

Higher order dual varieties: the toric case

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Introduction

The dual variety of a projective variety is the set of hyperplanes tangent to the variety, considered as a variety in the dual projective space. It can also be considered as the *discriminant* of the family of hyperplane sections of the variety.

The expected codimension of the dual of a smooth variety is 1; if it is higher, the variety is called *defective*.

Higher order dual varieties are defined as the set of hyperplanes osculating the variety to some higher order.

In joint work with [Alicia Dickenstein](#) and [Sandra Di Rocco](#) we consider higher dual varieties of *toric* varieties, find formulas for their degree in terms of the polytope, and use this to characterize 2-defective toric threefolds.

Dual varieties

Let $X \subset \mathbb{P}^m = \mathbb{P}_K^m$ be a projective variety. The *dual variety* $X^\vee \subset \mathbb{P}^{m^\vee}$ is the closure of set of hyperplanes tangent to X at smooth points.

Example

If $X \subset \mathbb{P}^2$ is a plane curve of degree d , then (if $\text{char } K = 0$) $X^\vee \subset \mathbb{P}^{2^\vee}$ is a curve of degree

$$d^\vee = d(d-1) - \sum e_P$$

where $e_P = \mu_P + m_P - 1$ denotes the multiplicity of the Jacobian ideal at a singular point of X .

The nodes of X correspond to bitangents of $X^\vee$ and the cusps correspond to inflection points of $X^\vee$, and vice versa.

Jet bundles

Assume $X \subset \mathbb{P}^m$ is smooth, of dimension n . Set $\mathcal{L} := \mathcal{O}_{\mathbb{P}^m}(1)|_X$. The k th *jet bundle* (or principal parts bundle of $\mathcal{L}$) is of rank $\binom{n+k}{n}$ and comes with a jet map

$$j_k: \mathcal{O}_X^{m+1} \rightarrow \mathcal{P}_X^k(\mathcal{L}),$$

whose fibers are given by Taylor expansions up to k th order of

$$s = (s_0, \dots, s_m): \mathcal{O}_X^{m+1} \rightarrow \mathcal{L}.$$

The exact sequences

$$0 \rightarrow S^i \Omega_X^1 \otimes \mathcal{L} \rightarrow \mathcal{P}_X^i(\mathcal{L}) \rightarrow \mathcal{P}_X^{i-1}(\mathcal{L}) \rightarrow 0$$

allow one to compute the Chern classes of the jet bundles in terms of those of X and $\mathcal{L}$.



Higher order dual varieties

The embedded tangent space to X at a point x is equal to

$$\mathbb{T}_{X,x} = \mathbb{P}(\operatorname{Im} j_{1,x}) = \mathbb{P}(\mathcal{P}_X^1(\mathcal{L})_x) \cong \mathbb{P}^n.$$

The k th osculating space to X at x is the linear space

$$\mathbb{T}_{X,x}^k := \mathbb{P}(\operatorname{Im} j_{k,x}).$$

Note that $j_{k,x}$ is surjective iff $\dim \mathbb{T}_{X,x}^k = \binom{n+k}{n} - 1$. If j_k is (generically) surjective, then X is called (generically) k -regular.

The k th dual variety $X^{(k)}$ is defined as:

$$X^{(k)} = \overline{\{H \in \mathbb{P}^{m^\vee} \mid H \supseteq \mathbb{T}_{X,x}^k \text{ for some } x \in X_{\text{sm}}\}}.$$

In particular, $X^{(1)} = X^\vee$, $X^{(k-1)} \supseteq X^{(k)}$, and $X^{(k)}$ is contained in the singular locus of $X^\vee$ for $k \geq 2$.



Twisted cubics

The twisted cubic is the curve $X \subset \mathbb{P}^3$ given by $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ sending $(s : t) \rightarrow (s^3 : s^2t : st^2 : t^3)$.

For $s = 1$ the first jet map j_1 is given by

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \end{pmatrix}$$

and the second jet map j_2 by

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \end{pmatrix}$$



The dual variety $X^\vee$ is a ruled (developable) surface of degree 4, and the second dual variety $X^{(2)}$ is the cuspidal edge of $X^\vee$. Indeed, $X^{(2)}$ consists of the planes $H = \mathbb{T}_{X,x}^2$ that intersect X in a point x with multiplicity 3, hence is a singular point of $X^\vee$. The curve $X^{(2)}$ is again a twisted cubic, and $X^\vee$ is its tangent developable.

Let $\overline{X} \subset \mathbb{P}^2$ be a projection of X from $P \in \mathbb{P}^3$. Then $\overline{X}^\vee \subseteq X^\vee \cap P^\perp$, with equality if P is general.

If $P = (0 : 1 : 0 : 0)$, then $X^\vee \cap P^\perp = \overline{X}^\vee \cup T$, where T is the tangent to X at $(1 : 0 : 0 : 0)$. Indeed, $\overline{X}$ is a cuspidal cubic, and $\overline{X}^\vee$ is again a twisted cubic.

This analysis generalizes to rational normal curves $\mathbb{P}^1 \rightarrow \mathbb{P}^m$ of degree m .

Dimension, defect and degree

If X is smooth (or not “too” singular) then the expected codimension of $X^\vee$ is 1. If $\text{codim } X^\vee \geq 2$, X is called *defective*.

Facts: The only defective curves are lines. The only defective surfaces are cones and tangent developables (and they are “too” singular). Defective varieties have been studied and classified for certain classes of varieties.

If X is smooth and generically k -regular, the expected codimension of X is $\binom{n+k}{k} - n$. If it is higher, X is called k -defective.

If X is not k -defective, then the degree of $X^{(k)}$ is given by $c_n(\mathcal{P}_X^k(\mathcal{L}))$. In particular, in this case,

$$c_n(\mathcal{P}_X^k(\mathcal{L})) = 0 \iff X \text{ is } k\text{-defective.}$$

Surfaces

Set $c_i := c_i(T_X) = (-1)^i c_i(\Omega_X)$ and $L := c_1(\mathcal{L})$.

For a k -regular surface, Lanteri and Mallavibarrena computed

$$c_2(\mathcal{P}_X^k(\mathcal{L})) = \frac{1}{3} \binom{k+3}{4} ((3L - kc_1)^2 + 3c_2 - c_1^2).$$

Using the fact that $3L - kc_1$ is very ample unless $(X, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}(2))$, and Miyaoka's inequality $3c_2 - c_1^2 \geq 0$, they concluded that the only k -defective smooth surface is $(\mathbb{P}^2, \mathcal{O}(2))$.

Threefolds

For a threefold,

$$c_3(\mathcal{P}_X^1(\mathcal{L})) = 4L^3 - 3c_1L^2 + 2c_2L - c_3,$$

$$c_3(\mathcal{P}_X^2(\mathcal{L})) = 120L^3 - 180c_1L^2 + 48c_2L + 72c_1^2L - 7c_1^3 - 20c_1c_2 - 8c_3.$$

Example

Consider $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ with embedding given by the line bundle 2ξ , where ξ is the tautological bundle.

For any $\mathbb{P}^2$ -bundle over $\mathbb{P}^1$, $c_3 = 6$, $c_1c_2 = 24$, $c_1^3 = 54$.

$$\deg X^\vee = c_3(\mathcal{P}_X^1(\mathcal{L})) = 6(2(a + b + c) - 1),$$

and for the second dual variety,

$$\deg X^{(2)} = c_3(\mathcal{P}_X^2(\mathcal{L})) = 6(8(a + b + c) - 7).$$



Toric embeddings and polytopes

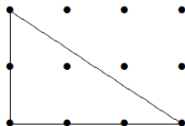
A convex lattice polytope P in $\mathbb{R}^n$ defines a toric embedding $X \subset \mathbb{P}^m$, where $m + 1$ is the number of lattice points in P .

Example

If $P = \text{convex hull of } \{(0, 0), (3, 0), (0, 2)\}$, then the embedding is given by

$$(x, y) \mapsto (1 : x : y : x^2 : xy : y^2 : x^3).$$

Then $X = \mathbb{P}(1, 2, 3)$, a *weighted projective plane*.



Toric threefolds

Theorem. [Dickenstein–Di Rocco–P.] *Let $(X, \mathcal{L})$ be a smooth, 2-regular toric threefold embedding corresponding to a lattice polytope P . Then X is 2-defective if and only if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}(2))$. Moreover:*

- (1) $\deg X^{(2)} = 120$ if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}(3))$
- (2) $\deg X^{(2)} = 6(8(a + b + c) - 7)$ if $(X, \mathcal{L}) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi)$, where ξ denotes the tautological line bundle,
- (3) In all other cases,

$$\deg X^{(2)} = 62V - 57F + 28E - 8v + 58V_1 + 51F_1 + 20E_1,$$

where V, F, E (resp. V_1, F_1, E_1) denotes the (lattice) volume, area of facets, length of edges of P (resp. $\text{int}(P)$), and v is the number of vertices of P .



Example

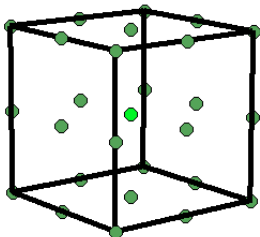
If P is a cube with edge lengths 2, then

$$(X_P, \mathcal{L}_P) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 2, 2)).$$

$$V = 3!8 = 48, F = 6 \cdot 2 \cdot 4 = 48, E = 12 \cdot 2 = 24, v = 8.$$

Since $\text{int}(P) = \{(1, 1, 1)\}$ is a point, $V_1 = F_1 = E_1 = 0$. Hence

$$\deg X^{(2)} = 62V - 57F + 28E - 8v = 848.$$



Sketch of proof

Recall that $V = L^3$, $F = c_1 L^2$, $E = c_2 L$, and $v = c_3$, and that $c_1 = -K_X = \sum_{\Phi \text{ facet}} [\Phi]$.

A line bundle (divisor) on a variety X is called *nef* if it intersects every curve non-negatively. For the proof, we shall consider two cases: (a) $L - c_1 = L + K_X$ is nef, (b) $L - c_1$ is not nef.

In case (a), recall that the adjoint polytope P_1 obtained by “moving each facet inwards by 1” is equal to the convex hull of $\text{int}(P)$, and the corresponding toric embedding is given by $L - c_1$.

Hence $V_1 = (L - c_1)^3$, $F_1 = c_1(L - c_1)^2$, and $E_1 = c_2(L - c_1)$. This allows us to express c_1^3 and $c_1^2 L$ in terms of volumes, and we obtain the formula in (3) from the Chern class formula for $c_3(\mathcal{P}_X^2(\mathcal{L}))$. Note that $V_1, F_1, E_1 \geq 0$.



Rewrite the formula as

$$5V + 57(V - F) + 28E - 8v + 58V_1 + 51F_1 + 20E_1.$$

Note that $V > 0$, $V - F = L^3 - c_1 L^2 = L^2(L - c_1) \geq 0$ (since $L - c_1$ is nef).

Since P is 2-regular, each edge of P has length ≥ 2 . It follows that we can “put” a simplex at each vertex of P such that the simplices don’t overlap on the edges of P . This implies that $E \geq 3v$. Therefore, $28E - 8v > 0$.

Finally, the last three terms are all non-negative.

Hence $c_3(\mathcal{P}_X^2(\mathcal{L})) > 0$, and so X is not 2-defective.



Assume now that $L - c_1$ is not nef. It follows from Fujita's classification, as explained in [Di Rocco–Haase–Nill–Paffenholz], that there is only a short list of possibilities.

In our case, since X is 2-regular, the only possibilities are

- ▶ $(\mathbb{P}^3, \mathcal{O}(2))$
- ▶ $(\mathbb{P}^3, \mathcal{O}(3))$
- ▶ $(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi).$

In the first case, it is easy to see that j_2 is an isomorphism, hence that $\mathcal{P}_X^2(\mathcal{L})$ is trivial and therefore $c_3(\mathcal{P}_X^2(\mathcal{L})) = 0$. Hence X is defective. In the second case, a computation yields (1). And the third case is the example we already computed.

Toric surfaces

Recall that for a k -regular surface, Lanteri and Mallavibarrena computed

$$c_2(\mathcal{P}_X^k(\mathcal{L})) = \frac{1}{3} \binom{k+3}{4} ((3L - kc_1)^2 + 3c_2 - c_1^2).$$

Using the fact that $3L - kc_1$ is very ample unless $(X, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}(2))$, and Miyaoka's inequality $3c_2 - c_1^2 \geq 0$, they concluded that the only k -defective smooth surface is $(\mathbb{P}^2, \mathcal{O}(2))$.

If X is toric, then

$$3c_2 - c_1^2 = 3c_2 - (12 - c_2) = 4(c_2 - 3).$$

But $c_2 = v$ is the number of vertices of the polygon P corresponding to X , hence $v \geq 3$ replaces Miyaoka.

Non- k -regular varieties

The typical examples are scrolls. They are not k -regular for $k \geq 2$.

Rational normal scrolls are toric varieties: they correspond to Cayley polytopes:

$$P = \text{Cayley}(d_1\Delta_1, \dots, d_n\Delta_1).$$

The higher order dual varieties of rational normal scrolls were completely described in [P.–Sacchiero].



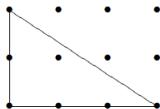
Singular toric varieties

The weighted projective plane $\mathbb{P}(1, 2, 3)$ has two singular points. It is the toric linear projection of $(X, \mathcal{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 3))$ from a 4-dimensional linear space intersecting the surface in a point, with high order of contact [Heidi Mork]. SINGULAR gives $\dim X^\vee = 5$ and $\deg X^\vee = 7$.

The “smooth” formula would have given

$$3V - 2E + v = 18 - 12 + 3 = 9.$$

The difference 2 (1 for each singular point) could be the Milnor number (of a hyperplane section), or weighted by the Euler obstruction [Matsui–Takeuchi].



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