Higher order dual and polar varieties

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Osculating spaces

Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a projective variety of dimension m, and $P \in X$. There is a sequence of osculating spaces to X at P:

$$\{P\} \subseteq T_P = \mathrm{Osc}_P^1 \subseteq \mathrm{Osc}_P^2 \subseteq \mathrm{Osc}_P^3 \subseteq \cdots \subseteq \mathbb{P}^n,$$

defined via the sheaves of principal parts: let $m_k + 1$ denote the generic rank of the k-jet map

$$j_k \colon V_X \to \mathcal{P}_X^k(1).$$

Then $m_0 = 0$ and $m_1 = m$, and $\dim \mathrm{Osc}_P^k = \min\{n, m_k\}$.

Polar varieties

Let $L_i \subset \mathbb{P}^n$ be a linear subspace of codimension $m - i + 2$.

The *ith polar variety* of of X (with respect to L_i) is

$$M_i := \overline{\{P \in X_{\rm sm} \mid \dim(T_P \cap L_i) \geq i - 1\}}.$$

The classes $[M_i]$ are *projective invariants* of X: the ith class of a (general) projection of X is the projection of the ith class of X, and the ith class of a (general) linear section is the linear section of the ith class.

Note that $\deg[M_m] = \deg X^\vee$, where $X^\vee \subset (\mathbb{P}^n)^\vee$ denotes the dual variety of X.

Higher order polar varieties

Assume $m_k < n$. Let $L_{k,i} \subset \mathbb{P}^n$ be a linear subspace of codimension $m_k - i + 2$.

The *ith polar variety of order k* of X (with respect to $L_{k,i}$) is

$$M_{k,i} := \overline{\{P \in X_{k-\mathrm{cst}} \mid \dim(\mathrm{Osc}_P^k \cap L_i) \geq i - 1\}},$$

where $X_{k-\mathrm{cst}} \subseteq X$ denotes the open where the rank of j_k is $m_k + 1$.

The classes $[M_{k,i}]$ are projective invariants of X, like the usual polar classes, and

$$\deg[M_{k,i}] = \deg c_i(\mathcal{P}^k),$$

where $\mathcal{P}^k$ denotes the kth osculating bundle of X.

Higher order dual varieties

The *kth dual variety* of X is

$$X^{(k)} := \overline{\{H \in \mathbb{P}(V^\vee) | H \supseteq \mathrm{Osc}_P^k, P \in X_{k-\mathrm{cst}}\}}.$$

Let $\nu^k \colon \widetilde{X}^k \to X$ denote the kth Nash map and $V_{\widetilde{X}^k} \to \mathcal{P}^k$ the corresponding $(m_k + 1)$-quotient. We call $\mathcal{P}^k$ *the kth order osculating bundle.* Then

$$\deg X^{(k)} = \deg[M_{k,m}] = c_m(\mathcal{P}^k)$$

(provided $X^{(k)}$ is of the expected dimension $n + m - m_k - 1$).

k-jet spanned varieties

A variety is *k-jet spanned* if the kth jet map is surjective, i.e., if $\mathcal{P}^k = \mathcal{P}^k_X(1)$.

Example

(Dickenstein–Di Rocco–P.)
Let

$$X = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \to \mathbb{P}^1$$

be a $\mathbb{P}^2$-bundle, with $a, b, c \geq 1$, embedded by $\mathcal{O}_X(2)$. Then X is 2-jet spanned, and

$$\deg X^\vee = 6(2(a+b+c)-1)$$

$$\deg X^{(2)} = 6(8(a+b+c)-7).$$

Polarity with respect to a quadratic form

Let V and V' be a vector spaces of dimensions $n+1$ and n, and $V \to V'$ a surjection.

Let $H_\infty := \mathbb{P}(V') \subset \mathbb{P}(V)$ be the hyperplane at infinity, so $\mathbb{P}(V) \smallsetminus H_\infty \cong V'$ is affine n-space.

A non-degenerate quadratic form on V' gives an isomorphism $V' \cong (V')^\vee$ and a non-singular quadric $Q_\infty \subset H_\infty$.

Let $L' = \mathbb{P}(W) \subset \mathbb{P}(V')$ be a linear space, and set

$$K := \operatorname{Ker}((V')^\vee \cong V' \to W).$$

Then $L'^\perp := \mathbb{P}(K^\vee) \subset \mathbb{P}(V')$ is the polar of L'.

Orthogonality

Given a linear space $L \subset \mathbb{P}(V)$, $L \not\subseteq H_\infty$, and $P \in L$. The *orthogonal space to L at P* is

$$L_P^\perp := \langle P, (L \cap H_\infty)^\perp \rangle.$$

Example

In $\mathbb{P}^2$, take $H_\infty : z = 0$, $Q_\infty : x^2 + y^2 = 0$, $L : x - 2y - 4z = 0$, $P = (2 : -1 : 1)$.

Then $L \cap H_\infty : (2 : 1 : 0)$, $(L \cap H_\infty)^\perp = (1 : -2 : 0)$, and $L_P^\perp : 2x + y - 3z = 0$.

Higher order Euclidean normal spaces

Assume $X \subset \mathbb{P}(V)$ with $X \not\subseteq H_\infty$, and $m_k < n$.

For $P \in X_{k-\mathrm{cst}} \setminus H_\infty$, define the *kth order normal space to X at P*:

$$N_P^k := (\mathrm{Osc}_P^k)_P^\perp.$$

Set $\mathcal{K}^k := \mathrm{Ker}(V_{\widetilde{X}^k} \to \mathcal{P}^k)$ so that

$$0 \to \mathcal{K}^k \to V_{\widetilde{X}^k} \to \mathcal{P}^k \to 0$$

is exact.

Higher order Euclidean normal bundle

Consider $0 \to V'' \to V \to V' \to 0$ ($\dim V'' = 1$).

Assuming $H_\infty = \mathbb{P}(V')$ is general with respect to X, we get

$$0 \to \mathcal{K}^k \to V'_{\widetilde{X}^k} \to \overline{\mathcal{P}}^k \to 0.$$

The polarity in H_∞ w.r.t. Q_∞ gives $V' \cong (V')^\vee$, so we have

$$V'_{\widetilde{X}^k} \cong V'^\vee_{\widetilde{X}^k} \to (\mathcal{K}^k)^\vee$$

whose fibers give the spaces polar to the spaces $\mathrm{Osc}^k_P \cap H_\infty$, and combining $V_{\widetilde{X}^k} \to V'_{\widetilde{X}^k}$ and $V_{\widetilde{X}^k} \to \mathcal{O}_{\widetilde{X}^k}(1)$, we get

$$V_{\widetilde{X}^k} \to \mathcal{E}^k := (\mathcal{K}^k)^\vee \oplus \mathcal{O}_{\widetilde{X}^k}(1)$$

whose fibers correspond to the kth order Euclidean normal spaces N^k_P. We call $\mathcal{E}^k$ the *kth order Euclidean normal bundle.*

Higher order reciprocal polar varieties

Impose conditions on the higher order Euclidean normal spaces instead of on the osculating spaces :

For $i = 0, \dots, m$, let $L_i \subset \mathbb{P}(V)$, $L_i \not\subseteq H_\infty$, have codimension $n - m_k + i$. Define *kth order reciprocal polar varieties*

$$M_{k,i}(L)^\perp := \overline{\{P \in X_{k-\mathrm{cst}} \setminus H_\infty | N_P^k \cap L_i \neq \emptyset\}}.$$

By Porteous' formula, $M_{k,i}(L)^\perp$ have classes

$$[M_{k,i}^\perp] = \nu_*^k(s_i(\mathcal{E}^k) \cap [\widetilde{X}^k]) = \nu_*^k\big([s((\mathcal{K}^k)^\vee)s(\mathcal{O}_{\widetilde{X}^k}(1))]_i \cap [\widetilde{X}^k]\big),$$

hence, since $s((\mathcal{K}^k)^\vee) = c(\mathcal{P}^k)$ and
$s(\mathcal{O}_{\widetilde{X}^k}(1)) = 1 + c_1(\mathcal{O}_{\widetilde{X}^k}(1)) + c_1(\mathcal{O}_{\widetilde{X}^k}(1))^2 + \cdots,$

$$[M_{k,i}^\perp] = \sum_{j=0}^{i} h^{i-j} \cap [M_{k,j}].$$

The Euclidean distance degree

Note that

$$M_{1,i}^{\perp} = M_i^{\perp}$$

and that

$$\deg[M_{1,m}^{\perp}] = \deg[M_m^{\perp}] = \sum_{j=0}^{m} \deg[M_j]$$

is the Euclidean distance degree.

Curves

Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a curve. At a general point $P \in X$ we have a complete flag:

$$\{P\} \subseteq T_P = \mathrm{Osc}_P^1 \subset \mathrm{Osc}_P^2 \subset \mathrm{Osc}_P^3 \subset \cdots \subset \mathrm{Osc}_P^{n-1} \subset \mathbb{P}^n.$$

In this case $\widetilde{X}^k = \widetilde{X}$ is the normalization of X, $m_k = k$, $\dim L_1 = n - k - 1$, and $M_{k,1}$ is the set of points which maps to kth hyperosculating points on the image of X under the linear projection $\mathbb{P}^n \to \mathbb{P}^k$. We get

- $\deg[M_{k,1}] = c_1(\mathcal{P}^k) = r_k$, the *$k$th rank* of X, which is also equal to the degree of the *kth associated curve* of X.
- $\deg[M_{k,1}^{\perp}] = \deg[M_{k,0}] + \deg[M_{k,1}] = r_0 + r_k$, where $r_0 = \deg X$.

Examples

- If $X \subset \mathbb{P}^n$ is a *rational normal curve*,
  $\deg[M_{k,1}] = r_k = (k+1)(n-k)$
  $\deg[M_{k,1}^{\perp}] = r_0 + r_k = n + (k+1)(n-k)$.

  Note that X is $(n-1)$-*self dual*: $X^{(n-1)} \subset (\mathbb{P}^n)^\vee$ is a rational normal curve.

- Dye's special curve: a curve of degree 8 and genus 5.

  Take $X = S_1 \cap S_2 \cap S_3 \subset \mathbb{P}^4$, with S_i Fermat quadrics. Then $r_0 = 8$, $r_1 = 24$, $r_2 = 48$, and $r_3 = 40$. The curve is *canonical* and has 40 Weierstrass points, all of weight 3.

Rational normal scrolls (Dickenstein, P., Sacchiero)

Let $X \subset \mathbb{P}^n$ be a *rational normal scroll* of dimension m and type $(d_1, \ldots, d_m)$, where $n + 1 = \sum_1^m (d_i + 1)$.

If $k \leq \min\{d_1, \ldots, d_m\}$, then $\dim X^{(k)} = n + 1 - km$ and $\deg X^{(k)} = \deg[M_{k,m}] = kd - k(k-1)m$, where $d = \sum_1^m d_i$ is the degree of X.

Example

Take $m = 3$ and $d_1 = d_2 = d_3 = 2$, so that $n = 8$ and $d = 6$. Then for $k = 2$, $\dim X^{(2)} = 3$ and $\deg X^{(2)} = \deg[M_{2,3}] = 6$.

Indeed, X is *2-self dual*: $X^{(2)}$ is a rational normal scroll of the same type as X. (This holds for any *balanced* rational normal scroll.)

Toric varieties (Dickenstein–Di Rocco–P.)

- $m = 2$: Convex smooth lattice polygon $\Pi \subset \mathbb{R}^2$, with (lattice) area a, edge lengths e, and number of vertices v. If all edge lengths are $\geq k$, then the corresponding projective toric variety X has

$$\deg[M_{k,2}] = \binom{k+3}{4}\left(3a - 2ke - \frac{1}{3}(k^2-4)v + 4(k^2-1)\right).$$

- $m = 3$: In a similar situation, we get (setting w to be the lattice volume of the polytope Π),

$$\deg[M_{2,3}] = 62w - 57a + 28e - 8v + 58w_0 + 51a_0 + 20e_0,$$

where w_0, a_0, e_0 are the lattice volumes for the adjoint polytope $\Pi_0 := \mathrm{Conv}(\mathrm{int}(\Pi) \cap \mathbb{Z}^3)$.

Singular toric varieties

If X is a projective toric variety (Matsui–Takeuchi),

$$[M_i] = \sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} h^{i-j} \cap \sum_{\alpha} \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

where the X_α are the orbits of codimension j and $\mathrm{Eu}_X(X_\alpha)$ denotes the value of the local Euler obstruction of X at a point in X_α.

Question: Find similar expressions for the higher order polar classes $[M_{k,i}]$.

Toric linear projections and sections

Let $\mathcal{A} = (a_0, \ldots, a_n) \subset \mathbb{Z}^m$ be a lattice point configuration and let $X_{\mathcal{A}} \subset \mathbb{P}^n$ denote the corresponding toric embedding. Let $\mathcal{A}'$ be a lattice point configuration obtained from $\mathcal{A}$ by removing r points.Then the toric embedding $X_{\mathcal{A}'} \subset \mathbb{P}^{n'}$, where $n' = n - r$, is the *toric linear projection* of $X_{\mathcal{A}}$ with center equal to the linear span of the "removed points".

A *toric hyperplane section* of $X_{\mathcal{A}}$ is obtained by taking a hyperplane in $\mathbb{Z}^m$ and "collapsing" the point configuration $\mathcal{A}$ into this lattice hyperplane in such a way that one point is "lost": two points map to the same point.

The degree 6 Del Pezzo surface

- As a *hyperplane section*:
  Let $\mathcal{A} \subset \mathbb{Z}^3$ be the vertices of the unit cube. Collapse the cube in a plane by identifying the opposite vertices $(1, 1, 1)$ and $(0, 0, 0)$. This gives a hexagon with one interior point. So this hyperplane section of $(\mathbb{P}^1)^3 \subset \mathbb{P}^7$ is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.
- As a *projection*:
  Let $\mathcal{A} \subset \mathbb{Z}^2$ be the lattice points of the square with sides of length 2. Project $X_{\mathcal{A}} \subset \mathbb{P}^8$ from the points corresponding to the vertices $(2, 0)$ and $(0, 2)$. The projected surface is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.

Togliatti's surface

The lattice points defining *Togliatti's surface* $\overline{X} \subset \mathbb{P}^5$ are those of the Del Pezzo hexagon, with the interior point deleted. The 2nd order osculating spaces to X all pass through one point, namely the point corresponding to the interior point of the hexagon. So the (general) 2nd order osculating spaces of $\overline{X}$ have dimension 4, not 5.

The Togliatti surface is 2-self dual, so

$$\deg[M_{2,2}] = \deg \overline{X}^{(2)} = \deg \overline{X} = 6.$$

Thank you for your attention!