

Classifying regular lattice polytopes using toric fibrations

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Lattice polytopes

Let $P \subset \mathbb{R}^n$ be a convex lattice polytope, i.e., P is the convex hull of a finite set of lattice points.

Write

$$P = \cap H_{\rho_i, -a_i}^+$$

where the $H_{\rho_i, -a_i}^+$ are halfplanes defined by the normal vectors ρ_i and the a_i 's are integers.

The *codegree* of P is defined (Batyrev–Nill) by

$$\text{codeg}(P) = \min_{\mathbb{N}} \{m \mid mP \text{ has interior lattice points}\}$$

For example, $\text{codeg}(\Delta_n) = n + 1$ and $\text{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil$.

The degree of P is $\deg(P) = n + 1 - \text{codeg}(P)$.

Cayley polytopes

A generalized *Cayley polytope* is a polytope of the form

$$P = \text{Cayley}^s(P_0, \dots, P_k)$$

where $P_i = \text{Conv}(\{p_j^i\}) \subset \mathbb{R}^m$, $e_0, \dots, e_k \in \mathbb{R}^k$ are the vertices of the standard simplex, and $P = \text{Conv}(\{(p_j^i, se_i)\}) \subset \mathbb{R}^{m+k}$.

If $s = 1$, $P = \text{Cayley}(P_0, \dots, P_k)$ is called a Cayley polytope.

Batyrev–Nill asked: Given d , does there exist an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

We believe that if P is regular, then $N(d) = 2d + 1$. We prove it under an additional assumption (which we believe always holds).

Note: $n \geq 2d + 1$ is the same as $\text{codeg}(P) \geq \frac{n+3}{2}$.

Codegree of polytopes

Assume from now on $P = \cap H_{\rho_i, -a_i}^+ \subset \mathbb{R}^n$ is regular, i.e., at any vertex there are precisely n edges, and the first lattice points on these edges form a basis for the lattice.

Set $P^{(r)} = \cap H_{\rho_i, -a_i+r}^+$. Because P is regular, $P^{(1)}$ is the convex hull of the interior lattice points of P . Note that $P^{(1)}$ need not be regular nor of the same dimension as P .

The $\mathbb{Q}$ -codegree of P is defined as

$$\text{codeg}_{\mathbb{Q}}(P) = \inf_{\mathbb{Q}} \left\{ \frac{a}{b} \mid (aP)^{(b)} \neq \emptyset \right\}.$$

Note that we have

$$\text{codeg}(P) \geq \text{codeg}_{\mathbb{Q}}(P) \geq \text{codeg}(P) - 1.$$

Example



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$$\text{codeg}_{\mathbb{Q}}(P) = \frac{n+1}{2} \text{ and } \text{codeg}(P) = \lceil \frac{n+1}{2} \rceil$$



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Nef value for polytopes

We say that $P^{(r)}$ is *spanned* if $P^{(r)} \neq \emptyset$ and for any vertex $m = (-a_1, \dots, -a_n)$ (in the lattice basis $\{\rho_1, \dots, \rho_n\}$) the lattice point $(-a_1 + r, \dots, -a_n + r)$ is in $P^{(r)}$.

Example

Let P be the polytope obtained from the simplex

$$m\Delta_3 = \text{Conv}\{(0 : 0 : 0), (m : 0 : 0), (0 : m : 0), (0 : 0 : m)\}$$

by removing the simplex

$$\Delta_3 = \text{Conv}\{(0 : 0 : 0), (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$$

Assume $m \geq 4$. Then $P^{(1)} \neq \emptyset$, but $P^{(1)}$ is not spanned.

In fact, the vertex $(1 : 0 : 0) = (y = 0) \cap (z = 0) \cap (x + y + z = 1)$ of P "goes to" the lattice point

$$(0 : 1 : 1) = (y = 1) \cap (z = 1) \cap (x + y + z = 2)$$

which is not a point in $P^{(1)}$. Similarly for the vertices $(0 : 1 : 0)$ and $(0 : 0 : 1)$.

Note that if we instead remove $2\Delta_3$, then the polytope is spanned: the vertices $(2 : 0 : 0)$, $(0 : 2 : 0)$, and $(0 : 0 : 2)$ all go to the same lattice point $(1 : 1 : 1)$, which is an interior point of the original polytope.



Define the *nef value* of the polytope P to be

$$\tau(P) = \inf_{\mathbb{Q}} \left\{ \frac{a}{b} \mid (aP)^{(b)} \text{ is spanned} \right\}.$$

Clearly, $\tau(P) \geq \text{codeg}_{\mathbb{Q}}(P)$, and if $\tau(P)$ is an integer, then $\tau(P) \geq \text{codeg}(P)$.

Example

Take $P = m\Delta_3 \setminus \Delta_3$, $m \geq 4$. Then $\text{codeg}_{\mathbb{Q}}(P) = \text{codeg}(P) = 1$ and $\tau(P) = 2$ (because $P^{(1)}$ is not spanned at the vertices of the small facet).

We believe that when the codegree is big enough, $\tau(P) > \text{codeg}_{\mathbb{Q}}(P)$ cannot happen.



The theorem

Theorem

Let P be a regular lattice polytope of dimension n , and assume that $\tau(P) = \text{codeg}_{\mathbb{Q}}(P)$. The following are equivalent

- (1) $\text{codeg}(P) \geq \frac{n+3}{2}$
- (2) $P = \text{Cayley}(P_0, \dots, P_k)$, where $k = \text{codeg}(P) - 1$ and $k > \frac{n}{2}$.

Note that a polytope as in (2) is *defective*, with defect $\delta = 2k - n$.



Adjunction on toric varieties

Let X be a nonsingular projective variety, L an ample line bundle. Assume the canonical line bundle K_X is not nef. The nef value of (X, L) is

$$\tau_L = \min_{\mathbb{R}} \{t \mid K_X + tL \text{ is nef} \}.$$

It is well known that τ_L is a positive rational number.

Assume X is a toric variety, and let $P = \cap H_{\rho_i, -a_i}^+$ denote the polytope defined by (X, L) . Hence $L = \sum a_i D_i$, where the D_i are the invariant divisors, and $K_X = -\sum D_i$. On a toric variety, a line bundle is nef if and only if it is spanned (generated by its global sections).

Since $P_{bK_X+aL} = (aP)^{(b)}$, we have $\tau_L = \tau(P)$.

Proof of the theorem

If $\tau = \frac{a}{b}$, the line bundle $bK_X + aL$ defines a morphism from X to a projective space. The Remmert–Stein factorization $\phi: X \rightarrow Y$ of this map is called the nef value map.

Lemma

Assume $\tau := \tau(P) = \text{codeg}_{\mathbb{Q}}(P)$. Then ϕ is not birational.

It follows from this lemma and $\tau \geq \frac{n+1}{2}$ that there exists a line C (with respect to L) on X which is contracted by ϕ , i.e., such that $(K_X + \tau L) \cdot C = 0$. It follows that τ is an integer, hence equal to $\text{codeg}(P)$.

By adjunction theory, the inequality $\tau \geq \frac{n+3}{2}$ implies that ϕ is the contraction of an extremal ray in the nef cone $\text{NE}(X)$.

By a result of Reid, such a contraction is a flat toric fibration, with Y smooth and toric and the general fiber $F = \mathbb{P}^k$, where $k = n - \dim Y$. Hence, $L|_F = \mathcal{O}_{\mathbb{P}}^k(s)$ for some s , and one shows that $s = 1$. This forces all fibers of ϕ to be $\mathbb{P}^k$, and therefore $X = \mathbb{P}(\phi_* L)$ is a projective bundle.

Taking a line ℓ in $\mathbb{P}^k = F$, we get

$$0 = (K_X + \tau L) \cdot \ell = K_X \cdot \ell + \tau s = -(k+1) + \tau s$$

so that

$$\frac{n+3}{2} \leq \tau = \frac{k+1}{s} \leq \frac{n+1}{s}.$$

So $s = 1$ and $\tau = k+1$, with $k > \frac{n}{2}$.

Since Y is toric, $\phi_* L$ splits as a sum of line bundles, and thus ϕ gives P the structure of a Cayley polytope.



For $n \leq 7$, it seems our assumption $\tau(P) = \text{codeg}_{\mathbb{Q}}(P)$ holds.

We obtain the following corollary (obtained by Batyrev–Nill in the more general case of not necessarily regular polytopes).

Corollary

Assume P is a regular n -dimensional lattice polytope.

- (1) $\text{codeg}(P) = \text{codeg}_{\mathbb{Q}}(P) = \tau(P) = n + 1$ if and only if $P = \Delta_n$
- (2) If $P \neq \Delta_n$, then $\text{codeg}_{\mathbb{Q}}(P) \leq \tau(P) \leq n$.



If $P = \text{Cayley}^s(P_0, \dots, P_k)$, then

$$\text{codeg}_{\mathbb{Q}}(P) = \max_{\mathbb{Q}} \left\{ \frac{k+1}{s}, \text{codeg}_{\mathbb{Q}}(P_i) \right\}$$

and

$$\tau(P) = \max_{\mathbb{Q}} \left\{ \frac{k+1}{s}, \tau(P_i) \right\}.$$



Corollary

Assume P is a regular n -dimensional lattice polytope.

- (1) $\deg(P) = 0$ if and only if $P = \Delta_n$,*
- (2) $\deg(P) = 1$ if and only if $P = \text{Cayley}(P_0, \dots, P_{n-1})$ is a Lawrence prism (the P_i are segments) or $P = 2\Delta_2$.*

