Higher order dual and polar varieties

RAGNI PIENE

32º Colóquio Brasileiro de Matemática
IMPA, Rio de Janeiro
July 30, 2019
Let \( X \subset \mathbb{P}(V) \cong \mathbb{P}^n \) be a projective variety of dimension \( m \), and \( P \in X \). There is a sequence of osculating spaces to \( X \) at \( P \):

\[
\{ P \} \subseteq T_P = \text{Osc}_P^1 \subseteq \text{Osc}_P^2 \subseteq \text{Osc}_P^3 \subseteq \cdots \subseteq \mathbb{P}^n,
\]

defined via the sheaves of principal parts: let \( m_k + 1 \) denote the generic rank of the \( k \)-jet map

\[
\dot{j}_k : V_X \to \mathcal{P}_X^k(1).
\]

Then \( m_0 = 0 \) and \( m_1 = m \), and \( \dim \text{Osc}_P^k = \min\{ n, m_k \} \).
Polar varieties

Let $L_i \subset \mathbb{P}^n$ be a linear subspace of codimension $m - i + 2$.

The \textit{i-th polar variety} of of $X$ (with respect to $L_i$) is

$$M_i := \{ P \in X_{\text{sm}} \mid \dim(T_P \cap L_i) \geq i - 1 \}.$$ 

The classes $[M_i]$ are \textit{projective invariants} of $X$: the \textit{i-th class} of a (general) projection of $X$ is the projection of the \textit{i-th class} of $X$, and the \textit{i-th class} of a (general) linear section is the linear section of the \textit{i-th class}.

Note that $\deg[M_m] = \deg X^\vee$, where $X^\vee \subset (\mathbb{P}^n)^\vee$ denotes the dual variety of $X$. 

UiO: University of Oslo
Higher order polar varieties

Assume $m_k < n$. Let $L_{k,i} \subset \mathbb{P}^n$ be a linear subspace of codimension $m_k - i + 2$.

The \textit{ith polar variety of order $k$ of $X$} (with respect to $L_{k,i}$) is

$$M_{k,i} := \{ P \in X_{k-\text{cst}} \mid \dim(\text{Osc}_P^k \cap L_i) \geq i - 1 \},$$

where $X_{k-\text{cst}} \subseteq X$ denotes the open where the rank of $j_k$ is $m_k + 1$.

The classes $[M_{k,i}]$ are projective invariants of $X$, like the usual polar classes, and

$$\deg[M_{k,i}] = \deg c_i(\mathcal{P}^k),$$

where $\mathcal{P}^k$ denotes the $k$th osculating bundle of $X$. 
Higher order dual varieties

The \( k \)th dual variety of \( X \) is

\[
X^{(k)} := \{ H \in \mathbb{P}(V^\vee) | H \supseteq \text{Osc}_P^k, P \in X_{k-\text{cst}} \}.
\]

Let \( \nu^k : \tilde{X}^k \to X \) denote the \( k \)th Nash map and \( V_{\tilde{X}^k} \to \mathcal{P}^k \) the corresponding \( (m_k + 1) \)-quotient. We call \( \mathcal{P}^k \) the \( k \)th order osculating bundle. Then

\[
\deg X^{(k)} = \deg[M_{k,m}] = c_m(\mathcal{P}^k)
\]

(provided \( X^{(k)} \) is of the expected dimension \( n + m - m_k - 1 \)).
$k$-jet spanned varieties

A variety is *$k$-jet spanned* if the $k$th jet map is surjective, i.e., if $\mathcal{P}^k = \mathcal{P}_X^k(1)$.

**Example**

(Dickenstein–Di Rocco–P.)

Let

$$X = \mathbb{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \to \mathbb{P}^1$$

be a $\mathbb{P}^2$-bundle, with $a, b, c \geq 1$, embedded by $\mathcal{O}_X(2)$. Then $X$ is 2-jet spanned, and

$$\deg X^\vee = 6(2(a + b + c) - 1)$$

$$\deg X^{(2)} = 6(8(a + b + c) - 7).$$
Polarity with respect to a quadratic form

Let $V$ and $V'$ be a vector spaces of dimensions $n + 1$ and $n$, and $V \to V'$ a surjection.

Let $H_\infty := \mathbb{P}(V') \subset \mathbb{P}(V)$ be the hyperplane at infinity, so $\mathbb{P}(V) \setminus H_\infty \cong V'$ is affine $n$-space.

A non-degenerate quadratic form on $V'$ gives an isomorphism $V' \cong (V')^\vee$ and a non-singular quadric $Q_\infty \subset H_\infty$.

Let $L' = \mathbb{P}(W) \subset \mathbb{P}(V')$ be a linear space, and set

$$K := \text{Ker}((V')^\vee \cong V' \to W).$$

Then $L'_{\perp} := \mathbb{P}(K^\vee) \subset \mathbb{P}(V')$ is the polar of $L'$. 
Orthogonality

Given a linear space $L \subset \mathbb{P}(V)$, $L \not\subset H_\infty$, and $P \in L$. The orthogonal space to $L$ at $P$ is

$$L_P^\perp := \langle P, (L \cap H_\infty)^\perp \rangle.$$ 

Example

In $\mathbb{P}^2$, take $H_\infty : z = 0$, $Q_\infty : x^2 + y^2 = 0$, $L : x - 2y - 4z = 0$, $P = (2 : -1 : 1)$.

Then $L \cap H_\infty : (2 : 1 : 0)$, $(L \cap H_\infty)^\perp = (1 : -2 : 0)$, and $L_P^\perp : 2x + y - 3z = 0$. 

UiO : University of Oslo
Higher order Euclidean normal spaces

Assume $X \subset \mathbb{P}(V)$ with $X \not\subseteq H_\infty$, and $m_k < n$.

For $P \in X_{k-cst} \setminus H_\infty$, define the $k$th order normal space to $X$ at $P$:

$$N^k_P := (\text{Osc}^k_P)^\perp_P.$$ 

Set $\mathcal{K}^k := \text{Ker}(V_{\tilde{X}^k} \to \mathcal{P}^k)$ so that

$$0 \to \mathcal{K}^k \to V_{\tilde{X}^k} \to \mathcal{P}^k \to 0$$

is exact.
Higher order Euclidean normal bundle

Consider $0 \to V'' \to V \to V' \to 0$ (dim $V'' = 1$).

Assuming $H_\infty = \mathbb{P}(V')$ is general with respect to $X$, we get

$$0 \to \mathcal{K}^k \to V'_{\tilde{X}_k} \to \mathcal{P}^k \to 0.$$  

The polarity in $H_\infty$ w.r.t. $Q_\infty$ gives $V' \cong (V')^\vee$, so we have

$$V'_{\tilde{X}_k} \cong V'_{\tilde{X}_k}^\vee \to (\mathcal{K}^k)^\vee$$

whose fibers give the spaces polar to the spaces $Osc^k_P \cap H_\infty$, and combining $V'_{\tilde{X}_k} \to V'_{\tilde{X}_k}$ and $V_{\tilde{X}_k} \to O_{\tilde{X}_k}(1)$, we get

$$V_{\tilde{X}_k} \to \mathcal{E}^k := (\mathcal{K}^k)^\vee \oplus O_{\tilde{X}_k}(1)$$

whose fibers correspond to the $k$th order Euclidean normal spaces $N^k_P$. We call $\mathcal{E}^k$ the $k$th order Euclidean normal bundle.
Higher order reciprocal polar varieties

Impose conditions on the higher order Euclidean normal spaces instead of on the osculating spaces:

For $i = 0, \ldots, m$, let $L_i \subset \mathbb{P}(V)$, $L_i \not\subset H_\infty$, have codimension $n - m_k + i$. Define $k$th order reciprocal polar varieties

$$M_{k,i}(L) \perp := \{ P \in X_{k-cst} \setminus H_\infty | N_P^k \cap L_i \neq \emptyset \}.$$  

By Porteous’ formula, $M_{k,i}(L) \perp$ have classes

$$[M_{k,i}] = \nu_* (s_i(\mathcal{E}^k) \cap [\tilde{X}^k]) = \nu_* ([s((\mathcal{K}^k)^\vee)s(\mathcal{O}_{\tilde{X}_k}(1))]_i \cap [\tilde{X}^k]),$$

hence, since $s((\mathcal{K}^k)^\vee) = c(\mathcal{P}^k)$ and $s(\mathcal{O}_{\tilde{X}_k}(1)) = 1 + c_1(\mathcal{O}_{\tilde{X}_k}(1)) + c_1(\mathcal{O}_{\tilde{X}_k}(1))^2 + \cdots$,

$$[M_{k,i}] = \sum_{j=0}^{i} h^{i-j} \cap [M_{k,j}].$$
The Euclidean distance degree

Note that

\[ M_{1,i}^\perp = M_i^\perp \]

and that

\[ \deg[M_{1,m}^\perp] = \deg[M_m^\perp] = \sum_{j=0}^{m} \deg[M_j] \]

is the Euclidean distance degree.
Curves

Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a curve. At a general point $P \in X$ we have a complete flag:

$$\{P\} \subseteq T_P = \text{Osc}_P^1 \subset \text{Osc}_P^2 \subset \text{Osc}_P^3 \subset \cdots \subset \text{Osc}_P^{n-1} \subset \mathbb{P}^n.$$

In this case $\tilde{X}^k = \tilde{X}$ is the normalization of $X$, $m_k = k$, $\dim L_1 = n - k - 1$, and $M_{k,1}$ is the set of points which maps to $k$th hyperosculating points on the image of $X$ under the linear projection $\mathbb{P}^n \to \mathbb{P}^k$. We get

- $\deg[M_{k,1}] = c_1(\mathcal{P}^k) = r_k$, the $k$th rank of $X$, which is also equal to the degree of the $k$th associated curve of $X$.

- $\deg[M_{k,1}^\perp] = \deg[M_{k,0}] + \deg[M_{k,1}] = r_0 + r_k$, where $r_0 = \deg X$. 

UiO: University of Oslo
Examples

- If $X \subset \mathbb{P}^n$ is a *rational normal curve*,
  \[
  \text{deg}[M_{k,1}] = r_k = (k + 1)(n - k)
  \]
  \[
  \text{deg}[M_{k,1}^\perp] = r_0 + r_k = n + (k + 1)(n - k).
  \]

  Note that $X$ is $(n - 1)$-*self dual*: $X^{(n-1)} \subset (\mathbb{P}^n)^\vee$ is a rational normal curve.

- Dye’s special curve: a curve of degree 8 and genus 5.

  Take $X = S_1 \cap S_2 \cap S_3 \subset \mathbb{P}^4$, with $S_i$ Fermat quadrics.
  Then $r_0 = 8$, $r_1 = 24$, $r_2 = 48$, and $r_3 = 40$. The curve is *canonical* and has 40 Weierstrass points, all of weight 3.
Rational normal scrolls (Dickenstein, P., Sacchiero)

Let $X \subset \mathbb{P}^n$ be a rational normal scroll of dimension $m$ and type $(d_1, \ldots, d_m)$, where $n + 1 = \sum_1^m (d_i + 1)$.

If $k \leq \min\{d_1, \ldots, d_m\}$, then $\dim X^{(k)} = n + 1 - km$ and $\deg X^{(k)} = \deg[M_{k,m}] = kd - k(k - 1)m$, where $d = \sum_1^m d_i$ is the degree of $X$.

Example

Take $m = 3$ and $d_1 = d_2 = d_3 = 2$, so that $n = 8$ and $d = 6$. Then for $k = 2$, $\dim X^{(2)} = 3$ and $\deg X^{(2)} = \deg[M_{2,3}] = 6$.

Indeed, $X$ is 2-self dual: $X^{(2)}$ is a rational normal scroll of the same type as $X$. (This holds for any balanced rational normal scroll.)
Toric varieties (Dickenstein–Di Rocco–P.)

- \( m = 2 \): Convex smooth lattice polygon \( \Pi \subset \mathbb{R}^2 \), with (lattice) area \( a \), edge lengths \( e \), and number of vertices \( v \). If all edge lengths are \( \geq k \), then the corresponding projective toric variety \( X \) has

\[
\deg[M_{k,2}] = \binom{k + 3}{4}(3a - 2ke - \frac{1}{3}(k^2 - 4)v + 4(k^2 - 1)).
\]

- \( m = 3 \): In a similar situation, we get (setting \( w \) to be the lattice volume of the polytope \( \Pi \)),

\[
\deg[M_{2,3}] = 62w - 57a + 28e - 8v + 58w_0 + 51a_0 + 20e_0,
\]

where \( w_0, a_0, e_0 \) are the lattice volumes for the adjoint polytope \( \Pi_0 := \text{Conv}(\text{int}(\Pi) \cap \mathbb{Z}^3) \).
Singular toric varieties

If $X$ is a projective toric variety (Matsui–Takeuchi),

$$[M_i] = \sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} h^{i-j} \cap \sum_{\alpha} \text{Eu}_X(X_{\alpha})[X_{\alpha}],$$

where the $X_{\alpha}$ are the orbits of codimension $j$ and $\text{Eu}_X(X_{\alpha})$ denotes the value of the local Euler obstruction of $X$ at a point in $X_{\alpha}$.

**Question:** Find similar expressions for the higher order polar classes $[M_{k,i}]$. 

---

University of Oslo
Toric linear projections and sections

Let $\mathcal{A} = (a_0, \ldots, a_n) \subset \mathbb{Z}^m$ be a lattice point configuration and let $X_{\mathcal{A}} \subset \mathbb{P}^n$ denote the corresponding toric embedding. Let $\mathcal{A}'$ be a lattice point configuration obtained from $\mathcal{A}$ by removing $r$ points. Then the toric embedding $X_{\mathcal{A}'} \subset \mathbb{P}^{n'}$, where $n' = n - r$, is the toric linear projection of $X_{\mathcal{A}}$ with center equal to the linear span of the “removed points”.

A toric hyperplane section of $X_{\mathcal{A}}$ is obtained by taking a hyperplane in $\mathbb{Z}^m$ and “collapsing” the point configuration $\mathcal{A}$ into this lattice hyperplane in such a way that one point is “lost”: two points map to the same point.
The degree 6 Del Pezzo surface

- As a hyperplane section:
  Let $A \subset \mathbb{Z}^3$ be the vertices of the unit cube. Collapse the cube in a plane by identifying the opposite vertices $(1,1,1)$ and $(0,0,0)$. This gives a hexagon with one interior point. So this hyperplane section of $(\mathbb{P}^1)^3 \subset \mathbb{P}^7$ is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.

- As a projection:
  Let $A \subset \mathbb{Z}^2$ be the lattice points of the square with sides of length 2. Project $X_A \subset \mathbb{P}^8$ from the points corresponding to the vertices $(2,0)$ and $(0,2)$. The projected surface is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.
Togliatti’s surface

The lattice points defining Togliatti’s surface $\overline{X} \subset \mathbb{P}^5$ are those of the Del Pezzo hexagon, with the interior point deleted. The 2nd order osculating spaces to $X$ all pass through one point, namely the point corresponding to the interior point of the hexagon. So the (general) 2nd order osculating spaces of $\overline{X}$ have dimension 4, not 5.

The Togliatti surface is 2-self dual, so

$$\deg[M_{2,2}] = \deg \overline{X}^{(2)} = \deg \overline{X} = 6.$$
Thank you for your attention!