

PLANAR POSITIVE GEOMETRIES AND THEIR ADJOINTS

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Scattering amplitudes

In physics, scattering amplitudes play a vital role in understanding the interactions between fundamental particles. The scattering amplitude is a probability amplitude.

There is an effort to reformulate quantum field theory in terms of geometric concepts, where all properties of scattering amplitudes are the answers to some natural questions in the context of *positive geometry*.

A positive geometry is a semi-algebraic set that comes with a meromorphic differential form, the *canonical form*. The scattering amplitude is the integral of this differential form. Hence one is interested in finding explicit expressions for the canonical form.



Positive geometries and their canonical forms

Let X be a projective complex variety of dimension n , $X_{\geq 0} \subset X(\mathbb{R})$ a closed semi-algebraic subset such that its Euclidean interior $X_{>0}$ is an open oriented n -dimensional manifold with closure $X_{\geq 0}$. Set $\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$. Let $C_1, \dots, C_k$ be the irreducible components of the Zariski closure of $\partial X_{\geq 0}$. Set $C_{i,\geq 0} := C_i \cap \partial X_{\geq 0} \subset C_i(\mathbb{R})$.

We say that $(X, X_{\geq 0})$ is a *positive geometry* if there exists a unique meromorphic n -form $\Omega(X, X_{\geq 0})$, called its canonical form, such that

- (a) if $n = 0$, $X_{\geq 0}$ is a point and $\Omega(X, X_{\geq 0}) = \pm 1$,
- (b) if $n > 0$, $(C_i, C_{i,\geq 0})$ is a positive geometry with canonical form $\Omega(C_i, C_{i,\geq 0}) = \text{Res}_{C_i}(\Omega(X, X_{\geq 0}))$,
- (c) $\Omega(X, X_{\geq 0})$ is holomorphic on $X \setminus \cup C_i$.



Examples of positive geometries

- 1) convex polytopes
- 2) toric varieties $(X_P, X_{P,\geq 0})$
- 3) Grassmann varieties $(\mathbb{G}(k, n), \mathbb{G}(k, n)_{\geq 0})$
- 4) generalized tree amplituhedra $\mathcal{A}_{n,k,m}$
- 5) moduli space of points on $\mathbb{P}^1$ $(\overline{M}_{0,n}, (\overline{M}_{0,n})_{\geq 0})$

In this talk I will focus on another type of positive geometries.¹ These are *polypols*, which originated with the work of Wachspress in the 1970's, in the context of geometric modeling and the finite element method for solving PDEs.

¹K. Kohn, R. Piene, K. Ranestad, F. Rydell, B. Shapiro, R. Sinn, M.-S. Sorea, S. Telen, “Adjoints and canonical forms of polypols”, Doc. Math. 30 (2025), 275–346.



Positive geometries in dimension 1

Observe that it follows from the definition that if $(X, X_{\geq 0})$ is a positive geometry, then X can have no non-zero holomorphic n -forms.

Example ($n = 1$)

The curve X must be rational, hence $X = \mathbb{P}^1$. Then $X_{\geq 0}$ is a finite union of closed segments in $\mathbb{P}^1(\mathbb{R})$.

Assume $X_{\geq 0} = [a, b] \subset \mathbb{R}$. Consider $\Omega := (\frac{1}{t-b} - \frac{1}{t-a})dt$. Then $\text{Res}_a \Omega = -1$ and $\text{Res}_b \Omega = 1$ and Ω is holomorphic on $\mathbb{P}^1 \setminus \{a, b\}$. Hence

$$\Omega(\mathbb{P}_{\mathbb{C}}^1, [a, b]) = \frac{t-a-t+b}{(t-a)(t-b)}dt = \frac{b-a}{(t-a)(t-b)}dt.$$



A positive geometry in dimension 2

Set $X = \mathbb{P}^2$ and let $X_{\geq 0} \subset \mathbb{R}^2 \subset X(\mathbb{R})$ be the triangle with vertices $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $v_3 = (0, 0)$.

The boundary components are

$C_1 : y = 0$, $C_2 : x + y - 1 = 0$, and $C_3 : x = 0$.

Consider

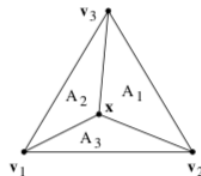
$$\Omega(X, X_{\geq 0}) = \frac{1}{y(x+y-1)x} dx \wedge dy = \frac{1}{(x+y-1)x} dx \wedge \frac{dy}{y}.$$

Then

$$\text{Res}_{C_1} \Omega(X, X_{\geq 0}) = \frac{1}{x(x-1)} dx = \Omega(C_1, C_{1,\geq 0}).$$



Barycentric coordinates



$$\phi_i(x) := A_i(x)/A$$

$$x = \phi_1(x)v_1 + \phi_2(x)v_2 + \phi_3(x)v_3$$

$$\phi_i(x) \geq 0$$

$$\phi_1(x) + \phi_2(x) + \phi_3(x) = 1$$

Barycentric coordinates were introduced by August Ferdinand Möbius in 1827.



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Generalized barycentric coordinates

Generalized barycentric coordinates for *convex polygons* were introduced by Eugene Wachspress, with a view towards applications for solving PDE's by the finite element method (further work by Warren, Floater, ...).

Let $v_1, \dots, v_k$ be the vertices of a convex polygon P . Define weights

$$w_i(v) := A(v_{i-1}, v_i, v_{i+1}) \prod_{j \neq i-1, i} A(v, v_j, v_{j+1})$$

One shows

$$\sum_{i=1}^k w_i(v) v_i = \left(\sum_{i=1}^k w_i(v) \right) v.$$



Wachspress coordinates (1975)

The *Wachspress coordinates* for $v \in P$ are

$$\phi_i(v) := \frac{w_i(v)}{\sum_{i=1}^k w_i(v)}.$$

These coordinates satisfy

- $\phi_i(v_j) = \delta_{i,j}$
- $v = \sum_{i=1}^k \phi_i(v) v_i$
- $\phi_i(v) \geq 0$
- $\sum_{i=1}^k \phi_i(v) = 1$



The Wachspress adjoint

Setting $v = (x, y)$, we get that $A(v, v_j, v_{j+1})$ is a linear polynomial in x and y and hence the $w_i(x, y)$ are polynomials of degree $k - 2$.

Since $\sum_{i=1}^k w_i(x, y)(x_i, y_i) = (\sum_{i=1}^k w_i(x, y))(x, y)$, the polynomial

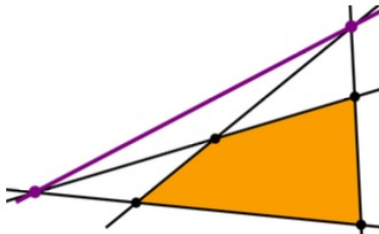
$$\alpha_P(x, y) := \sum_{i=1}^k w_i(x, y)$$

is a polynomial in x and y of degree $k - 3$. It has zeros at the intersections of the lines defining P other than the vertices of P .

The curve A_P defined by $\alpha_P = 0$ is called the *Wachspress adjoint* of P .

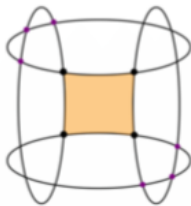


The adjoint of a quadrangle



Polypols

Wachspress defined *polypols* as “curved polygons” with sides being real segments of curves $C_1, \dots, C_k \subset \mathbb{P}^2$ and vertices $v_i \in C_i \cap C_{i+1}$.



The residual scheme of a polypol

Set $C := \cup C_i$, $d_i := \deg C_i$, and $d := \sum d_i = \deg C$.

The *residual scheme* $R(P)$ of P is $\text{Sing } C \setminus \{v_1, \dots, v_k\} \subset C$ fattened by the conductor ideal of C at each point.

Recall that if $\mathcal{O}$ is a local ring of dimension 1 and $\overline{\mathcal{O}}$ its normalization, then the conductor ideal is

$$\mathcal{C} := \{f \in \mathcal{O} \mid f\overline{\mathcal{O}} \subseteq \mathcal{O}\}.$$

The conductor ideal of a node is the maximal ideal which is also the Jacobian ideal – that of a cusp is also the maximal ideal, which in this case is different from the Jacobian ideal.



Adjoints of plane curves

Let $D \subset \mathbb{P}^2$ be a curve of degree d and $\text{Sing } D \subset D$ its singular locus (defined by the conductor ideal $\mathcal{C} \subset \mathcal{O}_D$).

Classically, an *adjoint curve* is any curve containing $\text{Sing } D$, and a *special adjoint* is an adjoint of degree $d - 3$.

Consider the restriction map

$$H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(D)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 3)) \rightarrow H^0(D, \omega_D)$$

The (special) adjoint curves are defined by sections that map to non-zero elements of $H^0(D, \mathcal{C}\omega_D)$.

Example

If D is a nonsingular quartic curve, then $\Omega_D^1 = \omega_D = \mathcal{O}_D(1)$, so the lines cut out the canonical divisor on D .



Adjoints of polypols

Definition

An adjoint curve A_P of a polypol P is a curve of degree $d - 3$ that contains $R(P)$.



Regular rational polypols

A polypol is *rational* if all curves C_i are rational. We show:
a rational polypol has precisely one adjoint curve.

Wachspress used the adjoint curve A_P to propose generalized barycentric coordinates on a rational polypol P . These coordinates should be rational functions on P that are positive on the interior of P and have poles on A_P .

A polypol is *real* if the C_i and v_i are real, with a choice of real segments (sides) $C_{i,\geq 0}$ from v_{i-1} to v_i and a closed semi-algebraic set $P_{\geq 0} \subset \mathbb{P}^2(\mathbb{R})$ with interior $P_{>0}$ a union of simply connected sets with boundary the union of the sides.

A real polypol is *regular* if all points on the sides of P except the vertices v_i are nonsingular on C and $C \cap P_{>0} = \emptyset$.



Wachspress coordinates for a polypol?

Let P be a rational regular polypol, and let $C_i: f_i = 0$. Set

$$p_i := \frac{f_1 \cdots f_k}{f_i f_{i+1}}.$$

The adjoint A_P is defined by a polynomial α_P of degree $d - 3$.

The proposed Wachspress coordinates are the rational functions with poles on A_P :

$$\phi_i(v) := \frac{\alpha_P(v_i)}{p_i(v_i)} \cdot \frac{p_i(v)}{\alpha_P(v)}.$$

They should satisfy, for $v \in P_{\geq 0}$,

$$\phi_i(v_j) = \delta_{i,j}, v = \sum_{i=1}^k \phi_i(v) v_i, \phi_i(v) \geq 0, \sum_{i=1}^k \phi_i(v) = 1.$$



Wachspress's conjecture

Let P be a regular rational polypol.

Wachspress (and we) showed that $A_P \cap \partial P_{\geq 0} = \emptyset$.

Wachspress conjectured: $A_P \cap P_{>0} = \emptyset$.

He claimed the conjecture holds

- (1) for convex polygons,
- (2) for polypols with $d \leq 5$.

We proved (1) and (2).

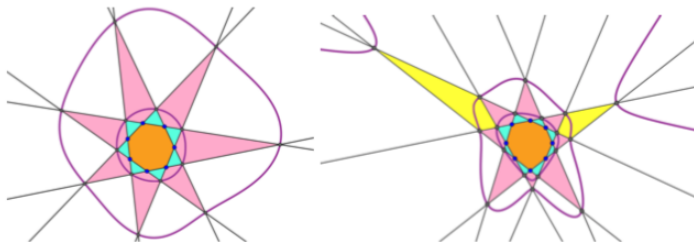
The first open case of the conjecture is a P formed by three ellipses. We show there are 44 configurations and prove the conjecture for 33 of these.



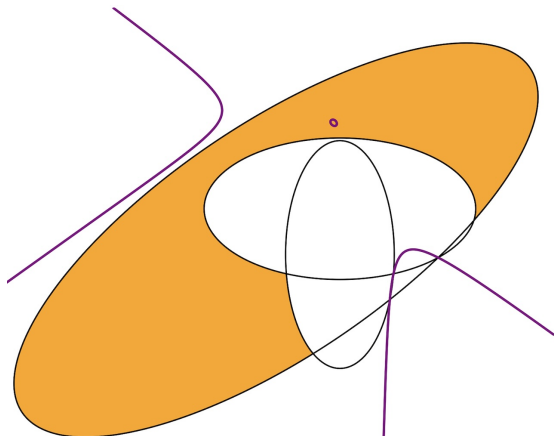
Convex polygons

Theorem

Let P be a convex real polygon with k sides. Then A_P is hyperbolic with respect to any $v \in P_{\geq 0}$. More precisely, $A_P(\mathbb{R})$ consists of $\lfloor \frac{k-3}{2} \rfloor$ nested ovals plus, if k is even, a pseudoline.



A counterexample to W's conjecture (C. Brüser²)



²C. Brüser, “The geometry of polycons and a counterexample to Wachspress’ conjecture”, arXiv:2603.18643



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Regular rational polypols give positive geometries

Theorem

A regular rational polypol P gives a positive geometry with canonical form

$$\Omega(\mathbb{P}^2, P_{\geq 0}) = \frac{\alpha_P}{f_1 \cdots f_k} dx \wedge dy.$$

Proof. We have

$$\frac{\alpha_P}{f_1 \cdots f_k} dx \wedge dy = \frac{\alpha_P}{f_1 \cdots \hat{f}_i \cdots f_k} dx \wedge \frac{dy}{f_i}$$

and $df_i = (f_i)_x dx + (f_i)_y dy$ and $dx \wedge df_i = (f_i)_y dx \wedge dy$, hence

$$\text{Res}_{C_i} \Omega(\mathbb{P}^2, P_{\geq 0}) = \frac{\alpha_P}{f_1 \cdots \hat{f}_i \cdots f_k \cdot (f_i)_y} \Big|_{f_i=0} dx.$$



Take a rational parameterization $t \mapsto (x(t), y(t))$ of $C_{i, \geq 0}$ with $v_{i-1} = (x(a_i), y(a_i))$ and $v_i = (x(b_i), y(b_i))$.

We get

$$\operatorname{Res}_{C_i} \Omega(\mathbb{P}^2, P_{\geq 0}) = \frac{F(t)}{G(t)(t - a_i)(t - b_i)} dt$$

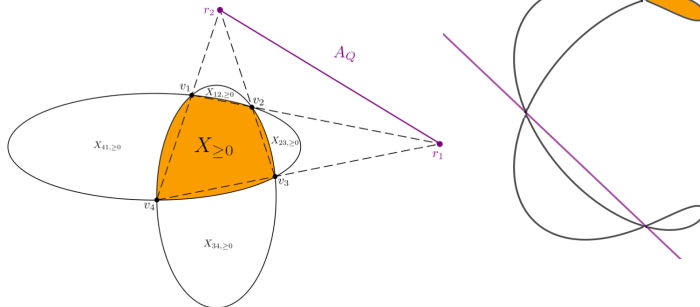
with $\deg F = \deg G = d_i(d - 1) - 2$. Check that F and G have the same roots with the same multiplicities (Jacobian ideal = conductor ideal $\cdot$ ramification ideal). Thus $F(t)/G(t)$ is a constant, and we get, for some constant γ_i ,

$$\operatorname{Res}_{C_i} \Omega(\mathbb{P}^2, P_{\geq 0}) = \gamma_i \Omega(\mathbb{P}^1, [a_i, b_i]) = \gamma_i \frac{b_i - a_i}{(t - a_i)(t - b_i)}.$$

With the correct scaling of α_P , we show $\gamma_i = 1$, for all i .



Other planar positive geometries



Functoriality

In order to compute canonical forms, one can use their properties.

Theorem (Brown–Dupont³)

Canonical forms satisfy

- *Linearity (additivity under triangulations)*
- *Recursion (reducing to lower dimension)*
- *Invariance under modifications (e.g., blow-ups)*
- *Compatibility with pushforward and pullback (e.g., pushdown from a simpler positive geometry)*
- *Multiplicativity (product of positive geometries)*

³F. Brown, C. Dupont, “Positive geometries and canonical forms via mixed Hodge theory,” Commun. Math. Phys. (2025), 406; 267.



Amplituhedra

Let Z be a $(k+m) \times n$ matrix with positive maximal minors (a positroid), so that $Z : \mathbb{G}(k, n)_{\geq 0} \rightarrow \mathbb{G}(k, k+m)_{\geq 0}$.

The *generalized tree amplituhedron* is

$$\mathcal{A}_{n,k,m}(Z) := Z(\mathbb{G}(k, n)_{\geq 0}).$$

For $k = 1$, it is a convex polytope and hence gives a positive geometry.

Example

$\mathcal{A}_{d+1,1,2}(Z) \subset \mathbb{G}(1, 3)_{\geq 0} \cong (\mathbb{P}^2)_{\geq 0}$, with

$$Z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & \cdots & d-3 & d-2 & d-1 & d \\ 0 & 0 & 1 & 3 & 6 & \cdots & \binom{d-3}{2} & \binom{d-2}{2} & \binom{d-1}{2} & \binom{d}{2} \end{pmatrix}.$$



The canonical form

The matrix Z defines a lattice polygon $P(d)$ with $d + 1$ vertices. To determine the canonical form, decompose

$$P(d) = \Delta(d) \cup P(d-1),$$

where $\Delta(d)$ is the triangle with vertices $(0, 0)$, $(d-1, \binom{d-1}{2})$, and $(d, \binom{d}{2})$.

$$\Omega(\Delta(d)) = \frac{2d(d-1)}{(2y-(d-2)x)(2y-2(d-1)x+d(d-1))((d-1)x-2y)} dx \wedge dy.$$

The additivity of the canonical form then gives

$$\Omega(P(d)) = \Omega(\Delta(d)) + \Omega(P(d-1))$$

$$\Omega(P(d)) = \sum_{i=2}^d \frac{2i(i-1)}{(2y-(i-2)x)(2y-2(i-1)x+i(i-1))((i-1)x-2y)} dx \wedge dy.$$



The toric variety defined by $P(d)$

The lattice polygon $P(d)$ defines a toric variety $X_{P(d)} \subset \mathbb{P}^d$:

$$(t_1, t_2) \mapsto (1 : t_1 : t_1^2 t_2 : t_1^3 t_2^2 : \cdots : t_1^{d-1} t_2^{\binom{d-1}{2}} : t_1^d t_2^{\binom{d}{2}}).$$

$X_{P(d)}$ is a surface that contains the rational normal curve Y_d of degree d (set $t_2 = 1$).

At each point $v \in Y_d$, the tangent space to $X_{P(d)}$ at v is equal to the osculating plane of Y_d at v

$X_{P(d)}$ is the *2nd toric interpolant*⁴ of the rational normal curve.

⁴A. Dickenstein, S. Di Rocco, R. Piene, “Interpolation of toric varieties,” New York. J. Math. 30 (2024), 1498–1516.



Open questions

- The adjoint curve of a convex polygon is hyperbolic. For convex polypols (or for polytopes in higher dimension) the adjoint can be hyperbolic or not. Find a condition for hyperbolicity.
- When is the adjoint curve of a polypol singular?
- The *adjoint map* that sends a planar polypol to its adjoint curve is a finite map in some cases (we found all). What is the degree of this map? For convex heptagons, we conjecture the degree is 864.



THANK YOU FOR YOUR ATTENTION!



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