Singular curves on a moving surface

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Plane curves

The space of plane curves of degree d is $\mathbb{P}^N$, $N = \binom{d+2}{2} - 1$. Let $n_r(d)$ denote the number of plane curves of degree d with r nodes that pass through $N - r$ points. Equivalently, $n_r(d)$ is the number of such curves contained in a subspace $\mathbb{P}^r \subseteq \mathbb{P}^N$.

Classical results:

Steiner (1848): $n_1(d) = 3d^2 - 6d + 3 = 3(d-1)^2$

Cayley (1863): $n_2(d) = \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11)$

Roberts (1875): $n_3(d) = \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525$

Itzykson (1994): $n_4(d)$

Vainsencher (1995): $n_r(d), r \leq 6$

Curves on a surface

Let S be a smooth, projective surface, $\mathcal{L}$ a line bundle,

$$m := c_1(\mathcal{L})^2, k := c_1(K_S) \cdot c_1(\mathcal{L}), s := c_1(K_S)^2, x := c_2(S)$$

the four Chern numbers.

An *r-nodal curve* is a curve with precisely r nodes.

Let N_r denote the number of r-nodal curves in $|\mathcal{L}|$ passing through $\dim |\mathcal{L}| - r$ points on S.

Zeuthen–Segre–Hirzebruch: $N_1 = 3m + 2k + x$

Vainsencher ($r \leq 6$) and Kleiman–P. ($r \leq 8$) expressed N_r as polynomials in m, k, s, x and conjectured that this could be done for all r, and similarly for other singularities than nodes.

Göttsche's conjecture

Göttsche (1998) gave a more precise conjecture, for the generating function:

$$\sum_{r\geq 0} N_r q^r = A_1(q)^m A_2(q)^k A_3(q)^s A_4(q)^x = \exp\left(\sum_{i\geq 1} a_i q^i/i!\right),$$

where the $A_i(q) \in \mathbb{Q}[[q]]$ are universal power series and the a_i are linear polynomials in m, k, s, x.

Proved in 2010 by Tzeng and by Kool–Shende–Thomas. For other singularities, existence of universal polynomials by Kazarian, Li–Tzeng, and Rennemo.

Remark

In addition to the existence and shape of the formulas for N_r, there is the question under which hypotheses the formulas are *valid*. I will not discuss this question today.

Curves on a *family* of surfaces – why?

- (Kleiman–P.) Let S be a surface, Y a linear system on S, $D \subset S \times Y \to Y$ the universal curve. By blowing up a (moving) point in S one gets a family over $S \times Y$, which is not of the form a fixed surface times the base $S \times Y$.
- (Vainsencher, Kleiman–P.) Fix a threefold in $\mathbb{P}^4$. Consider the family of planes in $\mathbb{P}^4$ and the family of curves obtained by intersecting each plane with the threefold.
- (Mukherjee et al., Laarakker) Consider the family of planes in $\mathbb{P}^3$ and the family of curves of fixed degree in each plane.

Partitions of a finite set

Let n be a positive integer.

A partition of $\{1, 2, \ldots, n\}$ is a way of writing it as a union of subsets.

$\{1\} = \{1\}$
$\{1, 2\} = \{1, 2\} = \{1\} \cup \{2\}$
$\{1, 2, 3\} = \{1, 2, 3\} = \{1, 2\} \cup \{3\} = \{1, 3\} \cup \{2\} = \{2, 3\} \cup \{1\} = \{1\} \cup \{2\} \cup \{3\}$

The number of *blocks* in the partition is the number of subsets.

Stirling and Bell numbers

The *Stirling* numbers $S_{n,k}$ count the number of partitions of a *set* with n elements into k blocks.

Example: $S_{4,2} = 7$

$\{1,2,3,4\} = \{1\} \cup \{2,3,4\} = \{2\} \cup \{1,3,4\} = \{3\} \cup \{1,2,4\} = \{4\} \cup \{1,2,3\} = \{1,2\} \cup \{3,4\} = \{1,3\} \cup \{2,4\} = \{1,4\} \cup \{2,3\}$

The *Bell* numbers count *all* partitions:

$$B_n := \sum_{k=1}^{n} S_{n,k}.$$

They satisfy a recursive relation: set $B_0 := 1$, then

$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i.$$

We get: $B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, \ldots$

Block partitions

Set $\Pi_n := \{\text{partitions of } \{1, \ldots, n\}\}$.

Given $\mathbf{k} = (k_1, \ldots, k_n)$, $k_i \geq 0$, $\sum_{i=1}^{n} i k_i = n$, we say $\pi \in \Pi_n$ has *type* $\mathbf{k}$ if π has k_i blocks of size i.

A partition of type $\mathbf{k}$ has $k := \sum_{i=1}^{n} k_i$ blocks.

Let $\beta_{\mathbf{k}}$ denote the number of partitions of type $\mathbf{k}$. Then

$$\beta_{\mathbf{k}} := \frac{n!}{k_1! \cdots k_n!} \Big(\frac{1}{1!}\Big)^{k_1} \cdots \Big(\frac{1}{n!}\Big)^{k_n}.$$

We have

$$S_{n,k} = \sum_{\mathbf{k},k} \beta_{\mathbf{k}} \text{ and } B_n = \sum_{\mathbf{k}} \beta_{\mathbf{k}}.$$

Example

$n = 4,\ k = 2$

$\mathbf{k} = (1, 0, 1, 0)$:

$\{1\} \cup \{2, 3, 4\}; \{2\} \cup \{1, 3, 4\}; \{3\} \cup \{1, 2, 4\}; \{4\} \cup \{1, 2, 3\}$

$\mathbf{k} = (0, 2, 0, 0)$:

$\{1, 2\} \cup \{3, 4\}; \{1, 3\} \cup \{2, 4\}; \{1, 4\} \cup \{2, 3\}$.

There are $\beta_{(1,0,1,0)} = \frac{4!}{1!1!}(\frac{1}{1!})^1(\frac{1}{3!})^1 = 4$ of the first type,

and $\beta_{(0,2,0,0)} = \frac{4!}{2!}(\frac{1}{2!})^2 = 3$ of the second type.

$S_{4,2} = 4 + 3 = 7$

$\sum_{k=1}^{4} S_{4,k} = 1 + 7 + 6 + 1 = 15 = B_4$

Polydiagonals

Let X be a space, and consider

$$X^n := X \times \cdots \times X = \{(x_1, \ldots, x_n) \,|\, x_i \in X\}.$$

For $\pi \in \Pi_n$, set

$$\Delta_\pi^{(n)} := \{(x_1, \ldots, x_n) \in X^n |\, x_i = x_j \text{ if } i, j \text{ in same block of } \pi\}.$$

If π has type $\mathbf{k}$, we say that $\Delta_\pi^{(n)}$ is a *polydiagonal* of type $\mathbf{k}$.

There are $\beta_{\mathbf{k}}$ polydiagonals of type $\mathbf{k}$, and $\sum_{\mathbf{k}} \beta_{\mathbf{k}} = B_n$ polydiagonals.

Example

The *small* diagonal: $\Delta_{\{1,\ldots,n\}}^{(n)} = \{(x, \ldots, x) \in X^n |\, x \in X\}$.

Bell polynomials

The *Bell polynomials* are

$$B_n(z_1, \ldots, z_n) := \sum_{\mathbf{k}} \beta_{\mathbf{k}} z_1^{k_1} \cdots z_n^{k_n}.$$

Note that $B_n(1, \ldots, 1) = B_n$.

$$\begin{aligned} B_1(z_1) &= z_1, \\ B_2(z_1, z_2) &= z_1^2 + z_2, \\ B_3(z_1, z_2, z_3) &= z_1^3 + 3z_1 z_2 + z_3 \\ B_4(z_1, z_2, z_3, z_4) &= z_1^4 + 6z_1^2 z_2 + 4z_1 z_3 + 3z_2^2 + z_4 \end{aligned}$$

Bell polynomials – other definitions

Recursively defined by $B_0 = 1$ and

$$B_{n+1}(z_1, \ldots, z_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(z_1, \ldots, z_{n-i}) z_{i+1},$$

or by the formal identity for the (exponential) *generating function*

$$\sum_{n\geq 0} \frac{1}{n!} B_n(z_1, \ldots, z_n) q^n = \exp\left(\sum_{j\geq 1} \frac{1}{j!} z_j q^j\right),$$

Note binomiality:

$$B_n(z_1+z_1', \ldots, z_n+z_n') = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(z_1, \ldots, z_{n-i}) B_i(z_1', \ldots, z_i').$$

Nodal curves on families of surfaces

Given a family of curves on a family of surfaces $D \subset F \xrightarrow{f} Y$, find an expression N_r for the class of curves that have r nodes.

Conjecture (Kleiman–P.): There exist universal polynomials b_i of weighted degree $i+2$ in the Chern classes $c_1(\mathcal{O}_F(D))$, $c_1(\Omega^1_f)$, and $c_2(\Omega^1_f)$ such that, setting $a_i := f_* b_i$,

$$N_r = \tfrac{1}{r!} B_r(a_1, \ldots, a_r) \cap [Y],$$

where B_r is the rth Bell polynomial.

Proved for $r \leq 8$, and gave an explicit algorithm for the computations, using the recursive property of the Bell polynomials.

Existence and shape of the polynomials

T. Laarakker (2018) proved part of our conjecture: there exist universal polynomials U_r such that N_r is equal to U_r evaluated on classes $f_* c_1(\mathcal{O}_F(D))^a c_1(\Omega^1_f)^b c_2(\Omega^1_f)^c$, with $a+b+2c \leq r+2$.

He also showed that the polynomials are multiplicative:

$$U_r(F \sqcup F') = \sum_i U_i(F) U_{r-i}(F').$$

Given the binomiality of the Bell polynomials:

$$\tfrac{1}{r!} B_r(a_1 + a_1', \dots) = \sum_i \tfrac{1}{i!} B_i(a_1, \dots, a_i) \tfrac{1}{(r-i)!} B_{r-i}(a_1', \dots, a_{r-i}'),$$

this gives evidence for our conjecture that $U_r = \frac{1}{r!} B_r$, but does not prove it. (However, it does so when $F = S \times Y$ is a trivial family.)

Why Bell polynomials? The recursion

Blow up the surfaces to get rid of one node in each curve, then use the formula for $(r-1)$-nodal curves on the blown up surfaces and push it down.This creates a "derivation formula" of the form

$$ru_r = u_{r-1}u_1 + \partial(u_{r-1})$$

$$r!u_r = (r-1)!u_{r-1}u_1 + \partial((r-1)!u_{r-1})$$

Set $a_1 := u_1$ and $a_i := \partial(a_{i-1})$. Then $u_1 = B_1(a_1)$ and

$$2!u_2 = a_1^2 + a_2 = B_2(a_1, a_2),$$

and, pretendig ∂ is a derivation: $\partial(2!u_2) = 2a_1a_2 + a_3$,

$$3!u_3 = (a_1^2 + a_2)a_1 + \partial(a_1^2 + a_2) = a_1^3 + 3a_1a_2 + a_3 = B_3(a_1, a_2, a_3).$$

Intersection theory (Fulton)

Recall the definition of intersection product:

Let $U \subset W$ be regularly embedded, of codimension c and normal bundle $\mathcal{N}$. If $V \subset W$ is of pure dimension k, then

$$U \cdot V := \{c(\mathcal{N}|_{U \cap V}) \cap s(U \cap V, V)\}_{k-c} \in A_{k-c}(U \cap V).$$

We can write

$$U \cdot V = \sum_{i=1}^{s} m_i \alpha_i,$$

where α_i is supported on the ith *distinguished variety* Z_i of the intersection product. If Z is a distinguished variety, then the sum of the $m_i \alpha_i$ such that $Z_i = Z$ is called the *equivalence* of Z for the intersection product.

The configuration space of singular points

Let $D \subset F \xrightarrow{f} Y$ be a family of curves on surfaces, and set

$$X := \{x \in D |\, x \in D_{f(x)} \text{ is singular}\}.$$

Let $\Delta \subset X^r = X \times_Y \cdots \times_Y X$ be the union of all diagonals: $X^r \setminus \Delta$ is the rth *configuration space* of X. Set $f^r : F^r \to Y$. Then

$$f^r_*[\overline{X^r \setminus \Delta}] = r! N_r$$

Let $p_j : F^r \to F$ be the projection maps. Then

$$N_r = \tfrac{1}{r!} f^r_*[\overline{X^r \setminus \Delta}] = \tfrac{1}{r!} f^r_* \Big(\prod_{j=1}^r p_j^*[X] - (p_1^* X \cdots p_r^* X)^\Delta \Big),$$

where the last term is the sum of the *equivalences of all distinguished irreducible varieties* in Δ.

Why Bell polynomials? Polydiagonals

Following N. Qviller:

$$\prod_{j=1}^r p_j^*[X] - (p_1^*X \cdots p_r^*X)^\Delta = \sum_{\pi \in \Pi_r} n_\pi^{(r)} (p_1^*X \cdots p_r^*X)^{\Delta_\pi^{(r)}},$$

where

$$n_\pi^{(r)} := \prod_{i=1}^r ((-1)^{i-1}(i-1)!)^{k_i}$$

and $\mathbf{k} = (k_1, \ldots, k_r)$ is the type of π.

Set $b_i := (-1)^{i-1}(i-1)!(p_1^*X \cdots p_i^*X)^{\Delta_{\{1,\ldots,i\}}^{(i)}}$ and $a_i := f_*^i b_i$.

Then

$$n_\pi^{(r)} f_*^r (p_1^*X \cdots p_r^*X)^{\Delta_\pi^{(r)}} = a_1^{k_1} \cdots a_r^{k_r}$$

and

$$N_r = \frac{1}{r!} \sum_\pi a_1^{k_1} \cdots a_r^{k_r} = \frac{1}{r!} \sum_{\mathbf{k}} \beta_{\mathbf{k}} a_1^{k_1} \cdots a_r^{k_r} = \frac{1}{r!} B_r(a_1, \ldots, a_r).$$

What is proved and what remains

Define

$$y(a,b,c) := f_* c_1(\mathcal{O}_F(D))^a c_1(\Omega_f^1)^b c_2(\Omega_f^1)^c.$$

By Laarakker, the N_r are universal polynomials in the classes $y(a,b,c)$, with $a+b+2c \le r+2$. By the above argument, the N_r are Bell polynomials in the classes a_i. By Kleiman–P., the a_i are linear polynomials in the $y(a,b,c)$, with $a+b+2c = i+2$ for $i \le 8$.

Conjecture

For all i, the classes a_i are linear polynomials in the classes $y(a,b,c)$, with $a+b+2c = i+2$.

The codimension of a singularity

The *codimension* of a planar curve singularity is the codimension of the equisinguar locus in the miniversal deformation space of the singularity.

It can also be defined using the Enriques diagram of the singularity.

Example

- A node has codimension 1, an ordinary cusp has codimension 2: an A_k-singularity has codimension k.
- An ordinary triple point has codimension 4: an ordinary m-uple point has codimension $\binom{m+1}{2} - 2$.

The contributions from the distinguished varieties

From the intersection product $p_1^*[X] \cdot \dots \cdot p_r^*[X]$ we subtracted the equivalences of the distinguished varieties. Some of these varieties are the polydiagonals, which have "excess" dimension, whereas others are subvarieties of the polydiagonals, representing embedded components of the intersection $p_1^* X \cap \dots \cap p_r^* X$. The latter correspond to singularities other than r nodes, but with the same codimension r.

Example

For $r = 2$, in $p_1^*[X] p_2^*[X] = X \times_Y X$ we have:

- pairs of distinct points (= two nodes on fibers of D)
- the diagonal (= one node on fibers of D)
- points in the diagonal that are cusps on the fibers of D.

The equivalence of the small diagonal

Let $e_i := (p_1^* X \cdots p_i^* X)_0^\Delta$ denote the equivalence of $\Delta := \Delta^{(i)}_{\{1,\ldots,i\}}$ *without* including the other distinguished varieties $Z \subsetneq \Delta$.

We have

$$e_i = \left(\prod_{j=1}^i c(p_j^* \mathcal{P}_f^1(D)|_\Delta) \cap s(\Delta, F^i)\right)_{\dim Y - i},$$

and, since $X \cong \Delta$, $c(p_j^* \mathcal{P}_f^1(D)|_\Delta) = c(\mathcal{P}_f^1(D)|_X)$ and

$$s(\Delta, F^i) = c(\mathcal{P}_f^1(D)|_X)^{-1} (c(T_f|_X)^{\oplus i-1})^{-1} \cap [X].$$

Note that $e_i \in A_{\dim Y - i}(F)$ and is a (computable!) polynomial in $c_1(\mathcal{O}_F(D))$, $c_1(\Omega_f^1)$, $c_2(\Omega_f^1)$ capped with the class $[X]$, which is also a polynomial in these Chern classes.

The linearity of the a_i

It follows from Laarakker's result that each a_i is a universal polynomial in the classes $y(a, b, c)$ with $a + b + 2c \leq i + 2$.

Set $\widetilde{b}_i := (-1)^{i-1}(i-1)!e_i$ and $\widetilde{a}_i := f_*^i \widetilde{b}_i$. From what we have seen, $\widetilde{a}_i$ is a universal, linear polynomial in the classes $y(a, b, c)$ with $a + b + 2c = i + 2$.

Remains to show:

$$a_i - \widetilde{a}_i = (-1)^{i-1}(i-1)! \sum\nolimits_{Z \subsetneq \Delta} (p_1^* X \cdots p_i^* X)^Z$$

is linear in the $y(a, b, c)$ with $a + b + 2c = i + 2$.

Codimension r singularities

Do the polynomials

$$\widetilde{N}_r := \tfrac{1}{r!} B_r(\widetilde{a}_1, \dots, \widetilde{a}_r)$$

give the codimension r multisingularities of the fibers of D?

Example

$\widetilde{N}_1 = \widetilde{a}_1 = a_1 = N_1$

$\widetilde{N}_2 = \frac{1}{2}(\widetilde{a}_1^2 + \widetilde{a}_2) = \frac{1}{2}(a_1^2 + a_2 + (\widetilde{a}_2 - a_2)) = N_2 + N_{A_2}$

$\widetilde{N}_3 = N_3 + N_{A_1+A_2} + N_{A_3}$???

Speculation – based on Kazarian and Qviller

We have seen:

$$\widetilde{N}_3 = \tfrac{1}{3!}(\widetilde{a}_1^3 + 3\widetilde{a}_1\widetilde{a}_2 + \widetilde{a}_3) = N_3 + N_1 N_{A_2} + \tfrac{1}{3!}(\widetilde{a}_3 - a_3)$$

According to Kazarian and Qviller,

$$\tfrac{1}{3!}(\widetilde{a}_3 - a_3) = S_{A_1A_2} + N_{A_3}$$

where $S_{A_1A_2}$ is supported on the small diagonal, as is also N_{A_3}, and

$$N_{A_1+A_2} = N_1 N_{A_2} + S_{A_1A_2},$$

Hence

$$\widetilde{N}_3 = N_3 + N_{A_1+A_2} + N_{A_3}.$$

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Thank you for your attention!