

Classifying regular lattice polytopes via toric fibrations

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Lattice polytopes

Let $P \subset \mathbb{R}^n$ be a convex lattice polytope, i.e., P is the convex hull of a finite set of lattice points.

The *codegree* of P is defined (Batyrev–Nill) by

$$\text{codeg}(P) = \min_{\mathbb{N}} \{m \mid mP \text{ has interior lattice points}\}.$$

The *degree* of P is $\deg(P) = n + 1 - \text{codeg}(P)$.

Example

Δ_n = the standard n -simplex. Then $\text{codeg}(\Delta_n) = n + 1$ and $\text{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil$.

Ehrhart series

Let $f_P(m)$ denote the number of lattice points in mP . The *Ehrhart series* is the generating function

$$F_P(t) = \sum_m f_P(m)t^m = \frac{h_0 + h_1t + \dots + h_d t^d}{(1-t)^{n+1}},$$

where d equal to the degree of P .

The normalized volume of P is equal to the sum of the coefficients h_i , and the leading coefficient is the number of lattice points in kP , where $k = n + 1 - d$ is the codegree. Moreover, the generating function for the number $g_P(m)$ of *interior* lattice points in mP is equal to

$$G_P(t) = \sum_m g_P(m)t^m = \frac{h_d t^{n+1-d} + \dots + h_0 t^{n+1}}{(1-t)^{n+1}}.$$

This gives an indication of the combinatorial and computational interest in these notions.

Cayley polytopes

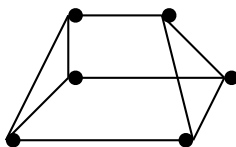
A *Cayley polytope* is a polytope of the form

$$P = \text{Cayley}(P_0, \dots, P_k),$$

where $k \geq 1$, $P_i = \text{Conv}(\{p_j^i\}) \subset \mathbb{R}^m$, $e_0, \dots, e_k$ are the vertices of $\Delta_k \subset \mathbb{R}^k$, and $P = \text{Conv}(\{(p_j^i, e_i)\}) \subset \mathbb{R}^{k+m}$.

A *generalized* Cayley polytope is a polytope P^s , where the e_i are replaced by se_i in the definition above.

Figure: A (non-regular) Cayley polytope



$$\text{Cayley}^2(6\Delta_1, 5\Delta_1, 3\Delta_1)$$



Batyrev–Nill asked: Given d , does there exist an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Recently, Haase, Nill, and Payne showed that for general polytopes, this holds, with

$$N(d) \leq (d^2 + 19d - 4)/2.$$

We conjecture that if P is *regular*, then $N(d) = 2d + 1$. We can prove it for regular polytopes satisfying an additional assumption.

(Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.)



Shrinking polytopes

Write

$$P = \cap H_{\rho_i, -a_i}^+,$$

where the $H_{\rho_i, -a_i}^+$ are halfspaces defined by the normal vectors ρ_i and the a_i 's are integers.

Assume from now on P is regular (aka Delzant, aka smooth), i.e., at any vertex there are precisely n edges, and the first lattice points on these edges form a basis for the lattice.

Set $P^{(r)} = \cap H_{\rho_i, -a_i+r}^+$. Observe that the lattice points of $P^{(1)}$ are the same as the interior lattice points of P . (Note that $P^{(r)}$ does not need to be a lattice polytope, and even if it is, it does not need to be regular nor of the same dimension as P .)



Spanned polytopes

We say that $P^{(r)}$ is *spanned* if $P^{(r)} \cap \mathbb{Z}^n \neq \emptyset$ and for any vertex $m = (-a_1, \dots, -a_n)$ (in the lattice basis $\{\rho_1, \dots, \rho_n\}$) the lattice point $(-a_1 + r, \dots, -a_n + r)$ is in $P^{(r)}$.

Example

Let P be the polytope obtained from the simplex

$$m\Delta_3 = \text{Conv}\{(0, 0, 0), (m, 0, 0), (0, m, 0), (0, 0, m)\}$$

by removing the simplex

$$\Delta_3 = \text{Conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

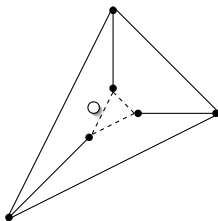
Assume $m \geq 4$. Then $P^{(1)} \cap \mathbb{Z}^n \neq \emptyset$, but $P^{(1)}$ is not spanned.

In fact, the vertex $(1, 0, 0) = (y = 0) \cap (z = 0) \cap (x + y + z = 1)$ of P “goes to” the lattice point

$$(0, 1, 1) = (y = 1) \cap (z = 1) \cap (x + y + z = 2)$$

which is not a point in $P^{(1)}$. Similarly for the vertices $(0, 1, 0)$ and $(0, 0, 1)$.

Note that if we instead remove $2\Delta_3$, then the polytope is spanned: the vertices $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$ all go to the same lattice point $(1, 1, 1)$, which is an interior point of the original polytope.



Nef value for polytopes

Define the *nef value* of the polytope P to be

$$\tau(P) = \inf_{\mathbb{Q}} \left\{ \frac{a}{b} \mid (aP)^{(b)} \text{ is spanned} \right\}.$$

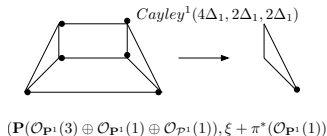
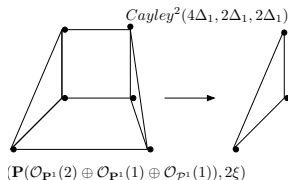
Note that, if $\tau(P)$ is an integer, then $\tau(P) \geq \text{codeg}(P)$.

We can refine the notion of codegree, corresponding to the *spectral value* in adjunction theory, so that e.g. the codegree of $2\Delta_n$ is $\frac{n+1}{2}$. With this notion, $\tau(P) \geq \text{codeg}(P)$ always.

Example

Take $P = m\Delta_3 \setminus \Delta_3$, $m \geq 4$. Then $\text{codeg}(P) = 1$ and $\tau(P) = 2$ (because $P^{(1)}$ is not spanned at the vertices of the small facet).

Example



The first polytope P^2 has $n = 3$, $s = 2$, $m = 1$, $k = 2$, and $\tau(P^2) = \frac{3}{2}$, $\text{codeg}(P^2) = 2$.

The second polytope P^1 has $n = 3$, $s = 1$, $m = 1$, $k = 2$, and $\text{codeg}(P^1) = \tau(P^1) = 3$.

The main result

Theorem

Let P be a regular lattice polytope of dimension n , and assume that $\tau(P) = (\text{refined}) \text{ codeg}(P)$. The following are equivalent

- (1) $\text{codeg}(P) \geq \frac{n+3}{2}$
- (2) $P = \text{Cayley}(P_0, \dots, P_k)$, where $k = \text{codeg}(P) - 1$ and $k > \frac{n}{2}$.

Note that a polytope as in (2) is *defective*, with defect $\delta = 2k - n$.

(This means that the polarized toric variety (X, L) corresponding to P has defect $\delta = 2k - n$, i.e., its dual variety has codimension $2k - n + 1$.)



Adjunction on toric varieties

Let X be a nonsingular projective variety, L an ample line bundle. Assume the canonical line bundle K_X is not nef (i.e., there is a curve C on X such that $K_X \cdot C < 0$). The *nef value* of (X, L) is

$$\tau_L = \min_{\mathbb{R}} \{t \mid K_X + tL \text{ is nef} \}.$$

It is well known that τ_L is a positive rational number.

Assume X is a toric variety, and let $P = \cap H_{\rho_i, -a_i}^+$ denote the polytope defined by (X, L) . Then $L = \sum a_i D_i$, where the D_i are the invariant divisors, and $K_X = -\sum D_i$. On a toric variety, a line bundle is nef if and only if it is spanned (generated by its global sections).

Since $P_{bK_X+aL} = (aP)^{(b)}$, we have $\tau_L = \tau(P)$.

Proof of the theorem

Let (X, L) be the toric variety corresponding to P . If $\tau(P) = \tau_L = \frac{a}{b}$, the line bundle $bK_X + aL$ defines a morphism from X to a projective space. The Remmert–Stein factorization $\varphi: X \rightarrow Y$ of this map is called the *nef value map*.

Assumption: φ is NOT birational.

This holds if $\tau(P)$ is equal to the refined codegree of P , and conjecturally this holds when the codegree is big enough.

By adjunction theory, the inequality $\tau_L \geq \frac{n+3}{2}$ implies that φ is the contraction of an extremal ray in the nef cone $\text{NE}(X)$.

By a result of Reid, such a contraction is a flat toric fibration, with Y smooth and toric and the general fiber $F = \mathbb{P}^k$, where $k = n - \dim Y$. Hence, $L|_F = \mathcal{O}_{\mathbb{P}}^k(s)$ for some s . Assume $s = 1$. Then all fibers of φ must be $\mathbb{P}^k$, and therefore $X = \mathbb{P}(\varphi_*L)$ is a projective bundle. Since Y is toric, φ_*L splits as a sum of line bundles, and thus φ gives P the structure of a Cayley polytope.

To show that $s = 1$, take a line ℓ in $\mathbb{P}^k = F$. We get

$$0 = (K_X + \tau L) \cdot \ell = K_X \cdot \ell + \tau s = -(k+1) + \tau s$$

so that

$$\frac{n+3}{2} \leq \tau = \frac{k+1}{s} \leq \frac{n+1}{s}.$$

So $s = 1$ and $\tau = k+1$, with $k > \frac{n}{2}$.

We obtain the following corollary (obtained by Batyrev–Nill in the case of not necessarily regular polytopes).

Corollary

Assume P is a regular n -dimensional lattice polytope.

- (1) $\deg(P) = 0$ if and only if $P = \Delta_n$,
- (2) $\deg(P) = 1$ if and only if $P = \text{Cayley}(P_0, \dots, P_{n-1})$ is a Lawrence prism (the P_i are segments) or $P = 2\Delta_2$.

Proof.

(1): $\tau(P) > \text{codeg}(P) - 1 = n \geq \frac{n+1}{2}$, so φ contracts a line; hence $\tau(P)$ is an integer and so $\tau(P) \geq n + 1$; therefore $\tau(P) = \text{codeg}(P) = n + 1$. The theorem implies that P is Cayley with $k = n$, which means that $P = \Delta_n$.

(2): For $n \geq 3$ the same argument applies and gives that P is Cayley with $k = n - 1$. For $n = 2$ a direct argument gives $P = \Delta_1 \times \Delta_1$ (Cayley) or $P = 2\Delta_2$. □

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