

Curves and surfaces

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What is algebraic geometry?
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Outline



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Motivation

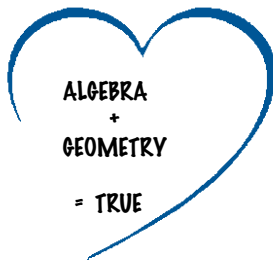
Algebraic geometry =
study of common solutions of a given set of polynomial equations.

The equations $3x + 5y = 1$ and $2x - 9y = 11$
have the common solution $x = 2$ and $y = -1$



What is so fascinating about solving equations?

The interplay between algebraic manipulations and geometric objects.



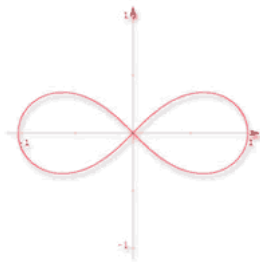
What is a curve?

Intuitively, a curve is a geometric object of dimension one:
a moving particle traces a curve.

For example, the lemniscate, given by

$$x = \frac{\sin t}{1+\cos^2 t} \text{ and } y = \frac{\sin t \cos t}{1+\cos^2 t}$$

Is this an algebraic curve?



Yes. As a point set, it is equal to the plane curve given by the equation

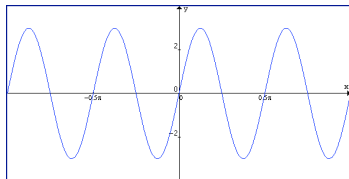
$$(x^2 + y^2)^2 = x^2 - y^2,$$

and it has the rational parameterization

$$x = \frac{u+u^3}{1+u^4} \text{ and } y = \frac{u-u^3}{1+u^4}.$$

The parameterized curve $x = t$, $y = \sin t$ is not algebraic.

It intersects the x -axis in infinitely many points.



Projective curves

Recall that an algebraic variety $V \subset \mathbb{P}^n(k)$ is the set of common zeros of finitely many homogeneous polynomials.

A *rational function* on V is the restriction to V of a rational function $\frac{f}{g}$ on $\mathbb{P}^n$, where f and g are polynomials of the same degree.

The function field $K = K(V)$ is the set of rational functions on V .

The variety V is a *curve* if the transcendence degree of K over k is 1.

Nonsingular projective curves with the same function field are isomorphic.

Any algebraic curve has a unique nonsingular projective model.

The arithmetic genus

The *Hilbert polynomial* $P_V(t)$ satisfies

$$P_V(m) = \dim(k[x_0, \dots, x_n]/I(V))_m, \text{ for } m \gg 0.$$

The *arithmetic genus* $p_a = p_a(V)$ is equal to

$$(-1)^{\dim V} (P_V(0) - 1).$$

If V is a curve, $P_V(t) = dt + 1 - p_a$, where d is the degree of V .

Example. Let $V \subset \mathbb{P}^3$ be the twisted cubic, given by
 $u \mapsto (1 : u : u^2 : u^3).$

Then the ideal $I(V)$ is generated by the 2-minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

We find $P_V(t) = 3t + 1$.



Plane curves

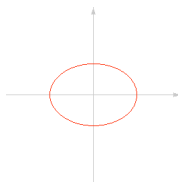
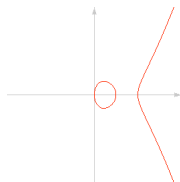
The simplest algebraic curves are the plane curves, $V \subset \mathbb{P}^2$, where $V = V(f)$.

The degree d of V is the degree of f .

$$P_V(t) = \binom{t+2}{2} - \binom{t-d+2}{2} = dt + 1 - \binom{d-1}{2}.$$

The arithmetic genus of a plane curve of degree d is $(d-1)(d-2)/2$.

Here are (real) curves of genus 1 and genus 0.



Topological genus

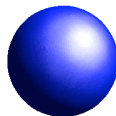
A complex algebraic curve can be viewed as a two-dimensional real manifold, a Riemann surface.

Topological genus: $g = \#$ holes in the Riemann surface.

Euler number: $e = \#$ vertices $- \#$ edges $+ \#$ faces in a triangulation.

Euler–Poincaré formula: $e = b_0 - b_1 + b_2 = 1 - 2g + 1 = 2 - 2g$.
(The b_i are the Betti numbers.)

The curve to the left has genus 1, the curve to the right has genus 0.



Hirzebruch–Riemann–Roch:

Topological genus = Arithmetic genus

Proof: Project $V \subset \mathbb{P}^n$ to a plane curve $\bar{V} \subset \mathbb{P}^2$ with δ double points.

Algebra. Compare the arithmetic genus of V with that of $\bar{V}$:

$$p_a(V) = p_a(\bar{V}) - \delta = \frac{(d-1)(d-2)}{2} - \delta.$$

Geometry. Compute the number d^* of tangents to $\bar{V}$ through a given point P :

$$d^* = d(d-1) - 2\delta$$

Project $\overline{V}$ from P to get a map $V \rightarrow \mathbb{P}^1$.

Topology. The map is a topological d -fold cover, with d^* ramification points. Compare the topological genus of V with that of $\mathbb{P}^1$:

Triangulate the sphere $\mathbb{P}^1$ and lift the triangulation to V :

$$e(V) = (d \# \{\text{vertices of } \mathbb{P}^1\} - d^*) - d \# \{\text{edges of } \mathbb{P}^1\} + d \# \{\text{faces of } \mathbb{P}^1\}$$

$$e(V) = d \cdot e(\mathbb{P}^1) - d^*$$

$$2 - 2g = d(2 - 2 \cdot 0) - d^* = 2d - d^*$$



Algebra:

$$p_a = \frac{1}{2}(d-1)(d-2) - \delta$$

Geometry:

$$d^* = d(d-1) - 2\delta$$

Topology:

$$2 - 2g = 2d - d^*$$

$$\text{Algebra} + \text{Geometry} + \text{Topology} \Rightarrow 1 - p_a = 1 - g$$



Classification of curves

All nonsingular projective curves with the same function field are isomorphic!

Any curve has a unique nonsingular projective model.

Classification problem: describe the “moduli space” of all curves of genus g .

The model of any rational curve is $\mathbb{P}^1$.

There is a one-dimensional family of curves of genus 1.

The moduli space of curves of genus g , $g \geq 2$, has dimension $3g - 3$.

Classification of curves in a fixed space

Classification of curves in a given projective space, of given degree and given arithmetic genus: Hilbert schemes.

The Hilbert scheme of degree d curves in $\mathbb{P}^2$ is the projective space $\mathbb{P}^{d(d+3)/2}$.

Indeed, a curve of degree d is given by a homogenous polynomial in three variables, of degree d . The space of such polynomials has dimension

$$\binom{d+2}{2} = \frac{d(d+3)}{2} + 1.$$

The Hilbert scheme of degree 3 curves of genus 0 in $\mathbb{P}^3$ is the union of a nonsingular 12-dimensional and a nonsingular 15-dimensional algebraic variety, intersecting along a nonsingular 11-dimensional variety.

Surfaces

A projective algebraic surface is a 2-dimensional variety $V \subset \mathbb{P}^n(k)$.

Dimension 2 means that the transcendence degree over k of the function field $K(V)$ is 2.

The arithmetic genus: $p_a = P_V(0) - 1$.

Topologically, a surface is a four-dimensional real variety.

The Euler number $e = \sum (-1)^i b_i$ is a topological invariant.

The genus of a “canonical curve” on V is another topological invariant. It is equal to $c_1^2 + 1$, where $c_1 = c_1(T_V)$.

The **Todd genus** of V is

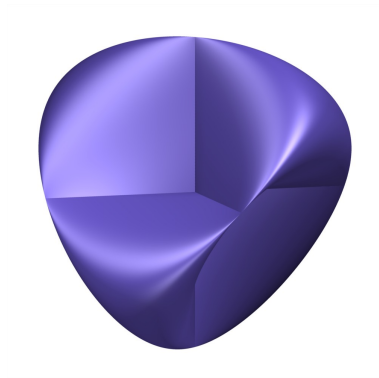
$$\tau = \frac{1}{12}(c_1^2 + e)$$



Noether's formula:

$$1 + p_a = \tau$$

Proof. Project V to a surface $\bar{V} \subset \mathbb{P}^3$ with generic singularities (a double curve and finitely many triple points and pinch points).



Algebra.

Compute the arithmetic genus of V in terms of the arithmetic genera of $\overline{V}$, the double curve, and the inverse image of the double curve. (Use the “conductor square.”)

Geometry. Compute the *rank* of V : the number of tangent planes to $\overline{V}$ at points in $H \cap \overline{V}$ through a given point P . Compute the *class* of V : the number of tangent planes through a given line L .

Topology. Compute the ramification curve of the projection $V \rightarrow \mathbb{P}^2$ in terms of topological invariants of V and $\mathbb{P}^2$.

Algebra + Geometry + Topology = OK

Classification of surfaces

Given a function field K of transcendence degree 2 over k , there are infinitely many surfaces V such that $K(V) = K$.

The reason for this is that one can blow up a point on a surface: replace the point by the set of all the tangent directions through that point. This set is a $\mathbb{P}^1$, and is called an *exceptional curve*.

A surface which contains no exceptional curves is called *minimal*.

Invariants: arithmetic genus is a birational invariant.

The Chern numbers: e and c_1^2 .

Kodaira dimension

Enriques classification



Families of curves on surfaces

Subvarieties of codimension 1 are called *divisors*.

Can “describe” a variety by looking at its divisors: “Tell me what your divisors are and I will tell you who you are.”

For example, a surface $V \subset \mathbb{P}^n$ that contains infinitely many lines is a ruled surface.

Can any surface V be “covered” by rational curves?



Rational curves on $\mathbb{P}^2$

A curve in $\mathbb{P}^2$ of degree d with only δ ordinary double points as singularities is rational (has genus 0) if and only if $\delta = (d-1)(d-2)/2$.

The set of plane curves of degree d can be identified with $\mathbb{P}^{d(d+3)/2}$.

The curves with $(d-1)(d+2)/2$ double points is a subvariety of dimension $d(d+3)/2 - (d-1)(d-2)/2 = 3d-1$.

Let N_d denote the number of plane rational curves of degree d passing through $3d-1$ given points.

$$N_1 = 1, N_2 = 1, N_3 = 12, N_4 = 620, \dots$$



Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

If we set $n_d := \frac{N_d}{(3d-1)!}$, then

$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} \frac{d_1 d_2 ((3d_1-2)(3d_2-2)(d+2) + 8(d-1))}{6(3d-1)(3d-2)(3d-3)}$$

Kontsevich's proof used a degeneration argument: mapping the one-dimensional family of curves through $3d-2$ points to $\mathbb{P}^1$ using the cross-ratio, and counting degrees above 0 and 1.

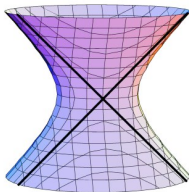


Quadric surfaces

A quadric surface $V \subset \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, embedded via the Segre map:

$$(s : t) \times (u : v) \mapsto (su : sv : tu : tv)$$

There are two families of lines on V . Through each point of V pass two lines, one from each family.



Lines on a cubic surface

A cubic surface $V \subset \mathbb{P}^3$ contains 27 lines!

Every cubic surface is isomorphic to $\mathbb{P}^2$ blown up in 6 points. The 6 exceptional curves are six of the 27 lines. The transforms of the 15 lines through two of the six points are also among the 27 lines. The last 6 lines are the transforms of the conics through 5 of the 6 points.

$$6 + \binom{6}{2} + \binom{5}{6} = 6 + 15 + 6 = 27$$



Sidestep: partitions

Let p denote the partition function: $p(n)$ is the number of ways of writing $n = n_1 + \dots + n_k$, where $n_1 \geq \dots \geq n_k \geq 1$

The generating function for p ,

$$\varphi(q) := \sum_{n \geq 0} p(n)q^n,$$

is equal to the formal power series

$$\prod_{m \geq 1} (1 - q^m)^{-1} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \dots$$



Rational curves on a quartic surface

Let $V \subset \mathbb{P}^3$ be a quartic surface.

Let $C \subset V$ be a curve, r an integer such that $C^2/2 = r - 1$, and set $N_r := \#\{r\text{-nodal rational curves in } |C|\}$

The generating function is $f(q) = \sum_r N_r q^r$.

One can show:

$$f(q) = \Pi(1 - q^m)^{-24} = \varphi(q)^{24}$$

where φ is the generating function of the partition function.

Proof (Bryan and Leung). Pass to symplectic geometry!

Degenerate the surface to an elliptic fibration over $\mathbb{P}^1$, with 24 nodal fibres. Reduce to combinatorics.

