

Ideals associated to a desingularization.

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1. Introduction.

Let X be a reduced algebraic curve with normalization $f : Z \rightarrow X$. Let J denote the jacobian ideal of X , I the ramification ideal of f , and C the conductor of Z in X . By comparing certain invertible sheaves on Z I showed in ([P], § 2) the equality

$$(*) \quad JO_Z = C \cdot I ,$$

when X is a local complete intersection.

As an example — where these three ideals can be easily computed — consider the plane curve

$$f(x, y) = x^b - y^a = 0 , \quad (a, b) = 1 , \quad a < b .$$

Its normalization $k[t]$ is given by $x \mapsto t^a$, $y \mapsto t^b$. The jacobian ideal is $J = (bx^{b-1}, ay^{a-1})$, the ramification ideal $I = (bt^{b-1}, at^{a-1})$, and the conductor, as an ideal in $k[t]$, is $C = (t^{(a-1)(b-1)})$.

When X is a variety of dimension ≥ 1 , whose normalization Z is smooth, I also showed that $(*)$ holds on the variety obtained by blowing up the ideal I in Z . However, examples indicated that $(*)$ should hold already on Z . We shall prove this below, as a corollary of a more general result:

Consider a desingularization $f : Z \rightarrow X$ of a Cohen-Macaulay variety X , of dimension r . Define the ω -jacobian ideal J of X by

$$J = \text{Ann} \left(\text{Coker} \left(\varphi : \Omega_X^r \rightarrow \omega_X \right) \right) ,$$

where φ is the canonical map from r -differentials to dualizing sheaf.

If X is a local complete intersection (l.c.i.) we show that J is equal

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to the (usual) jacobian ideal

$$\mathfrak{J} = F^r(\Omega_X^1),$$

defined as the r -th Fitting ideal of the module of differentials.

Inspired by ([M - T]) we consider the adjunction conductor (or ideal of adjoint functions) $A \subseteq \mathcal{O}_X$, defined by

$$A = \text{Ann}(\text{Coker}(\text{Tr}_f : f_*\omega_Z \rightarrow \omega_X)).$$

If f is finite (and X is Gorenstein), this ideal is equal to the usual conductor C .

We introduce an ideal $B \subseteq \mathcal{O}_Z$ measuring where

$$\rho : f_*f^*\Omega_Z^r \rightarrow \Omega_Z^r$$

is not surjective,

$$B = \text{Ann}(\text{Coker } \rho).$$

Note that $B = \mathcal{O}_Z$ if f is finite, but that $B \neq \mathcal{O}_Z$ for example if f factors through another desingularization of X .

Finally, let $I = F^0(\Omega_{Z/X}^1)$ denote the ramification ideal of f . Then we prove (Thm. 1): Suppose X is a Gorenstein variety. There is an equality of ideals,

$$(**) \quad B \cdot J\mathcal{O}_Z = I \cdot A\mathcal{O}_Z.$$

We also give some examples of the relations (*) and (**).

In the last section we look at Nash transformations and jacobian blow-ups: If X is a l.c.i., it is known ([N]) that the Nash transformation $\pi : \tilde{X} \rightarrow X$ is equal to the blow-up $\tilde{\pi}$ of the jacobian ideal $\mathfrak{J}$. Nobile ([N]) also showed that the Nash transformation is locally the blow-up of an ideal which is contained in the jacobian ideal, but whose associated subscheme might have support strictly bigger than the singular locus of X .

Here we show that, for a Gorenstein variety X , the Nash transformation π is (globally) equal to the blow-up of the ω -jacobian ideal,

hence to the blow-up of a subscheme whose support is contained in the singular locus.

Moreover, π is equal to the blow-up $\tilde{\pi}$ of the jacobian ideal $\mathfrak{J}$ if and only if $\dim \text{proj } \pi^* \Omega_X^1 \leq 1$ holds.

We know no explicit way of computing the ω -jacobian ideal J of a Gorenstein, non l.c.i. variety X . Therefore we can give no examples where $J \neq \mathfrak{J}$ (or $\pi \neq \tilde{\pi}$) holds, though such seem likely to exist.

2. Relations among the ideals.

Let k be an algebraically closed field. A variety is a reduced, equidimensional algebraic k -scheme. If X is a variety of dimension r , we let $\Omega_X^r = \wedge^r \Omega_X^1$ denote its sheaf of r -differentials. If X is Cohen-Macaulay, it has a dualizing sheaf ω_X , and if $f : Z \rightarrow X$ is a proper map between Cohen-Macaulay varieties of the same dimension, there is a trace homomorphism ([H], VII, 3.4)

$$\text{Tr}_f : f_* \omega_Z \rightarrow \omega_X .$$

Suppose X is Cohen-Macaulay, of dimension r , and let

$$n : X' \rightarrow X$$

denote its normalization. Let $i : U \hookrightarrow X'$ denote the inclusion of the smooth locus U of X' . Then $i^* \omega_{X'} = i^* \Omega_{X'}^r$. Since $\text{depth } \omega_{X'} \geq 2$ at all points of $X' - U$, we get

$$\omega_{X'} = i_* i^* \Omega_{X'}^r ,$$

and hence a natural homomorphism

$$\varphi_{X'} : \Omega_{X'}^r \rightarrow \omega_{X'} .$$

Define

$$\varphi_X : \Omega_X^r \rightarrow \omega_X$$

as the composition

$$\Omega_X^r \longrightarrow n_* \Omega_{X'}^r \xrightarrow{n_* \varphi_{X'}} n_* \omega_{X'} \xrightarrow{\text{Tr}_n} \omega_X .$$

(The map φ_X can be defined in much greater generality, see [E].) For any proper birational map $f : Z \rightarrow X$ between Cohen-Macaulay varieties, there is a commutative diagram

$$\begin{array}{ccc}
 \Omega_X^r & \xrightarrow{\varphi_X} & \omega_X \\
 \downarrow & f_*\varphi_Z & \uparrow \text{Tr}_f \\
 f_*\Omega_Z^r & \xrightarrow{\quad} & f_*\omega_Z
 \end{array}
 \quad (***).$$

For the rest of this section we shall assume that X is a Gorenstein variety of dimension r , given with a desingularization

$$f : Z \rightarrow X,$$

i.e., Z is smooth, f is proper and birational. As above we let

$$n : X' \rightarrow X$$

denote the normalization of X .

Note that ω_X is invertible, and that we have

$$\omega_Z = \Omega_Z^r.$$

The jacobian ideal $\mathfrak{J} \subseteq \mathcal{O}_X$ of X is the r th Fitting ideal of Ω_X^1 ,

$$\mathfrak{J} = F^r(\Omega_X^1).$$

We define the ω -jacobian ideal $J \subseteq \mathcal{O}_X$ of X by

$$J = \text{Ann}(\text{Coker } \varphi_X) = \text{Im}(\varphi_X \otimes \omega_X^{-1} : \Omega_X^r \otimes \omega_X^{-1} \rightarrow \mathcal{O}_X).$$

The conductor (of X' in X) is the ideal $C \subseteq \mathcal{O}_X$,

$$C = \text{Hom}_X(n_*\mathcal{O}_{X'}, \mathcal{O}_X) = \text{Ann}(\text{Coker } \mathcal{O}_X \rightarrow n_*\mathcal{O}_{X'}).$$

Note that the ideal $C^\sim \subseteq \mathcal{O}_X$, induced by C is equal to C .

Define the adjunction conductor $A \subseteq \mathcal{O}_X$ by

$$A = \text{Ann}(\text{Coker } \text{Tr}_f) = \text{Im}(\text{Tr}_f \otimes \omega_X^{-1} : f_*\omega_Z \otimes \omega_X^{-1} \rightarrow \mathcal{O}_X).$$

The ramification ideal $I \subseteq \mathcal{O}_Z$ of f is the 0-th Fitting ideal of

the sheaf of relative differentials $\Omega_{Z/X}^1$,

$$I = F^0(\Omega_{Z/X}^1) .$$

Finally we define the ideal $B \subseteq O_Z$ by

$$B = \text{Ann} (\text{Coker } \rho) = \text{Im} (\rho \otimes \omega_Z^{-1}: f^*f_*\omega_Z \otimes \omega_Z^{-1} \rightarrow O_Z) .$$

Remark that we always have inclusions

$$J \subseteq A \quad \text{and} \quad I \subseteq B .$$

The first is an immediate consequence of (***), while the second holds because of the equality

$$I = \text{Im} (f^*\Omega_X^r \otimes \omega_Z^{-1} \rightarrow O_Z)$$

and the commutativity of

$$\begin{array}{ccc} f^*\Omega_X^r & \longrightarrow & f^*f_*\omega_Z \\ & \searrow & \downarrow \\ & & \omega_Z \end{array} .$$

Moreover, if X itself is smooth, φ_X and Tr_f are isomorphisms, so in that case

$$J = A = O_X \quad \text{and} \quad I = B$$

hold.

General duality theory for finite maps implies that Tr_n induces an isomorphism

$$n_*\omega_X \simeq \underline{\text{Hom}}_X(n_*O_{X'}, O_X) \otimes \omega_X = C \otimes \omega_X .$$

Since φ_X factors through Tr_n , we obtain the inclusion

$$A \subseteq C .$$

The conductor C is the ideal of subadjoint functions, while A is the ideal of adjoint functions on X (see [Z], [M - T]). Locally, this can be formulated as follows. Suppose α is a meromorphic differential which generates ω_X . Then $\psi \in C$ iff $\psi\alpha$ is regular at all points of codimension ≤ 1 on X' , and $\psi \in A$ iff $\psi\alpha$ becomes regular on Z . One says that the

singularities of X do not affect the adjunction conditions if $A = C$ holds. This occurs for example when f is finite (i.e., $Z = X'$). The other classical example of such singularities are the rational double points (Du Val, [DV]). In that case, $f_*\omega_Z \xrightarrow{\sim} \omega_X$ holds, hence we get $A = C = \mathcal{O}_X$.

Remark: Suppose $g : Y \rightarrow X$ is a finite map between Gorenstein schemes of the same dimension. In ([P], 2.9) we used the duality isomorphism

$$\omega_Y \cong \underline{\mathrm{Hom}}_X(g_*\mathcal{O}_Y, \mathcal{O}_X)^\sim \otimes g^*\omega_X$$

to show that any such birational map is equal to the blow-up of the conductor

$$C_{Y/X} = \underline{\mathrm{Hom}}_X(g_*\mathcal{O}_Y, \mathcal{O}_X)$$

of Y in X . A special case — X is a hypersurface and g its normalization — has been proved independently by Wilson ([W], 2.7). ■

When X is a l.c.i., we can give an explicit description of the map $\varphi_X : \Omega_X^r \rightarrow \omega_X$. This we shall use to prove the following.

Proposition 1: If X is a l.c.i., then

$$J = \tilde{J}$$

holds, i.e., the jacobian and ω -jacobian ideals are equal.

Proof: The question is local, so assume $X \subseteq Y$, Y smooth of dimension n , and X is defined by $f_1, \dots, f_{n-r}$. Choose a system of coordinates $y_1, \dots, y_n$ on Y around $x \in X$. We know

$$\omega_X = \underline{\mathrm{Ext}}_Y^{n-r}(\mathcal{O}_X, \Omega_Y^n)$$

and, since X is a l.c.i. ([H], III, § 7),

$$\underline{\mathrm{Ext}}_Y^{n-r}(\mathcal{O}_X, \Omega_Y^n) = \underline{\mathrm{Hom}}_X(\wedge^{n-r} N, \Omega_Y^n|_X),$$

where $N = N_{X/Y}$ denotes the conormal bundle of X in Y ,

$$N = (f_1, \dots, f_{n-r}) / (f_1, \dots, f_{n-r})^2.$$

The choices $(f_1, \dots, f_{n-r})$ and $(y_1, \dots, y_n)$ give a generator

$$\begin{bmatrix} dy_1 \wedge \dots \wedge dy_n \\ f_1 \dots f_{n-r} \end{bmatrix}$$

for ω_X at x , and the map

$$\Omega_Y^r|_X \rightarrow \Omega_X^r \xrightarrow{\varphi_X} \omega_X$$

is given by

$$\begin{aligned} dy_{i_1} \wedge \dots \wedge dy_{i_r} &\mapsto \begin{bmatrix} dy_{i_1} \wedge \dots \wedge dy_{i_r} \wedge df_1 \wedge \dots \wedge df_{n-r} \\ f_1 \dots f_{n-r} \end{bmatrix} \\ &= (-1)^\sigma \left| \frac{\partial(f_1, \dots, f_{n-r})}{\partial(y_{j_1}, \dots, y_{j_{n-r}})} \right| \cdot \begin{bmatrix} dy_1 \wedge \dots \wedge dy_n \\ f_1 \dots f_{n-r} \end{bmatrix} \end{aligned}$$

where $\{j_1, \dots, j_{n-r}\} = \{1, \dots, n\} - \{i_1, \dots, i_r\}$.

Hence J is the ideal generated by the $(n-r)$ -minors of the jacobian matrix $\frac{\partial f}{\partial y}$, which is precisely the jacobian ideal

$$\mathfrak{J} = F^r(\Omega_X^1)$$

(since $N \xrightarrow{\frac{\partial f}{\partial y}} \Omega_Y^1|_X \rightarrow \Omega_X^1 \rightarrow 0$ is a presentation). ■

Theorem 1: There is an equality of ideals in O_Z

$$(**) \quad B \cdot JO_Z = I \cdot AO_Z.$$

Moreover, AO_Z is invertible if and only if B is, and JO_Z is invertible if and only if I is.

Corollary 1: Suppose f is finite. Then $C = A$ is invertible, and there is an equality

$$(*) \quad JO_Z = I \cdot C.$$

Corollary 2: Suppose X is a l.c.i. and $\rho : f^*f_*\omega_Z \rightarrow \omega_Z$ is surjective. Then the jacobian ideal $\mathfrak{J} = F^r(\Omega_X^1)$ decomposes in O_Z as

$$\tilde{J}O_Z = I \cdot AO_Z ,$$

and AO_Z is invertible.

Proof of the theorem: Consider the diagram

$$\begin{array}{ccc}
 & f^*\varphi_X & \\
 & \nearrow & \\
 f^*\Omega_X^r & \xrightarrow{\quad} & f^*f_*\omega_Z \\
 & \searrow & \\
 & \Lambda^r df & \\
 & \nearrow & \\
 & f^*\text{Tr} & \\
 & \searrow & \\
 & \omega_Z &
 \end{array}
 \quad
 \begin{array}{c}
 f^*\omega_X \\
 \rho \\
 \omega_Z
 \end{array}
 .$$

Since torsionfree quotients of the same sheaf which are generically isomorphic are everywhere isomorphic, we deduce

$$B \otimes \omega_Z = \text{Im}(\rho) = \text{Im}(f^*\text{Tr}) = AO_Z \otimes f^*\omega_X$$

and

$$I \otimes \omega_Z = \text{Im}(\Lambda^r df) = \text{Im}(f^*\varphi_X) = JO_Z \otimes f^*\omega_X .$$

■

Observe that if $\rho : f^*f_*\omega_Z \rightarrow \omega_Z$ is surjective, then AO_Z is invertible (in particular, f factors through the blow up of A). More generally, whenever $f^*\text{Tr}_f$ factors through ρ , we get

$$\omega_Z \cong \tilde{A} \otimes f^*\omega_X$$

for some invertible ideal $\tilde{A}$, with $AO_Z \subseteq \tilde{A}$ and $B \cdot \tilde{A} = AO_Z$.

3. Examples.

Assume $f : Z \rightarrow X$ is as in Cor. 2. We shall explicit the homomorphisms occurring in the definitions of the ideals and thus obtain (locally) a generator for the invertible ideal AO_Z in terms of generators of $\tilde{J}$ and I .

Take $x \in X$ and $z \in f^{-1}(x)$. We assume $X \subseteq Y$ as in the proof of Prop. 1, and in addition we choose a system of coordinates $(z_1, \dots, z_r)$ on Z around z . Look at

$$\begin{array}{ccccc}
 f^* \Omega_Y^r|_X & \longrightarrow & f^* \Omega_X^r & \xrightarrow{f^* \varphi_X} & f^* \omega_X \\
 & & \downarrow & & \uparrow f^* \text{Tr} \\
 & & f^* f_* \Omega_Z^r = f^* f_* \omega_Z & & \\
 & \searrow \alpha & & \nearrow \rho & \\
 & & & & \omega_Z
 \end{array}$$

The dotted arrow exists since ρ is surjective.

We have seen (proof of Prop. 1) that the element $dy_{i_1} \wedge \dots \wedge dy_{i_r}$ goes to

$$(-1)^\sigma \left| \frac{\partial(f_1, \dots, f_{n-r})}{\partial(y_{j_1}, \dots, y_{j_{n-r}})} \right| \begin{bmatrix} dy_1 \wedge \dots \wedge dy_n \\ f_1 \dots f_{n-r} \end{bmatrix}$$

in ω_X . The same element goes, via α , to

$$\left| \frac{\partial(y_{i_1}, \dots, y_{i_r})}{\partial(z_1, \dots, z_r)} \right| dz_1 \wedge \dots \wedge dz_r.$$

Hence $dz_1 \wedge \dots \wedge dz_r$ goes to

$$(-1)^\sigma \frac{\left| \frac{\partial(f_1, \dots, f_{n-r})}{\partial(y_{j_1}, \dots, y_{j_{n-r}})} \right|}{\left| \frac{\partial(y_{i_1}, \dots, y_{i_r})}{\partial(z_1, \dots, z_r)} \right|} \begin{bmatrix} dy_1 \wedge \dots \wedge dy_n \\ f_1 \dots f_{n-r} \end{bmatrix}$$

in $f^* \omega_X$. So the element

$$a = \frac{(-1)^\sigma \left| \frac{\partial(f_1, \dots, f_{n-r})}{\partial(y_{j_1}, \dots, y_{j_{n-r}})} \right|}{\left| \frac{\partial(y_{i_1}, \dots, y_{i_r})}{\partial(z_1, \dots, z_r)} \right|}$$

is a generator for $A\mathcal{O}_Z$ at z . (The element a is independent of the particular rearrangement

$$\{i_1, \dots, i_r, j_1, \dots, j_{n-r}\} \text{ of } \{1, 2, \dots, n\}.)$$

Example 1: Higher order pinchpoint on surface in 3-space.

Let $x \in X$ be a point on a surface in 3-space with smooth normalization Z such that $f^{-1}(x) = \{z\}$, and such that in the completion of the local rings the situation is as follows:

$$\hat{O}_{X,x} = k[[x, y, z]]/f \rightarrow k[[u, v]] = \hat{O}_{Z,z},$$

with $f(x, y, z) = y^{ab} - x^b z^a$, $x \mapsto u^a$, $y \mapsto uv$, $z \mapsto v^b$. ($a = 1$, $b = 2$ gives an ordinary pinch point ($\text{char } k \neq 2$).)

Then $A O_Z = C$ is generated in $O_{Z,z}$ by

$$-\frac{\frac{\partial f}{\partial x}}{\left| \frac{\partial(y, z)}{\partial(u, v)} \right|} = \frac{\frac{\partial f}{\partial y}}{\left| \frac{\partial(x, z)}{\partial(u, v)} \right|} = -\frac{\frac{\partial f}{\partial z}}{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|} = u^{a(b-1)} v^{b(a-1)}.$$

(In the case $a = 1$, $b = 2$, one sees directly that C is generated in $\hat{O}_{X,x}$ by x and y , hence in $\hat{O}_{Z,z}$ by u .)

Example 2: A cone over a smooth plane curve.

Let X be the cone over a smooth plane curve of degree d , with vertex $x \in X$. Take $f: Z \rightarrow X$ to be the blow-up of the maximal ideal m_x .

In this case, the map $\rho: f^* f_* \omega_Z \rightarrow \omega_Z$ is surjective.

An easy argument shows that, whenever ρ is surjective, the ideal A satisfies

$$A = f_*(A O_Z).$$

Set $E = f^{-1}(x)$, $m_x O_Z = O_Z(-E)$. Clearly

$$\tilde{J} O_Z = O_Z(-(d-1)E)$$

and

$$I = O_Z(-E).$$

Thus we obtain

$$A O_Z = O_Z(-(d-2)E),$$

$$A = f_*(A O_Z) = m_X^{d-2}.$$

Remark that the method of ([M - T], Th. 2.1.1) applies to compute A in this situation also.

Example 3: Different and discriminant.

Let X be a branch of a locally plane curve, with normalization $f : Z \rightarrow X$. Suppose $p : X \rightarrow Y$ is a projection onto a smooth curve satisfying

$$\Omega_{Z/X}^1 = \Omega_{Z/Y}^1 .$$

At a point $z \in Z$ we have the following situation ($x = f(z)$, $y = p(x)$)

$$\begin{array}{c} \mathcal{O}_{Z,z} = k[t]_{(t)} \leftarrow (k[x, y]/f)_{(x,y)} = \mathcal{O}_{X,x} \\ \uparrow \\ k[x]_{(x)} = \mathcal{O}_{Y,y} . \end{array}$$

Assume $I = (\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t})$ is generated by $\frac{\partial x}{\partial t}$. Then $\tilde{J}\mathcal{O}_Z$ is generated by $\frac{\partial f}{\partial y}$. Since

$$I = F^0(\Omega_{Z/X}^1) = F^0(\Omega_{Z/Y}^1) ,$$

our equality (*) can be written (at z)

$$\frac{\partial f}{\partial y} \cdot \mathcal{O}_{Z,z} = C \cdot F^0(\Omega_{Z/Y}^1) ,$$

where C is the conductor of Z in X .

In the present case, $\frac{\partial f}{\partial y} \mathcal{O}_{X,x}$ is the different of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$ ([Z], § 15), and $F^0(\Omega_{Z/Y}^1) = \text{Ann}(\Omega_{Z/Y}^1)$ is the different of $\mathcal{O}_{Z,z}$ over $\mathcal{O}_{Y,y}$ ([S], III, Prop. 14). So here (*) turns out to be the formula relating differentials and conductor ([S], III, Cor. 1, p. 65).

4. Nash transformation and jacobian blow-ups.

Let X be a r -dimensional variety. We let $\pi : \tilde{X} \rightarrow X$ denote its Nash transformation, with Nash bundle

$$\pi^* \Omega_X^1 \rightarrow \Omega .$$

Recall that $\tilde{X}$ is the closure of the rational section

$$X \dashrightarrow \text{Grass}_r(\Omega_X^1)$$

defined on the smooth locus of X . Let $\tilde{\pi} : \tilde{X} \rightarrow X$ denote the blow-up of

the jacobian ideal

$$\tilde{J} = F^r(\Omega_X^1)$$

of X . Since $\tilde{\pi}^*\Omega_X^1$ admits a r -quotient ([L], Lemma 1), $\tilde{\pi}$ factors through π , the latter being minimal with respect to that property.

If X is a l.c.i., then $\pi = \tilde{\pi}$ holds ([N]). We shall show that $\pi = \tilde{\pi}$ is equivalent to

$$\dim \text{proj } \pi^*\Omega_X^1 \leq 1 .$$

(Note that $\dim \text{proj } \tilde{\pi}^*\Omega_X^1 \leq 1$ always holds, by ([L], Lemma 1).)

Proposition 2: The Nash transformation π is equal to the jacobian blow-up $\tilde{\pi}$ if and only if

$$\dim \text{proj } \pi^*\Omega_X^1 \leq 1$$

holds.

Proof: Suppose (locally) we have a presentation

$$0 \rightarrow E \rightarrow F \rightarrow \pi^*\Omega_X^1 \rightarrow 0 .$$

Set E' equal to the kernel of the composition

$$F \rightarrow \pi^*\Omega_X^1 \rightarrow \Omega .$$

Then E' is locally free, with the same rank as E . Hence

$$F^0(\text{Coker } (E \rightarrow E'))$$

is invertible, therefore so is

$$\tilde{J}O_{\tilde{X}} = F^r(\pi^*\Omega_X^1) = F^0(\text{Ker } (\pi^*\Omega_X^1 \rightarrow \Omega)) = F^0(\text{Coker } (E \rightarrow E')) .$$

■

The following proposition is a slight generalization of ([P], 1.1). The proof carries over — we need only replace the Grassmann variety of the bundle E by the Grassmann scheme representing m -quotients of the coherent sheaf E .

Proposition 3: Let Y be a scheme, E a coherent O_Y -module, F a locally free rank m O_Y -module, and $\alpha : E \rightarrow F$ a homomorphism.

The ideal $I = F^0(\text{Coker } \alpha)$ is invertible if and only if $\text{Im}(\alpha)$ is

locally free with rank m .

We shall apply this to compare the Nash transformations and jacobian blow-ups of two varieties Z and X , with respect to a map $f : Z \rightarrow X$.

Proposition 4: Let $f : Z \rightarrow X$ be a proper, surjective, generically unramified map. Denote by $\pi_Z : \bar{Z} \rightarrow Z$ and $\pi_X : \bar{X} \rightarrow X$ (resp. $\tilde{\pi}_Z : \tilde{Z} \rightarrow Z$ and $\tilde{\pi}_X : \tilde{X} \rightarrow X$) the Nash transformation (resp. the blow-up of the jacobian ideal) of Z and of X . Set $\bar{Q}$ (resp. $\tilde{Q}$) equal to the cokernel of the composition

$$\begin{aligned} \bar{\alpha} : \pi_Z^* f^* \Omega_X^1 &\rightarrow \pi_Z^* \Omega_Z^1 \rightarrow \bar{\Omega}_Z \\ (\text{resp. } \tilde{\alpha} : \tilde{\pi}_Z^* f^* \Omega_X^1 &\rightarrow \tilde{\pi}_Z^* \Omega_Z^1 \rightarrow \tilde{\Omega}_Z) \end{aligned}$$

where $\bar{\Omega}_Z$ (resp. $\tilde{\Omega}_Z$) denotes the canonical quotient on $\bar{Z}$ (resp. $\tilde{Z}$). Finally, let $\bar{\psi} : Z' \rightarrow \bar{Z}$ (resp. $\tilde{\psi} : Z'' \rightarrow \tilde{Z}$) denote the blow-up of $F^\circ(\bar{Q})$ (resp. $F^\circ(\tilde{Q})$).

Then there exist factorizations $\bar{g} : Z' \rightarrow \bar{X}$ and $\tilde{g} : Z'' \rightarrow \bar{X}$ of $f \circ \pi_Z \circ \bar{\psi}$ and $f \circ \tilde{\pi}_Z \circ \tilde{\psi}$ through π_X .

Proof: Apply Prop. 3 to the maps $\bar{\psi}^*(\bar{\alpha})$ and $\tilde{\psi}^*(\tilde{\alpha})$ to get the existence of quotients of Ω_X^1 on Z' and Z'' . Then use the minimality property of π_X . ■

Corollary 1: Suppose $\pi_X = \tilde{\pi}_X$ holds. Then $\bar{\psi}$ (resp. $\tilde{\psi}$) is equal to the blow-up of the ideal $\tilde{J}\bar{O}_{\bar{Z}}$ (resp. $\tilde{J}\tilde{O}_{\tilde{Z}}$), where $\tilde{J} = F^r(\Omega_X^1)$ denotes the jacobian ideal of X . In other words, the ideals $\tilde{J}\bar{O}_{\bar{Z}}$ and $F^\circ(\bar{Q})$ (resp. $\tilde{J}\tilde{O}_{\tilde{Z}}$ and $F^\circ(\tilde{Q})$) become invertible at the same time.

Proof: Consider

$$\pi_Z^* f^* \Omega_X^1 \xrightarrow{\bar{\alpha}} \bar{\Omega}_Z \rightarrow \bar{Q} \rightarrow 0.$$

By Prop. 3, $F^\circ(\bar{Q})$ invertible $\iff \text{Im}(\bar{\alpha})$ is locally free with rank $r = \dim X = \dim Z$.

This is equivalent to the existence of a r -quotient of $\pi_Z^* f^* \Omega_X^1$, therefore to $\tilde{J}\bar{O}_{\bar{Z}}$ being invertible, since $\pi_X = \tilde{\pi}_X$ holds. ■

Corollary 2: Suppose $\pi_X = \tilde{\pi}_X$ holds, and that Z is smooth. Then the blow-up $\psi : Z' \rightarrow Z$ of $I = F^0(\Omega_{Z/X}^1)$ is equal to the blow-up of $\tilde{J}O_Z$.

Earlier we showed (Thm. 1) that if X is Gorenstein, then I and JO_Z become invertible at the same time (here J is the ω -jacobian ideal). Hence, when X is a l.c.i., we have already proved Cor. 2 above.

Theorem 2: The Nash transformation $\pi : \bar{X} \rightarrow X$ of a Gorenstein variety X is equal to the blow-up of the ω -jacobian ideal J of X .

Proof: We want to show that, on any Y with $Y \rightarrow X$, J is invertible if and only if Ω_X^1 admits a r -quotient.

Assume $\Omega_X \rightarrow \Omega$ is a r -quotient. Then $\Omega_X^r \rightarrow \Lambda^r \Omega$ is a 1-quotient which is generically isomorphic to the torsion-free quotient $\Omega_X^r \rightarrow J \otimes \omega_X$, hence everywhere isomorphic – therefore J is invertible.

If J is invertible there is a 1-quotient $\Omega_X^r \rightarrow \Omega^r = J \otimes \omega_X$. Consider (as in the proof of Prop. 3, see [P], 1.1)

$$\begin{array}{ccc} \text{Grass}_r(\Omega_X^1) & \xrightarrow{i} & \mathbb{P}(\Omega_X^r) \\ \sigma' \uparrow & \searrow \sigma & \\ & X & \end{array}$$

where σ is the section defined by $\Omega_X^r \rightarrow \Omega^r$, and σ' is the obvious rational section defined on the smooth locus of X . Clearly σ restricts to σ' , so since the Plücker embedding i is closed, σ' extends to X . Hence there exists a r -quotient $\Omega_X^1 \rightarrow \Omega$ (with $\Lambda^r \Omega = \Omega^r$). ■

Remarks. 1) We are unable to give any examples of a Gorenstein, non l.c.i. variety X with $J \neq \tilde{J}$, because we do not know how to compute J in this case. Probably such varieties exist, however: Suppose $X \subseteq Y$, Y smooth of dimension n , and X defined by m equations. Then the jacobian ideal $\tilde{J}$ is generated by $\binom{n}{r} \binom{m}{n-r}$ elements (the $(n-r)$ -minors of the jacobian matrix), while the ω -jacobian ideal J is generated by $\binom{n}{r}$ elements (it is the image of $\Omega_{Y|X}^r \otimes \omega_X^{-1} \rightarrow \mathcal{O}_X$).

2) Since $\varphi_X : \Omega_X^r \rightarrow \omega_X$ is an isomorphism wherever X is smooth, there is an inclusion $\text{supp}(\mathcal{O}_X/J) \subseteq \text{sing}(X) = \text{supp}(\mathcal{O}_X/\tilde{J})$. It seems

reasonable to believe $J \subseteq \tilde{J}$ holds – this would imply the other inclusion and also that φ_X surjective implies X smooth (and hence φ_X an isomorphism).

Suppose φ_X is surjective, so that $J = \mathcal{O}_X$ (since $J \subseteq \mathcal{C}$, this implies X normal). Then there exists (Thm. 2) a r -quotient of Ω_X^1 . If $\text{char } k = 0$, this implies X is smooth ([L]). We do not know whether this is still true in positive characteristic. The usual positive characteristic counter-examples to " Ω_X^1 has a r -quotient $\Rightarrow X$ is smooth" do not apply as counter-examples to the above, because they do not have surjective φ_X (though the image of φ_X is invertible).

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