

Generating functions and enumerative geometry

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“Abel in Holland”

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“A generating function is a clothesline on which we hang up a sequence of numbers for display.”

H.S.Wilf
generatingfunctionology
Academic Press Inc., 1990



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Partitions

Let $p(n)$ denote the number of partitions of the integer n , i.e., $p(n)$ is the number of ways we can write $n = n_1 + n_2 + \dots$, where $n_1 \geq n_2 \geq \dots > 0$.



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$$(1 + q + q^2 + \dots)(1 + q^2 + q^4 + \dots) \cdots (1 + q^m + q^{2m} + \dots) \cdots$$

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The function $f(q)$ displays the integer sequence

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, 77, 101, \dots$$

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$$\pi(1) = 1$$

$$\pi(2) = 3 \leftrightarrow 2, \quad 1 \mid 1, \quad \frac{1}{1}$$

$$\pi(3) = 6 \leftrightarrow 3, \quad 2 \mid 1, \quad \frac{2}{1}, \quad \frac{1}{1}, \quad 1 \mid 1 \mid 1, \quad \frac{1}{1} \mid \frac{1}{1}$$



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MacMahon: the generating function for the $\pi(n)$ is

$$M(q) = \prod_{m \geq 1} (1 - q^m)^{-m}$$



Degree zero Donaldson–Thomas invariants

A plane partition of n can be viewed as a set of n monomials in three variables such that all the other monomials generate an ideal in $\mathbb{C}[x, y, z]$.



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Or to the length n fixed points of $\mathbb{C}^3$ under the torus action: the virtual count of the degree zero Donaldson–Thomas invariants:

$$DT_0(\mathbb{C}^3)(q) = M(-q) = \prod_{m \geq 1} (1 - (-q)^m)^{-m}$$

Modulo q^4 :

$$\begin{aligned}M(x) &\equiv (1 + q + q^2 + q^3)(1 + q^2)^2(1 + q^3)^3 \\&\equiv 1 + q + q^2 + 2q^2 + q^3 + q \cdot 2q^2 + 3q^3\end{aligned}$$



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$$q \leftrightarrow 1 \leftrightarrow \langle 1 \rangle$$

$$q^2 \leftrightarrow 2 \leftrightarrow \langle 1, z \rangle$$

$$2q^2 \leftrightarrow 1 \mid 1 \quad \text{and} \quad \frac{1}{1} \leftrightarrow \langle 1, y \rangle \quad \text{and} \quad \langle 1, x \rangle$$

$$q^3 \leftrightarrow 3 \leftrightarrow \langle 1, z, z^2 \rangle$$

$$q \cdot 2q^2 \leftrightarrow 2 \mid 1 \quad \text{and} \quad \frac{2}{1} \leftrightarrow \langle 1, z, y \rangle \quad \text{and} \quad \langle 1, z, x \rangle$$

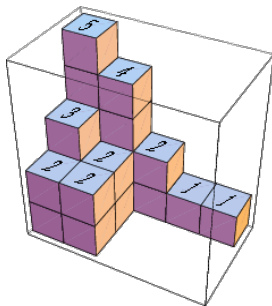
$$3q^3 \leftrightarrow 1 \mid 1 \mid 1, \frac{1}{1} \mid \frac{1}{1}, \frac{1}{1} \leftrightarrow \langle 1, y, y^2 \rangle, \langle 1, x, y \rangle, \langle 1, x, x^2 \rangle$$



A plane partition of 22

5	4	2	1	1
3	2			
2	2			

 $\longleftrightarrow \langle 1, z, z^2, z^3, z^4, y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2z \rangle$



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From this one deduces the equation $q C(q)^2 - C(q) + 1 = 0$ for the generating function:

$$C(q) := \sum_{n=0}^{\infty} C_n q^n = \frac{1 - \sqrt{1 - 4q}}{2q}$$



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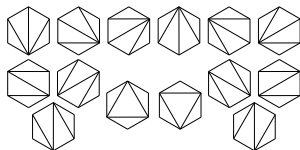
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- ▶ The ways of joining $2n$ points on a circle to form n nonintersecting chords.
- ▶ The number of ways to triangulate a regular $(n + 2)$ -gon.
For $n = 4$ there are 14 ways:



Schubert calculus



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- How many lines intersect four given lines in $\mathbb{P}^3$?



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Degenerate the four lines to two pairs of intersecting lines. Then the line through the two points of intersection, and the line which is the intersection of the two planes are the lines meeting all four lines – the answer is 2.



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The function $C(q) = \frac{1-\sqrt{1-4q}}{2q}$ displays the sequence

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \dots$$



Storm P. on H.G. Zeuthen (1839–1920) in 1946

- What is it about?
- It is about the theory of algebraic curves and surfaces, from which enumerative geometry originated!



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Counting curves on surfaces

Given a projective smooth surface S and a linear system $Y = |\mathcal{L}|$ of curves, how many curves of given genus g (or with given number r of singular points) pass through a given number of points on S ?



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Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$



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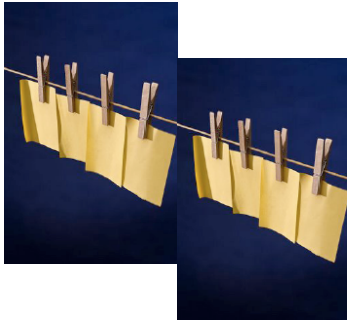
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No explicit generating function is known.

To display on the clothesline?



... the sequence of numbers:

$$N_1 = 1$$

$$N_2 = 1$$

$$N_3 = 12$$

$$N_4 = 620$$

$$N_5 = 87304$$

$$N_6 = 26312976$$

$$N_7 = 14616808192$$

$$N_8 = 13525751027392$$

$$N_9 = 19385778269260800$$

$$N_{10} = 40739017561997799680$$

$$N_{11} = 120278021410937387514880$$

$$N_{12} = 482113680618029292368686080$$



Göttsche's conjectures

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Note that there are three aspects:

- ▶ *shape* of generating function
- ▶ *explicit computation*
- ▶ *validity* of the formulas

Yau–Zaslow’s formula

Let S be a K3 surface and $\mathcal{L}$ a line bundle. The number N_r of rational (genus 0) curves in $|\mathcal{L}|$ is the same as the number of curves in $|\mathcal{L}|$ with $r = (\mathcal{L}^2 + 2)/2$ nodes and is independent of S and $\mathcal{L}$.



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The generating function is

$$\sum_{r=0}^{\infty} N_r q^r = \prod_{m \geq 1} (1 - q^m)^{-24} = (\sum p(n) q^n)^{24}$$



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Geometric reason (Bryan–Leung): The Euler number $c_2(S)$ is 24, and S can be degenerated (in symplectic geometry) to an elliptic fibration over $\mathbb{P}^1$ with 24 nodal (hence rational) fibers. There are $p(j_i)$ degree j_i maps from a stable rational curve (a bunch of $\mathbb{P}^1$ ’s) to a nodal fiber, hence

$$n_r = \sum_{j_1 + \dots + j_{24} = r} p(j_1) \cdots p(j_{24}).$$

Bell numbers

The Stirling numbers $S_{n,k}$ count the number of partitions of a set with n elements into k blocks. The Bell numbers count *all* partitions: $B_n = \sum_{k=1}^n S_{n,k}$. They can be defined recursively:

$$B_0 = 1 \text{ and } B_{n+1} = \sum_{i=0}^n \binom{n}{i} B_i$$



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Their *exponential* generating function

$$B(q) := \sum_{n=0}^{\infty} B_n q^n / n!$$

satisfies the differential equation

$$\frac{B'(q)}{B(q)} = e^q,$$

and hence

$$B(q) = e^{e^q - 1}.$$



Polydiagonals



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Polydiagonals

Let X be a space, and consider the set of ordered n -tuples of elements of X , denoted $X \times \cdots \times X = X^n$. For each $k \leq n$ and each partition $\mathbf{i}_k$ of $\{1, 2, \dots, n\}$ into k disjoint subsets $\{i_1^j, i_2^j, \dots, i_{l(j)}^j\}$, $j = 1, \dots, k$, $\sum l(j) = n$, the corresponding *polydiagonal* is defined as

$$\Delta(\mathbf{i}_k) = \{(x_1, \dots, x_n) \mid x_{i_r^j} = x_{i_s^j} \text{ for all } r, s, j\}$$



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$$\frac{r!}{j_1! \cdots j_r!} \left(\frac{1}{1!}\right)^{j_1} \cdots \left(\frac{1}{r!}\right)^{j_r}$$

polydiagonals of each type ($j_i = \#$ blocks of i elements).



Bell polynomials

The partial Bell polynomials are

$$B_{n,k}(a_1, \dots, a_{n-k+1}) = \sum \binom{n}{j_1 \dots j_{n-k+1}} \left(\frac{a_1}{1!}\right)^{j_1} \dots \left(\frac{a_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

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Recursively defined by $B_0 = 1$ and

$$B_{n+1}(a_1, \dots, a_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(a_1, \dots, a_{n-i}) a_{i+1}$$



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summing over $j \vdash n, \sum j_i = k$.

The complete Bell polynomials are

$$B_n(a_1, \dots, a_n) = \sum_{k=1}^n B_{n,k}(a_1, \dots, a_{n-k+1}).$$

Then $S_{n,k} = B_{n,k}(1, \dots, 1)$ and $B_n = B_n(1, \dots, 1)$.

Recursively defined by $B_0 = 1$ and

$$B_{n+1}(a_1, \dots, a_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(a_1, \dots, a_{n-i}) a_{i+1}$$

or by the formal identity

$$\sum_{i \geq 0} B_i(a_1, \dots, a_i) q^i / i! = \exp(\sum_{j \geq 1} a_j q^j / j!).$$



Faà di Bruno's formula

Let $h(t) = f(g(t))$ be a composed function.

Differentiate once: $h'(t) = f'(g(t))g'(t)$

and twice: $h''(t) = f''(g(t))g'(t)^2 + f'(g(t))g''(t)$.



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and indeed

$$h_n = \sum_{k=1}^n f_k B_{n,k}(g_1, \dots, g_{n-k+1})$$



Nodal curves

Let S be a smooth, projective surface, $Y = |\mathcal{L}|$ a linear system, $m = \mathcal{L}^2$, $k = K_S \cdot \mathcal{L}$, $s = K_S^2$, $x = c_2(S)$ the four Chern numbers.



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Theorem (Kleiman–P)

There exist universal linear polynomials $a_1, a_2, \dots$ in four variables, computable by an explicit algorithm, such that, for $r \leq 8$,

$$N_r = B_r(a_1(m, k, s, x), \dots, a_r(m, k, s, x))/r!$$

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The advantage of knowing the Bell form of the node polynomials N_r is that in order to compute N_{r+1} from the previous N_i one only needs to compute one new term, a_{r+1} :

$$N_{r+1} = \frac{1}{(r+1)!} \sum_{i=0}^r \binom{r}{i} (r-i)! N_{r-i} a_{i+1} = \dots + \frac{1}{(r+1)!} a_{r+1}$$



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$$ru_r = u_{r-1}u_1 + \partial(u_{r-1})$$

or

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and, pretending ∂ is a derivation:

$$3!u_3 = (a_1^2 + a_2)a_1 + \partial(a_1^2 + a_2) = a_1^3 + a_1a_2 + 2a_1a_2 + a_3.$$

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Let $X \subset S \times Y$ be the set of pairs (x, C) , where x is a node on C . Consider the fibered product $(S \times Y)^r = S^r \times Y$ and let X_j denote the pullback of X via the j th projection. Then a_r is equal to $(-1)^{r-1}(r-1)!$ times the contribution of all “distinguished varieties” of the scheme intersection product $X_1 \cap \dots \cap X_r$ supported on the small diagonal.

THANK YOU FOR YOUR ATTENTION!



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