

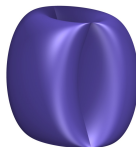
Singularities of algebraic curves and surfaces — classification and detection

An overview of Work Packages 4.1 and 4.2
in the GAIA II project

presented by

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Outline

Introduction

Classification of surfaces

Visualization

Detection of singularities



Algebraic geometry and CAGD

In order to use algebraic curves and surfaces efficiently in CAGD, we need to know about their shape:

- ▶ number of connected components
- ▶ selfintersections
- ▶ other singularities

The aim would be to have a “catalogue” of surfaces (or surface patches) from which a CAGD person could choose models, or candidates for approximate solutions to implicitization problems.

In short, we need to classify algebraic curves and surfaces. What are the tools provided by algebraic geometry?



Classification problems are classical in algebraic geometry :-)
but

- ▶ most known results are only valid for projective varieties defined over the complex numbers :-(
▶ classification over the real numbers is much harder :-(
▶ classification in affine space is in some sense harder than in projective space :-(

Example: the classification of conic sections.

The curves $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$ are not equivalent in $\mathbb{R}^2$, but they are equivalent (via $y \mapsto iy$) in $\mathbb{C}^2$ and (via $x \leftrightarrow z$) in $\mathbb{P}^2(\mathbb{R})$.



Algebraic surfaces

The (only?) interesting algebraic surfaces from a CAGD point of view are *parameterizable* (i.e., rational).

Examples are

- ▶ triangle (= Veronese) surfaces
- ▶ tensor (= Segre) surfaces (cf. Thi Ha Lê's talk)
- ▶ monoid surfaces (cf. Johansen's talk)
- ▶ Hirzebruch and Del Pezzo surfaces (e.g. cubic surfaces)
- ▶ other toric and almost toric surfaces
- ▶ rational scrolls
- ▶ tangent developables of rational space curves.



Singularities of triangle surfaces

A triangle (or Veronese) surface is the image of

$$\varphi: \mathbb{P}^2 \rightarrow \mathbb{P}^3,$$

where $\varphi = (f_0, f_1, f_2, f_3)$, with f_i homogeneous of degree d .

The degree of the (implicit) equation of the image surface is $\leq d^2$ (equality if no base points).

Curve of selfintersection has degree

$$m \leq (d^4 - 4d^2 + 3d)/2.$$

In general, this is an equality, and additional singularities are triple points and pinch points, whose numbers are also determined by d .

Examples

1. The Steiner surfaces ($d = 2$)

There are 6 different real Steiner surfaces. Can have $m = 1, 2, 3$.

A. $x^2y^2 + y^2z^2 + z^2x^2 - xyz = 0$

B. $x^2y^2 - y^2z^2 + z^2x^2 - xyz = 0$

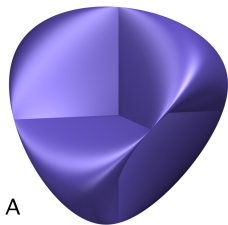
C. $xyz^2 + xy - x^2 - z^4 - 2z^2 - 1 = 0$

D. $xz^2 - y^2 + z^4 = 0$

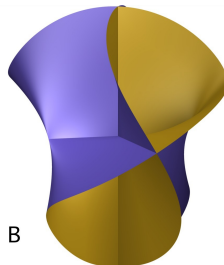
E. $x^4 + y^2 + z^2 - 2x^2y - 2x^2z + 2yz - 4yz = 0$

F. $y^2 + 2yz^2 + z^4 - x = 0$

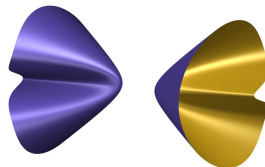




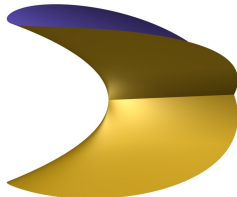
A



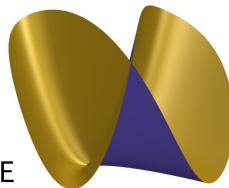
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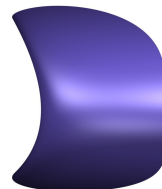
C



D



E



F



A: Steiner's Roman surface. Three real double lines meeting in a triple point. Each line has two real pinchpoints. ($d = 4$, $m = 3$, $t = 1$, $\nu_2 = 6$)

B: Three real double lines meeting in a triple point. One line has two real pinch points. ($d = 4$, $m = 3$, $t = 1$, $\nu_2 = 2$)

C: One real double line. The line has one real pinch point. ($d = 4$, $m = 1$, $\nu_2 = 1$)

D: One simple and one double double lines meeting in a triple point. The simple line has two real pinch points. ($d = 4$, $m = 2$, $t = 1$, $\nu_2 = 2$)

E: Somewhat similar to D.

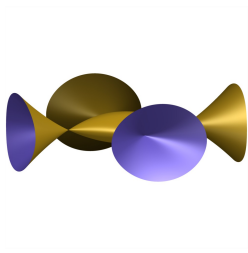
F: One threefold double line containing a triple point.



2. The double pillow ($d = 3$)

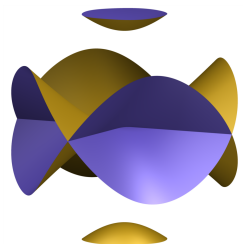
$$x^4 - 2x^2y^2 + y^4 - 2x^2w^2 - 2y^2w^2 - 16z^2w^2 + w^4 = 0$$

Picture in xyz -space ($w = 1$):



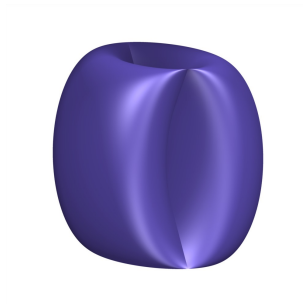
$$x^4 - 2x^2y^2 + y^4 - 2x^2 - 2y^2 - 16z^2 + 1 = 0$$

Picture in xyw -space ($z = 1$):



$$x^4 - 2x^2y^2 + y^4 - 2x^2w^2 - 2y^2w^2 - 16w^2 + w^4 = 0$$

The dual surface of the double pillow:



$$16x^2 - y^4 + 2y^2z^2 - 8x^2y^2 - z^4 - 8x^2z^2 - 16x^4 = 0$$

Singularities of tensor surfaces

A tensor (or Segre) surface is given by

$$\varphi: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3,$$

where $\varphi = (f_0, f_1, f_2, f_3)$, with f_i bihomogeneous of bidegree (a, b) .

The degree of the (implicit) equation of the image surface is $\leq 2ab$. Equality if no base points.

The curve of selfintersection has degree

$$m \leq 2a^2b^2 - 4ab + a + b.$$

In general, this is an equality, and additional singularities are triple points and pinch points, whose numbers are also determined by a and b .

For the case $a = 1, b = 2$, cf. the talk by Thi Ha Lê.



Real versus complex

The formulas for the degree of the singular loci are essentially generalized Plücker formulas. Need refined versions, treating only the *real* singular loci.

Curves: Klein showed that at most one third of the flexes of a plane curve can be real. Generalizations by Shuh, Wall.

Surfaces: Viro has some results for surfaces, but harder to interpret and to use. Generalizations by Ernström.

Reality questions: the number of real solutions is bounded by the number of complex solutions — when is the maximum achieved?



Singularities and deformations

Deformation theory exists mainly for complex varieties. From CAGD point of view, the interesting, but more difficult, cases are the real case, especially the affine case and the bounded case.

When a curve or surface acquires or loses a singularity, both local and global shape change.

Classification of isolated singularities: start with some coarse numerical invariant, like multiplicity or modularity.

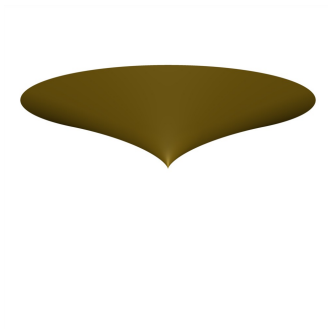
Modularity tells about how many different singularities can be “the same” without being equivalent. The simplest are the *simple* singularities: A_k , D_k , E_6 , E_7 , E_8 .

There are several definitions of “equivalent”: e.g. topological or analytical equisingularity, and in the real case, there are different types.

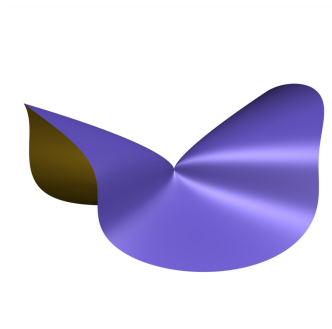
Arnold has classified all singularities up to a certain complexity. Each has a *normal form* (cf. Johansen's talk).

For example, the normal form for A_k is $x^2 \pm y^2 - z^{k+1}$.

The A_2 singularity can have two *real* types:

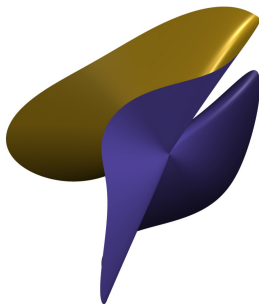


$$A_2^+ : x^2 + y^2 - z^3 = 0$$

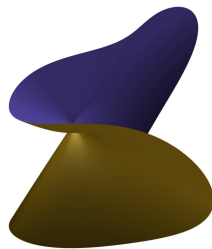


$$A_2^- : x^2 - y^2 - z^3 = 0$$

Here are two Q_{10} singularities:



$$x^3 + y^2z + z^4 + xz^3 = 0$$



$$x^3 + y^2z - z^4 + xz^3 = 0$$

The index 10 is the Milnor number. A finer invariant is the Tjurina number.

Moduli spaces

The space of all plane curves, or all surfaces in 3-space, of a given degree can be identified with the space of the coefficients of their equations.

E.g., the space of conics has dimension 5, the space of cubic surfaces has dimension 20. Over the complex numbers, all conics are equivalent, so the dimension of the *moduli space* of complex conics is 0. But the moduli space is more interesting for real affine conics — and gets more interesting, and much more complicated, for curves and surfaces of higher degree.



For parameterizable varieties (not necessarily hypersurfaces) we can look at the space of parameterizations instead of the space of (implicit) equations. Taking equivalence classes of maps, we get a smaller dimensional *moduli space*.

This can e.g. be done for patches of tensor surfaces (cf. Thi Ha Lê's talk).

More generally, want to find small dimensional families containing enough interesting objects. For example, when looking for possible solutions to (approximate) implicitization problems, want to get (very) sparse matrices in the elimination process.



Fast visualization of surfaces

To get a quick understanding of the shape of a given surface of degree d , make its equation $f(x, y, z)$ to be monic in z . Compute the resultant $R_0(x, y)$ and certain subdeterminants R_i of the resultant matrix, $i = 1, \dots, d - 2$. These are the Sturm–Habicht coefficients.

The arrangement of the curves $R_i = 0$ in the wanted region in the xy -plane gives regions (surface patches, curves, and points) where the number of real solutions in z to $f(x, y, z) = 0$ is constant. The curve $R_0(x, y) = 0$ is the projection of the *contour* of the surface viewed from the center of projection.

The contour curve is the intersection of the surface with its first polar surface w.r.t. the projection center. The theory of *real* polar varieties should be pursued, both in the global and local case.

Example: the double pillow

In the $w = 1$ case: The arrangement is given by the curve $R_0 = 0$, which is

$$x^4 - 2x^2y^2 + y^4 - 2x^2 - 2y^2 + 1 = 0$$

This gives the four lines $\pm x \pm y = 1$.

For points (x, y) on the lines, there is one solution in w , for points in the four cones, there are two solutions, and for points in the rest of the plane there are no real solutions.

In the $z = 1$ case: $R_0 = 0$ gives the lines $x + y = 0$ and $x - y = 0$, whereas neither $R_1 = 0$ nor $R_2 = 0$ has real solutions.

For points (x, y) on the lines there are three real solutions in w , for points outside the lines there are four.



Early detection of singularities

Parameterized surfaces have singularities essentially of two kinds; cuspidal edges and selfintersections. If the parameterization has base points, there might also be isolated singularities.

The first can be detected on the parameterization (computing the Jacobian of the parameterization), the second requires a global analysis.

For computing selfintersections of a Bézier bicubic surface, give two contributions:

- ▶ a specific sparse bivariate resultant adapted to the corresponding elimination problem
- ▶ the use of a semi-numeric polynomial solver able to deal with large systems of equations with floating point coefficients.

(For bidegree $(2, 2)$ patches, cf. the talk by S. Chau.)



Early detection of singularities is important for approximate implicitization of a parameterized surface patch. Require that the implicitly defined surface has (nearly) the same shape (and singularities) as the patch it tries to approximate.

A CAGD designer will not create selfintersections or other singularities on purpose. Singular surfaces arise from built-in functions: offset or draft or sweep. These *procedural* surfaces cannot be avoided.

The loops or folds may be very small and hence difficult to detect.

Can represent a procedural surface via a sampling of the parameterization domain. The parameterization can be a priori computed on such a sampling and eventually refined (cf. thesis of J.-P. Pavone).

Fast detection of singularities

Exploit the properties of polynomial representations in the Bernstein basis. This representation is much more numerically stable than the monomial representation and has a direct geometric meaning in terms of control points.

Subdivision approach: based on convex hull property, for checking the existence of solutions in the search domain.

Output: *no* or *maybe*. If *maybe*, subdivide the domain.

Continue until a termination criterion is satisfied.

Reduction approaches contract the domain where solutions are sought. Can concentrate on the parts of the domain where the roots are. (Cannot replace completely subdivision.)

Propose a general scheme for comparing and evaluating these methods. New reduction technique and new preconditioning steps
⇒ new reduction-subdivision solver.