X = \text{the projective tangent space to } X \text{ at } P. \]

- Affine variety $X \subset \mathbb{A}^n$, $\dim X = m$. The Gauss map is
  \[ \gamma: X \rightarrow \mathbb{G}(m - 1, n - 1); P \mapsto t_P X \]
  \[ t_P X = \text{the affine tangent space to } X \text{ at } P \text{ (considered as a subspace of } k^n). \]
Polar varieties

The polar varieties of $X \subset \mathbb{P}^n$ are the inverse images

$$P(L\bullet) := \gamma^{-1}\Omega(L\bullet)$$

of the Schubert varieties via the Gauss map.

Example

- $X \subset \mathbb{P}^2$ (resp. $\mathbb{P}^3$) is a curve ($m=1$). Then $P(0, 2)$ (resp. $P(1, 3)$) is the set of points $P \in X$ such that $T_P$ meets a given point (resp. line), i.e., the ramification points of the projection map $X \to \mathbb{P}^1$.

- $X \subset \mathbb{P}^5$ is a surface ($m = 2$). Then $P(1, 4, 5)$ is the ramification locus of the projection map $X \to \mathbb{P}^3$ with center a line $L_0$.

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Polar varieties and Chern classes

Let $L_k \subset \mathbb{P}^n$ be a linear subspace of codimension $m - k + 2$.

The $k$th polar variety of of $X \subset \mathbb{P}^n$ with respect to $L_k$ is

$$ M_k := \{ x \in X \mid \dim(T_{X,x} \cap L_k) \geq k - 1 \}. $$

Its class is $[M_k] = c_k(\mathcal{P}^1_X(1)) \cap [X]$, hence we get the Todd–Eger relation

$$ [M_k] = \sum_{i=0}^{k} (-1)^i \binom{m - i + 1}{m - k + 1} h^{k-i} c_i(T_X) \cap [X], \quad (1) $$

where $h = c_1(\mathcal{O}_X(1))$ is the class of a hyperplane.
Singular varieties and Nash transform

If $X$ is singular, we take its Nash transform $\pi : \overline{X} \to X$ and replace $\Omega^1_X$ by the Nash bundle $\Omega$ on $\overline{X}$.

The Mather–Chern classes of $X$ are

$$c^M_i(X) = \pi_*(c_i(\Omega^\vee) \cap [\overline{X}]),$$

and we get for the polar varieties:

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m - i + 1}{m - k + 1} h^{k-i} \cap c^M_i(X). \quad (2)$$
Affine space as Euclidean space

Coxeter:

“Kepler’s invention of points at infinity made it possible to regard the projective plane as the affine plane plus the line at infinity. A converse relationship was suggested by Poncelet (1822) and von Staudt (1847): regard the affine plane as the projective plane minus an arbitrary line \( \ell \), and then regard the Euclidean plane as the affine plane with a special rule for associating pairs of points on \( \ell \) (in “perpendicular directions”).”

This way we can consider affine space with an added notion of orthogonality (or perpendicularity) as “Euclidean space” (no distance function).
Euclidean normal spaces

Consider $X \subset \mathbb{P}^n = \mathbb{P}(V)$, fix a hyperplane $H_\infty \subset \mathbb{P}^n$ at infinity and a smooth quadric $Q$ in $H_\infty$.

Use the polarity in $H_\infty \cong \mathbb{P}^{n-1} = \mathbb{P}(V')$ induced by $Q$ to define *Euclidean normal spaces* at each smooth point $P \in X \setminus H_\infty$:

$$N_PX = \langle P, (T_PX \cap H_\infty)^\perp \rangle$$

Consider $0 \to V'' \to V \to V' \to 0$ and

$$0 \to \mathcal{N}_X(1) \to V_X \to \mathcal{P}_X^1(1) \to 0.$$

Assume (transversality of tangent spaces and $H_\infty$) this induces

$$0 \to V'' \to \mathcal{P}_X^1(1) \to \mathcal{P} \to 0.$$
Then we get
\[ 0 \rightarrow \mathcal{N}_X(1) \rightarrow V'_X \rightarrow \mathcal{P} \rightarrow 0. \]

The polarity in \( H_\infty \) given by \( Q \), gives \( V'^\lor \cong V' \), so we have
\[ V'_X \rightarrow \mathcal{N}_X(1)^\lor \]

corresponding to the spaces orthogonal to the spaces \( T_P X \cap H_\infty \), and combining with \( V_X \rightarrow V'_X \) and \( V_X \rightarrow \mathcal{O}_X(1) \), we get (assuming transversality of \( X \) and \( Q \))
\[ V_X \rightarrow \mathcal{N}_X^\lor(-1) \oplus \mathcal{O}_X(1) \]

whose fibers correspond to the Euclidean normal spaces \( N_{PX} \).
We call \( \mathcal{E} := \mathcal{N}_X^\vee(-1) \oplus \mathcal{O}_X(1) \) the Euclidean normal bundle of \( X \).

The normal spaces are the fibers of the projective bundle

\[
\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n \to X.
\]
Reciprocal polar varieties

There are several (essentially equivalent) ways of defining reciprocal polar varieties of projective and affine varieties. We can mimic the definition of classical polar varieties, by exchanging the tangent spaces with the normal spaces.

Consider the normal map $\nu : X \to \mathbb{G}(n - m, n)$ given by $P \mapsto N_P X$. The reciprocal polar varieties are

$$R(L_\bullet) := \nu^{-1}(\Omega(L_\bullet)),$$

where $\Omega(L_\bullet) \subset \mathbb{G}(n - m, n)$ is the Schubert variety corresponding to the flag $L_\bullet$.
Applications

Polar and reciprocal polar varieties have been applied to study

- singularities
- the topology of real affine varieties
- real solutions of polynomial equations
- complexity questions
- foliations
- finding nonsingular points on every component of a real affine plane curve (Banks et al. for smooth curves, Mork–P for compact curves with ordinary multiple points, counterexamples for curves with worse singularities).
Figure: A sextic and its polar.
Figure: A sextic curve with its polar with its reciprocal polar.
There exist compact singular real affine plane curves such that no polar variety contains a point from each connected component, e.g. this sextic with eight cusps:
The Euclidean endpoint map

Consider $\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n$.

Let $p : \mathbb{P}(\mathcal{E}) \to X$ and $q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ denote the projections on the first and second factor. The map $q$ is called the endpoint map.

Let $A \in \mathbb{P}^n \setminus H_\infty$. Then $p(q^{-1}(A))$ is a reciprocal polar variety:

$$p(q^{-1}(A)) = \{ P \in X \mid A \in \langle P, (T_P X \cap H_\infty)^\perp \rangle \}$$

so the degree of $q$ is the degree of the reciprocal polar variety.
Euclidean distance degree

The (general) Euclidean distance degree is the degree of the map \( q : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n \). Hence

\[
E \deg X = \deg p_* c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^n \cap [X] = \deg s_m(\mathcal{E}),
\]

where \( m = \dim X \). We can compute:

\[
s(\mathcal{E}) = s(\mathcal{N}_{X/\mathbb{P}^n}(-1)) s(\mathcal{O}_X(1)) = c(\mathcal{P}^1_X(1)) c(\mathcal{O}_X(-1))^{-1}
\]

We conclude:

\[
E \deg X = \sum_{k=0}^m \mu_k,
\]

where \( \mu_k \) is the degree of the \( k \)th polar variety \([M_k]\) of \( X \).

\(^1\text{Draisma–Horobet–Ottaviani–Sturmfels–Thomas.}\)
Hypersurfaces with isolated singularities

If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $\mu_0 = d$, then

$$\mu_k = d(d - 1)^k.$$  

If $X$ has only isolated singularities, then only $\mu_{n-1}$ is affected, and we get (from Teissier’s formula and the Plücker formula for hypersurfaces with isolated singularities (Teissier, Laumon))

$$\text{ED deg } X = \frac{d((d - 1)^n - 1)}{d - 2} - \sum_{P \in \text{Sing}(X)} (\mu_P^{(n)} + \mu_P^{(n-1)}),$$

where $\mu_P^{(n)}$ is the Milnor number and $\mu_P^{(n-1)}$ is the sectional Milnor number of $X$ at $P$. 

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Surface with ordinary singularities

Assume $X \subset \mathbb{P}^3$ is a generic projection of a smooth surface of degree $\mu_0 = d$, so that $X$ has *ordinary* singularities: a double curve of degree $\epsilon$, $t$ triple points, and $\nu_2$ pinch points. Then (using known formulas for $\mu_1$ and $\mu_2$)

$$\text{ED deg } X = \mu_0 + \mu_1 + \mu_2 = d^3 - d^2 + d - (3d - 2)\epsilon - 3t - 2\nu_2.$$
The focal locus $\Sigma_X$ is the branch locus of the map $q$. It is the image of the subscheme $R_X$ given by the ideal $F^0(\Omega^1_{\mathbb{P}(E)/\mathbb{P}n})$, so its class is

$$[\Sigma_X] = q_*((c_1(\Omega^1_{\mathbb{P}(E)}) - q^*c_1(\Omega^1_{\mathbb{P}n})) \cap [\mathbb{P}(E)]).$$
Example

$X \subset \mathbb{P}^2$ is a (general) plane curve of degree $d$. Then the focal locus is the *evolute* (or caustic) of $X$. Its degree is the degree of the class

$$q_*(\left( c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^* c_1(\Omega^1_{\mathbb{P}^2}) \right) c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E})])$$

hence is given by $\deg \Sigma_X = 3d(d - 1)$.

In the case that $X$ is a “Plücker curve” of degree $d = \mu_0$ and having only $\delta$ nodes and $\kappa$ ordinary cusps as singularities, and $\iota$ ordinary inflection points, then we obtain the classical formula due to Salmon

$$\deg \Sigma_X = 3d(d - 1) - 6\delta - 8\kappa.$$
The focal locus of a hypersurface

Let $X \subset \mathbb{P}^n$ be a general hypersurface ($m = n - 1$) of degree $\mu_0$. It is known that in this case $R_X \to \Sigma_X$ is birational. We compute

$$\deg \Sigma_X = (n - 1)\mu_{n-1} + 2(\mu_0 - 1) \sum_{i=0}^{n-2} \mu_i.$$ 

For a smooth hypersurface of degree $d$ in $\mathbb{P}^n$, we have $\mu_i = d(d - 1)^i$. Hence

$$\deg \Sigma_X = (n - 1)d(d - 1)^{n-1} + 2d(d - 1)((d - 1)^{n-1} - 1)(d - 2)^{-1},$$

which checks with the formula found by Trifogli.
Happy birthday, Boris!