Discriminants and polytopes in toric geometry

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Resultants and discriminants

Il faut éliminer la théorie de l'élimination.

J. Dieudonné (1969)

Eliminate, eliminate, eliminate
Eliminate the eliminators of elimination theory.

S. S. Abhyankar (1970)

Résultant, discriminant

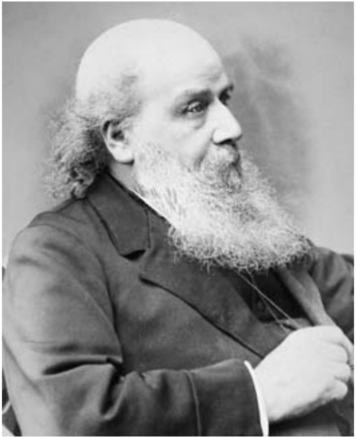
M. Demazure (2011) – à J.-P. Serre pour son 85-ième anniversaire

Question: For which $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ do

$$f(x) = a_m x^m + \cdots + a_0 \text{ and } g(x) = b_n x^n + \cdots + b_0$$

have a common root?

James Joseph Sylvester (1814–1897)



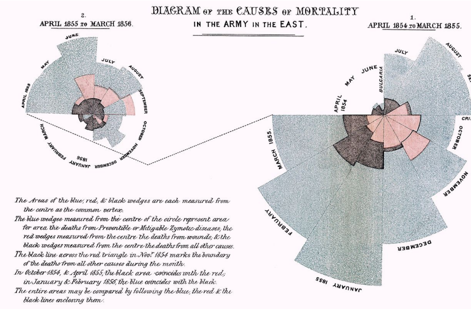
The *Sylvester matrix* is the $(m+n) \times (m+n)$-matrix

$$\begin{pmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & \cdots \\ 0 & a_m & a_{m-1} & a_{m-2} & \cdots \\ \vdots & & & \vdots & \\ b_n & b_{n-1} & b_{n-2} & \cdots & \cdots \\ 0 & b_n & b_{n-1} & b_{n-2} & \cdots \\ \vdots & & & \vdots & \end{pmatrix}$$

The *resultant* $\mathrm{Res}(f, g)$ is the determinant of this matrix.

A student of Sylvester?

Florence Nightingale (1820-1910)



"Diagram of the Causes of Mortality in the Army in the East"

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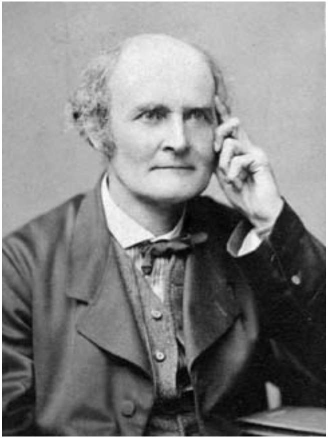
Arthur Cayley (1821–1895)



Set

$$h(x, y) := f(x) + yg(x).$$

If α is a common root of f and g, then

$$(\alpha, -\frac{f_x(\alpha)}{g_x(\alpha)})$$

is a common zero of h, h_x, h_y.

The Cayley trick

Consider

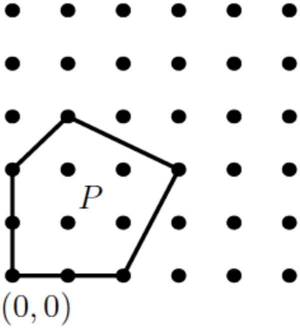
$$h(x_1, \ldots, x_{n-k}, y_1, \ldots, y_k) :=$$
$$f_0(x_1, \ldots, x_{n-k}) + y_1 f_1(x_1, \ldots, x_{n-k}) + y_k f_k(x_1, \ldots, x_{n-k}).$$

The *discriminant* $\Delta(h)$ of h is obtained by eliminating the x_i's and y_j's from the $n+1$ equations

$$h = 0, \partial h/\partial x_i = 0, \partial h/\partial y_j = f_j = 0.$$

Hence $\Delta(h) \sim \mathrm{Res}(f_0, \ldots, f_k)$.

Convex lattice polytopes



Cayley polytopes

Let $P_0, \dots, P_k \subset \mathbb{R}^{n-k}$ be convex lattice polytopes, and $e_0, \dots, e_k$ the vertices of $\Delta_k \subset \mathbb{R}^k$.

The polytope

$$P = \mathrm{Conv}\{e_0 \times P_0, \dots, e_k \times P_k\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n,$$

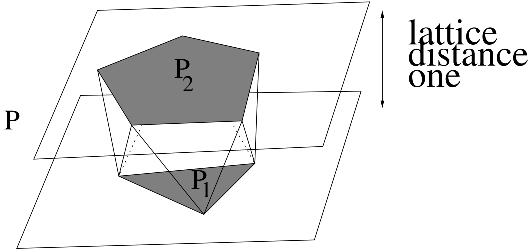
is called a *Cayley polytope*.

We write

$$P = P_0 \star \cdots \star P_k$$

A Cayley polytope is "hollow": it has no interior lattice points.

An example



The codegree and degree of a polytope

$$\operatorname{codeg}(P) = \min\{m \mid mP \text{ has interior lattice points}\}.$$

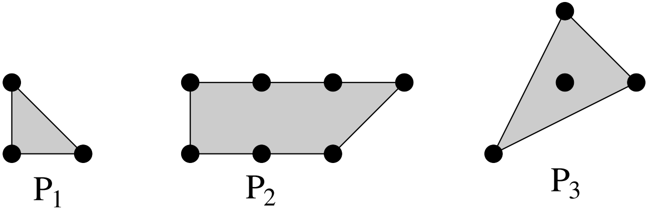
$$\deg(P) = n + 1 - \operatorname{codeg}(P)$$

Example (1)

$$\operatorname{codeg}(\Delta_n) = n + 1 \text{ and } \operatorname{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil.$$

Example (2)

$$P = P_0 \star \cdots \star P_k \text{ implies } \operatorname{codeg}(P) \geq k + 1.$$



$\mathrm{codeg}(P_1) = 3$ $\mathrm{codeg}(P_2) = 2$ $\mathrm{codeg}(P_3) = 1$

The Cayley polytope conjecture

Question (Batyrev–Nill): Is there an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Answer (Haase–Nill–Payne): Yes, and $N(d) \leq (d^2 + 19d - 4)/2$

Question: Is $N(d)$ linear in d?

Answer (Dickenstein–Di Rocco–P.): Yes, $N(d) = 2d + 1$ (if P is smooth and $\mathbb{Q}$-normal).

Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.

Lattice polytopes and toric embeddings

A convex lattice polytope $P \subset \mathbb{R}^m$ gives a *toric embedding*

$$\varphi\colon (\mathbb{C}^*)^m \to \mathbb{P}^N,$$

where $N+1$ is the number of lattice points in P, as follows:

Let $a_0, \dots, a_N \in \mathbb{Z}^m$ denote the latttice points in P. Then send $\mathbf{t} = (t_1, \dots, t_m)$ to $(\mathbf{t}^{a_0} : \dots : \mathbf{t}^{a_N})$.

The closure of the image is a *toric variety* $X_P \subset \mathbb{P}^N$.

Example

The polytope $P \subset \mathbb{R}$: • —— • —— •
corresponds to the toric embedding $\mathbb{C}^* \to \mathbb{P}^2$ given by $t \mapsto (1 : t : t^2)$; its closure X_P is a plane conic curve.

Defective polytopes

The *dual* projective space $(\mathbb{P}^N)^\vee$ is the space consisting of all hyperplanes in $\mathbb{P}^N$. A point $(\xi_0 : \cdots : \xi_N) \in (\mathbb{P}^N)^\vee$ corresponds to the hyperplane $H : \xi_0 x_0 + \cdots + \xi_N x_N = 0$.

The intersection of X_P with the hyperplane is given by $f(\mathbf{t}) := \xi_0 \mathbf{t}^{a_0} + \cdots \xi_N \mathbf{t}^{a_N} = 0$.

The intersection is *singular* precisely when H is tangent to X_P. The *dual* variety $X_P^\vee$ is the locus of these hyperplanes. The equation(s) of $X_P^\vee$ is the discriminant $\Delta(f)$.

We say P is *defective* if $X_P^\vee$ is not a hypersurface.

Example

$\Delta(\xi_0 + \xi_1 t + \xi_2 t^2) = \xi_1^2 - 4\xi_0\xi_2$.

Characterizing Cayley polytopes

Theorem (Dickenstein, Di Rocco, P., Nill)

Let P be a smooth lattice polytope of dimension n. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = P_0 \star \cdots \star P_k$ *is a smooth Cayley polytope with* $k+1 = \operatorname{codeg}(P)$ *and* $k > \frac{n}{2}$.

(3) P *is defective, with defect* $2k - n > 0$.

The proof is algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

An example with $n = 3$, $k = 2$

Let $P_0 = P_1 = P_2$: $\bullet$ —— $\bullet$ —— $\bullet$

and set $P := P_0 \star P_1 \star P_2$. Then P corresponds to the embedding

$$(\mathbb{C}^*)^3 \to \mathbb{P}^8$$

given by

$$(x, y) \mapsto (1 : x : y : z : xy : y^2 : yz : xy^2 : y^2z);$$

its closure X_P is a rational normal scroll of type $(2, 2, 2)$.

We have $\operatorname{codeg} P = 3 = \frac{3+3}{2} = 2 + 1$ and $2 > \frac{3}{2}$.

The polytope P is defective, with defect $2 \cdot 2 - 3 = 1$.

A geometric interpretation

An ordering of the lattice points in $P = P_0 \star \cdots \star P_k$ gives toric embeddings

$$\varphi_i \colon (\mathbb{C}^*)^{n-k} \to \mathbb{P}^{N_i} \subset \mathbb{P}^N.$$

The Cayley trick says that the hyperplanes $H \subset \mathbb{P}^N$ that are tangent to X_P are those that contain the points $\varphi_0(\mathbf{t}), \ldots, \varphi_k(\mathbf{t})$, for some $\mathbf{t} \in (\mathbb{C}^*)^{n-k}$.

Therefore, the dimension of the dual variety $X_P^\vee$ is equal to $n - k + N - 1 - k = N - 1 - (2k - n)$, so P is defective with defect $2k - n$ iff $2k - n > 0$.

Chern classes of toric varieties

Let $\{X_\alpha\}_\alpha$ denote the torus invariants orbits of X_P. Each orbit X_α corresponds to a face F_α of P of the same dimension.

By Ehler's formula, the *Schwartz–MacPherson* Chern classes are

$$c^{\mathrm{SM}}(X_P) = \sum_\alpha [\overline{X}_\alpha].$$

It follows that the *Mather* Chern classes are

$$c^{\mathrm{M}}(X_P) = \sum_\alpha \mathrm{Eu}_{X_P}(X_\alpha)[\overline{X}_\alpha],$$

where $\mathrm{Eu}_{X_P}(X_\alpha)$ denotes the value of the local Euler obstruction of X_P at a point of X_α.

The degree of the dual variety

Theorem (Gel'fand–Kapranov–Zelevinski)

If X_P is smooth and non defective,

$$\deg X_P^\vee = \sum_\alpha (-1)^{\operatorname{codim} F_\alpha} (\dim F_\alpha + 1) \mathrm{Vol}_{\mathbb{Z}}(F_\alpha).$$

Proof. $\deg X_P^\vee = c_n(\mathcal{P}^1(L_P))$ is a polynomial in the Chern classes of X_P and the hyperplane bundle L_P.

$c_1(L_P)^n = \mathrm{Vol}_{\mathbb{Z}}(P) = \deg X_P$

$c_i(T_{X_P}) c_1(L_P)^{n-i} = \sum_{\operatorname{codim} F_\alpha = i} \mathrm{Vol}_{\mathbb{Z}}(F_\alpha)$.

$c_n(T_{X_P}) = \#$ vertices of P

Theorem (Matsui–Takeuchi)

If X_P is non defective,

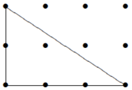
$$\deg X_P^\vee = \sum_\alpha (-1)^{\operatorname{codim} F_\alpha} (\dim F_\alpha + 1)\, \mathrm{Eu}_{X_P}(X_\alpha) \mathrm{Vol}_{\mathbb{Z}}(F_\alpha).$$

Example

$P = \mathrm{Conv}\{(0,0),(3,0),(0,2)\}$.

The weighted projective plane $X_P = \mathbb{P}(1,2,3)$ has

$$\deg X_P^\vee = 3 \cdot 6 - 2(2+3+1) - (1+0+0) = 7.$$



Higher order dual varieties

Let $X \subset \mathbb{P}^N$ be a projective variety. The kth order dual variety is

$$X^{(k)} := \overline{\{H \in (\mathbb{P}^N)^\vee \mid H \text{ is tangent to } X \text{ to the order } k\}}$$
$$= \overline{\{H \in \mathbb{P}^{N\vee} \mid H \supseteq \mathbb{T}^k_{X,x} \text{ for some } x \in X_{\text{smooth}}\}},$$

where $\mathbb{T}^k_{X,x} =$ is the kth osculating space to X at x.

Note that $\dim \mathbb{T}^k_{X,x} \leq \binom{n+k}{k} - 1$, $n = \dim X$, and

$X^{(1)} = X^\vee$ and $X^{(k-1)} \supseteq X^{(k)}$

The *expected dimension* of $X^{(k)} = n + N - \binom{n+k}{k}$.

X is *k-defective* if $\dim X^{(k)} < n + N - \binom{n+k}{k}$.

Toric threefolds

Theorem (Dickenstein–Di Rocco–P.)

$(X, P) = (X_P, L_P)$ smooth, 2-regular toric threefold embedding is 2-defective if and only if $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. Moreover:

(1) $\deg X^{(2)} = 120$ *if* $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$

(2) $\deg X^{(2)} = 6(8(a + b + c) - 7)$ *if* $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)), 2\ell)$, *where ℓ denotes the tautological line bundle,*

(3) *In all other cases,* $\deg X^{(2)} = 62V - 57F + 28E - 8v + 58V_1 + 51F_1 + 20E_1$, *where V, F, E (resp. V_1, F_1, E_1) denote the (lattice) volume, area of facets, length of edges of P (resp. of the adjoint polytope $\mathrm{Conv}(\mathrm{int} P)$), and $v = \#\{$vertices of $P\}$.*

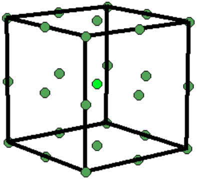
Example

If P is a cube with edge lengths 2, then $(X_P, L_P) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2,2))$.

$V = 3!8 = 48$, $F = 6 \cdot 2 \cdot 4 = 48$, $E = 12 \cdot 2 = 24$, $v = 8$.

$V_1 = F_1 = E_1 = 0$ ($\mathrm{int}(P) = \{(1,1,1)\}$ is a point)

$$\deg X^{(2)} = 62V - 57F + 28E - 8v = 848.$$



k-selfdual toric varieties (joint with A. Dickenstein)

$\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ a lattice point configuration, and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric embedding.

Form the matrix A by adding a row of 1's to the matrix $(a_0|\cdots|a_N)$. Denote by $\mathbf{v}_0 = (1, \ldots, 1)$, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^{N+1}$ the row vectors of A.

For any $\alpha \in \mathbb{N}^{n+1}$, denote by $\mathbf{v}_\alpha \in \mathbb{Z}^{N+1}$ the vector obtained as the coordinatewise product of α_0 times the row vector $\mathbf{v}_0$ times $\ldots$ times α_n times the row vector $\mathbf{v}_n$.

Order the vectors $\{\mathbf{v}_\alpha : |\alpha| \leq k\}$. Let $A^{(k)}$ be the $\binom{n+k}{k} \times (N+1)$ integer matrix with these rows.

Rational normal curve

Take $\mathcal{A} = \{0, \ldots, d\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \end{pmatrix},$$

and

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 1 & 4 & 9 & \cdots & d^2 \\ 0 & 1 & 8 & 27 & \cdots & d^3 \end{pmatrix}.$$

Note that

$$A^{(3)} \cong \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 3 & \cdots & \binom{d}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{d}{3} \end{pmatrix}.$$

Non-pyramidal configurations

The configuration $\mathcal{A}$ is *non-pyramidal* (nap) if the configuration of columns in A is not a pyramid (i.e., no basis vector e_i of $\mathbb{R}^{N+1}$ lies in the rowspan of the matrix).

The configuration $\mathcal{A}$ is $k$nap if the configuration of columns in $A^{(k)}$ is not a pyramid.

Example

A is a pyramid:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$

Characterization of k-self dual configurations

$X_{\mathcal{A}}$ is *k-selfdual* if $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$ for some $\phi\colon \mathbb{P}^N \cong (\mathbb{P}^N)^\vee$.

Theorem (Dickenstein–P.)

1. *$X_{\mathcal{A}}$ is k-selfdual if and only if* $\dim X_{\mathcal{A}} = \dim X_{\mathcal{A}}^{(k)}$ *and $\mathcal{A}$ is knap.*
2. *If $\mathcal{A}$ is knap and* $\dim \operatorname{Ker} A^{(k)} = 1$*, then $X_{\mathcal{A}}$ is k-selfdual.*
3. *If $\mathcal{A}$ is knap and k-selfdual, and* $\dim \operatorname{Ker} A^{(k)} = r > 1$*, then* $\mathcal{A} = e_0 \times \mathcal{A}_0 \cup \ldots \cup e_{r-1} \times \mathcal{A}_{r-1}$ *is r-Cayley.*

The proof generalizes [Bourel–Dickenstein–Rittatore] ($k = 1$).

A surface in $\mathbb{P}^3$

$$\mathcal{A} = \{(0,0),(1,0),(1,1),(0,2)\}$$

gives

$$X_{\mathcal{A}} : (x,y) \mapsto (1 : x : xy : y^2)$$

and

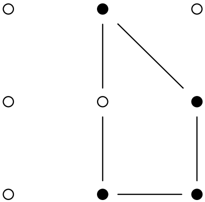
$$X_{\mathcal{A}}^{\vee} \cong X_{\mathcal{A}^{\vee}} : (x,y) \mapsto (-y^2 : 2x^{-1}y^2 : -2x^{-1}y : 1),$$

with

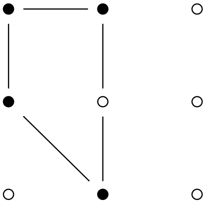
$$\mathcal{A}^{\vee} = \{(0,2),(-1,2),(-1,1),(0,0)\}.$$

This surface is self dual.

The corresponding polygons



$\mathcal{A}$



$\mathcal{A}^\vee$

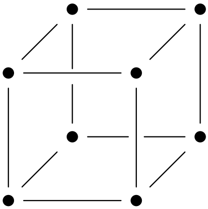
Toric linear projections and sections

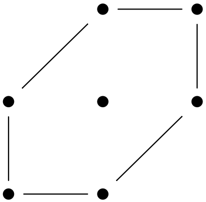
Let $\mathcal{A} = (a_0, \ldots, a_N) \subset \mathbb{Z}^n$ be a lattice point configuration and let $X_\mathcal{A} \subset \mathbb{P}^N$ denote the corresponding toric embedding. Let $\mathcal{A}'$ be a lattice point configuration obtained from $\mathcal{A}$ by removing r points.Then the toric embedding $X_{\mathcal{A}'} \subset \mathbb{P}^{N'}$, where $N' = N - r$, is the *toric linear projection* of $X_\mathcal{A}$ with center equal to the linear span of the "removed points".

A *toric hyperplane section* of $X_\mathcal{A}$ is obtained by taking a hyperplane in $\mathbb{Z}^n$ and "collapsing" the point configuration $\mathcal{A}$ into this lattice hyperplane in such a way that one point is "lost": two points map to the same point.

The degree 6 Del Pezzo surface

- As a *hyperplane section*:
  Let $\mathcal{A} \subset \mathbb{Z}^3$ be the vertices of the unit cube. Collapse the cube in a plane by identifying the opposite vertices $(1,1,1)$ and $(0,0,0)$. This gives a hexagon with one interior point. This hyperplane section of $(\mathbb{P}^1)^3 \subset \mathbb{P}^7$ is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.





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- As a *projection*:
  Let $\mathcal{A} \subset \mathbb{Z}^2$ be the lattice points of the square with sides of length 2. Project $X_{\mathcal{A}} \subset \mathbb{P}^8$ from the points corresponding to the vertices $(2, 0)$ and $(0, 2)$. The projected surface is the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6.

Togliatti's surface

The lattice points defining *Togliatti's surface* $\overline{X} \subset \mathbb{P}^5$ are those of the Del Pezzo hexagon, with the interior point deleted. The 2nd order osculating spaces to X all pass through one point, namely the point corresponding to the interior point of the hexagon. So the (general) 2nd order osculating spaces of $\overline{X}$ have dimension 4, not 5.

The Togliatti surface is 2-self dual, so

$$\deg \overline{X}^{(2)} = \deg \overline{X} = 6.$$

Thank you for your attention!

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