

ENVELOPES OF FAMILIES OF LINEAR SPACES

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Foliation theory and algebraic geometry

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UiO : **University of Oslo**

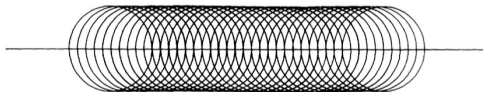
Envelope of plane curves

Given a family of plane curves $\{C_t\}$ in the plane, where C_t is given by $F(x, y, t) = 0$. The *envelope* of the family is the set of intersections of two “consecutive” curves:

$$\{(x, y) | F(x, y, t) = \frac{\partial F}{\partial t}(x, y, t) = 0\}$$

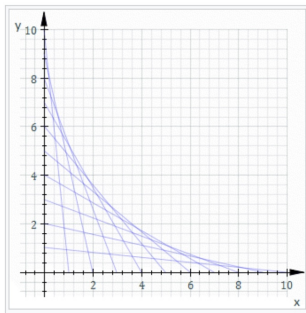
Example

$F(x, y, t) = (x - t)^2 + y^2 - 1$ gives the two lines $y = \pm 1$.



Envelopes of lines in the plane

Consider instead a family of *lines* in the plane. Its envelope is the locus of the intersection points of two “consecutive” lines.



For example, the envelope of the family of tangents to a plane curve is the curve itself.



Apollonius of Perga, 262–200 BC

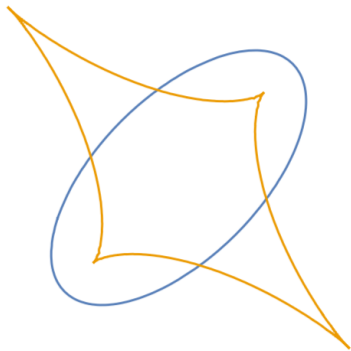
In his Treatise on Conic Sections, Book V, Apollonius constructs and studies *normals* to a conic – he finds these more interesting than the tangents:

In the fifth book I have laid down propositions relating to maximum and minimum straight lines. You must know that our predecessors and contemporaries have only superficially touched upon this investigation of the shortest lines, and have only proved what straight lines touch the sections . . .

He spends 16 pages finding the number – 4 – of normals through a given point (the Euclidean distance degree!) and considers the points “where two normals fall together” – the evolute!



An ellipse and its evolute



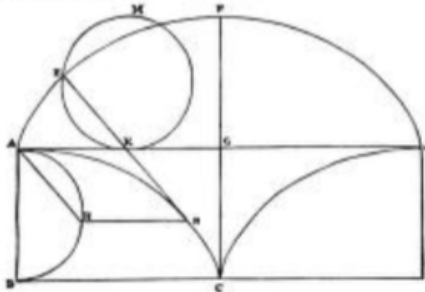
Christian Huygens: Horologium Oscillatorium, 1673

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CHRISTIANI HUGENII.

Repetita enim figura precedenti: cum possit totam semicycloidem ABC evolutam, filum occupet rectam CV , quæ dupla est AD , propterea quod axes cycloidum ABC , AET sunt æquales; appareat semicycloidem ipsam ABC , filo sibi circum applicito æqualem, duplicem esse sui axis AD , ac totam proinde cycloidem axis sui quadruplam.



“The evolute of a cycloid is a cycloid.”



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Families of linear spaces

Let $X \subset \mathbb{P}^n$ be a projective variety, $r = \dim X$, $\mathcal{F}$ a vector bundle of rank $n - r + 1$, and $\mathcal{O}_X^{n+1} \rightarrow \mathcal{F}$ a surjective map. Set $\pi : \mathbb{P}(\mathcal{F}) \rightarrow X$.

This gives a family of linear $(n - r)$ -spaces

$$\psi : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}^n \rightarrow \mathbb{P}^n.$$

The *envelope* $E_{\mathcal{F}} \subset \mathbb{P}^n$ is the branch locus of ψ :

$$[E_{\mathcal{F}}] = \psi(c_1(\psi^*T_{\mathbb{P}^n}) - c_1(T_{\mathbb{P}(\mathcal{F})})) \cap [\mathbb{P}^n].$$



Osculating hyperplanes of curves

Let $f: X \rightarrow \mathbb{P}^n$ be a curve.

The osculating hyperplanes to X are given by the Nash modification $\mathcal{P}^{n-1}$ of the $(n-1)$ th order principal parts bundle $\mathcal{P}_X^{n-1}(1)$. The envelope is the branch locus of

$$\psi: \mathbb{P}(\mathcal{P}^{n-1}) \rightarrow \mathbb{P}^n,$$

which is equal to the $(n-2)$ th osculating developable of X (union its hyperosculating hyperplanes).

Example

For $n=3$: The envelope is the tangent developable (union its hyperosculating planes).



Perpendicularity in projective space

Fix a hyperplane $H_\infty = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and a quadric $Q_\infty \subset H_\infty$.

Let L be a linear space of dimension r and let $(L \cap H_\infty)^\perp \subset H_\infty$ denote the *polar* with respect to Q_∞ . For $p \in L$, the space *perpendicular* to L at p is

$$\langle p, (L \cap H_\infty)^\perp \rangle.$$

Example

In $\mathbb{P}_{\mathbb{C}}^2$, take $H_\infty = \{z = 0\}$, and $Q_\infty = \{x^2 + y^2 = 0\} \subset H_\infty$. This gives the ordinary Euclidean perpendicularity in

$$\mathbb{R}^2 \subset \mathbb{C}^2 = \mathbb{P}_{\mathbb{C}}^2 \setminus H_\infty.$$



The Euclidean normal bundle

Let T_p be the tangent space to $X \subset \mathbb{P}^n$ at $p \in X$. The *normal* space to X at p is the linear $(n - r)$ -space

$$N_p := \langle p, (T_p \cap H_\infty)^\perp \rangle.$$

These normal spaces are the (projective) fibres of the *Euclidean normal bundle*

$$\mathcal{E} := \mathcal{K}^\vee \oplus \mathcal{O}_X(1),$$

where $\mathcal{K} := \text{Ker}(\mathcal{O}_X^{n+1} \rightarrow \mathcal{P}_X^1(1))$.

The map $\mathcal{O}_X^{n+1} \rightarrow \mathcal{E} = \mathcal{K}^\vee \oplus \mathcal{O}_X(1)$ is the sum of the maps

$$\mathcal{O}_X^{n+1} \rightarrow \mathcal{O}_X^n \cong (\mathcal{O}_X^n)^\vee \rightarrow \mathcal{K}^\vee \text{ and } \mathcal{O}_X^{n+1} \rightarrow \mathcal{O}_X(1).$$



Evolutes of hypersurfaces

Assume $X \subset \mathbb{P}^n$ is a hypersurface. The *evolute* E_X of X is the envelope of its family of normal lines

$$\psi : \mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n \rightarrow \mathbb{P}^n,$$

hence is equal to the branch locus of ψ .

If X has degree d and is nonsingular and in general position with respect to H_∞ and Q_∞ , then we recover Trifogli's formula:

$$\deg E_X = d(d-1)((n-1)(d-1)^{n-2} + 2 \sum_{i=0}^{n-2} (d-1)^i).$$

For $d = 2$, $\deg E_X = 6(n-1)$.



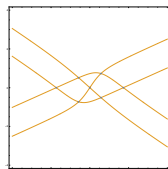
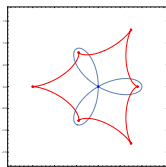
Evolutes of plane curves

For a plane curve $X \subset \mathbb{P}^2$ we may replace $\mathcal{O}_X^3 \rightarrow \mathcal{P}_X^1(1)$ by the Nash quotient $\mathcal{O}_{\tilde{X}}^3 \rightarrow \mathcal{P}^1$ on the normalization $\tilde{X}$.

If X has degree d and δ nodes and k_0 cusps, we have:

$$\deg E_X = 3d(d-1) - 6\delta - 8k_0$$

Real curves (with Cordian Riener and Boris Shapiro):



$M_X \subset (\mathbb{P}^2)^\vee$ is the curve of normals:

$$\deg M_X = d^2 - 4\delta - 5k_0.$$



Thom polynomials

Consider a family of linear $(n - r)$ -spaces

$$\psi : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}^n \rightarrow \mathbb{P}^n.$$

Its envelope $E_{\mathcal{F}}$ is the image of the singularity locus Σ^1 of ψ . The cuspidal locus $C_{\mathcal{F}}$ of $E_{\mathcal{F}}$ is given by $\Sigma^{1,1}$, and its cuspidal locus $\kappa_{\mathcal{F}}$ by $\Sigma^{1,1,1}$. Set $\bar{c}_i := c_i(\psi^*T_{\mathbb{P}^n} - T_{\mathbb{P}(\mathcal{F})})$. Then

$$\begin{aligned}[E_{\mathcal{F}}] &= \psi_* \bar{c}_1 \cap [\mathbb{P}^n] \\ [C_{\mathcal{F}}] &= \psi_* (\bar{c}_1^2 + \bar{c}_2) \cap [\mathbb{P}^n] \\ [\kappa_{\mathcal{F}}] &= \psi_* (\bar{c}_1^3 + 3\bar{c}_1\bar{c}_2 + 2\bar{c}_3) \cap [\mathbb{P}^n]\end{aligned}$$

The *degree* of these classes can be expressed in terms of the Chern classes of T_X , $\mathcal{O}_X(1)$, and $\mathcal{F}$.



The envelope of the normals

Set $E_X := E_{\mathcal{E}}$ and $C_X := C_{\mathcal{E}}$, where $\mathcal{E} = \mathcal{K}^\vee \oplus \mathcal{O}_X(1)$ is the normal bundle.

The class of the envelope $E_{\mathcal{E}}$ is

$$[E_X] = \psi_* (\pi^* c_1(\Omega_X^1) + \pi^* c_1(\mathcal{E}) + r c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))) \cap [\mathbb{P}^n],$$

and its degree is

$$\deg E_X = (c_1(\Omega_X^1) s_{r-1}(\mathcal{E}) + c_1(\mathcal{E}) s_{r-1}(\mathcal{E}) + r s_r(\mathcal{E})) \cap [X].$$



The cuspidal edge of the envelope

The “cuspidal edge” $C_X \subset \mathbb{P}^n$ of the envelope E_X has class

$$\begin{aligned}[C_X] = & \psi_* \left(\pi^* (2c_1(\Omega_X^1)^2 + 3c_1(\Omega_X^1)c_1(\mathcal{E}) + 2c_1(\mathcal{E})^2 - c_2(\Omega_X^1) - c_2(\mathcal{E})) \right. \\ & + (3r-1)\pi^*c_1(\mathcal{E})c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) + 3r\pi^*c_1(\Omega_X^1)c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \\ & \left. + \frac{3r^2-r}{2}c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^2 \right) \cap [\mathbb{P}^n]\end{aligned}$$

and its degree is

$$\begin{aligned}\deg C_X = & ((2c_1(\Omega_X^1)^2 + 3c_1(\Omega_X^1)c_1(\mathcal{E}) \\ & + 2c_1(\mathcal{E})^2 - c_2(\Omega_X^1) - c_2(\mathcal{E}))s_{r-2}(\mathcal{E}) + ((3r-1)c_1(\mathcal{E}) \\ & + 3rc_1(\Omega_X^1))s_{r-1}(\mathcal{E}) + \frac{3r^2-r}{2}s_r(\mathcal{E})) \cap [X].\end{aligned}$$



Evolute of space curves

Let $X \subset \mathbb{P}^3$ be a curve. The envelope of the family of its Euclidean normal planes was denoted the *polar developable* by Monge (1771) and the *polar surface* by Darboux (1887).

The *evolute* of X is the cuspidal edge C_X of the developable surface E_X , so that E_X is the tangent developable of C_X . The evolute is the locus of the *centers of spherical curvature*.

The osculating planes to the evolute are the normal planes to X (Blaschke and Leichtweiss).

Salmon (1862) considered curves in $\mathbb{P}^3$ and found formulas for the degrees of the polar developable and the evolute.



Numerical formulas for space curves

Let $X \subset \mathbb{P}^3$ be a curve of degree d , genus g , and with k_0 cusps. Evaluating the Thom polynomials with $\mathcal{F} = \mathcal{E}$ and taking degrees, gives

$$\deg E_X = 6(d + g - 1) - 2k_0$$

$$\deg C_X = 3(3d + 4g - 4 - k_0)$$

$$\deg \kappa_X = 4(3d + 5g - 5 - k_0).$$

Salmon found that the degree of the polar developable E_X is equal to $3d + d^*$, where d^* is the *class* of X (the degree of its strict dual). He found the degree of the evolute C_X to be $5d + 2k_1 + k_2$, where k_1 is the number of flexes and k_2 is the number of hyperosculating planes to X . His formulas agree with the ones above.



The “développées” of a space curve



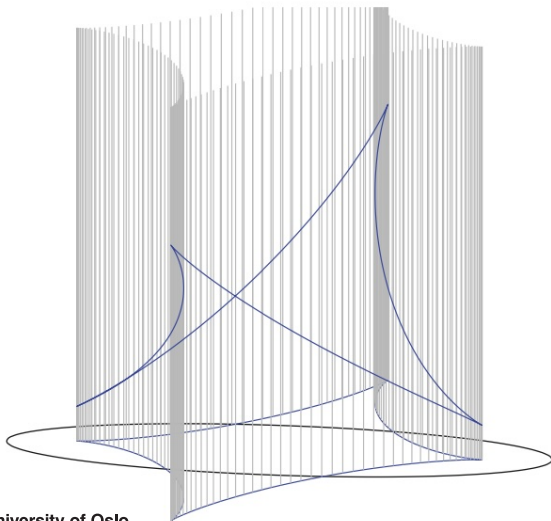
Passing from a plane curve to a space curve, Monge replaced the centers of curvature by the “axes polaires”: the lines through the center of an osculating circle perpendicular to the osculating plane.

These lines generate a developable surface – the polar developable. He showed the existence of infinitely many curves – “développées” – on this surface, each having the property that their tangents are normals to the given curve.

The polar developable is thus *foliated* by the “développées”.



Foliation of the polar developable



Evolute of surfaces

Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d . The Thom polynomials give

$$[E_X] = \psi_* (\pi^* c_1(\Omega_X^1) + \pi^* c_1(\mathcal{E}) + 2c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))) \cap [\mathbb{P}^3]$$

$$[C_X] = \psi_* (2\pi^* c_1(\Omega_X^1)^2 + \dots) \cap [\mathbb{P}^3]$$

$$[\kappa_X] = \psi_* (22\pi^* c_1(\Omega_X^1)^2 c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) + \dots) \cap [\mathbb{P}^3].$$

In terms of the degree d of X this gives

$$\deg E_X = 2d(d-1)(2d-1)$$

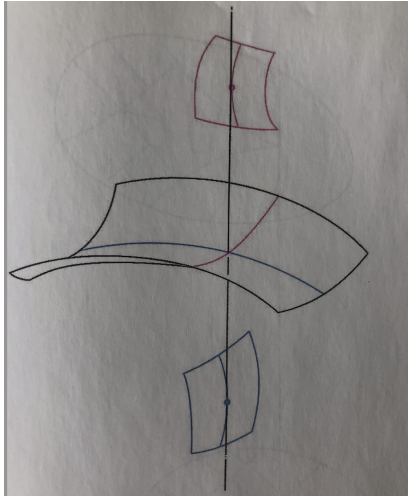
$$\deg C_X = 2d(d-1)(11d-16)$$

$$\deg \kappa_X = 4d(30d^2 - 97d + 78).$$

The first formula agrees with Salmon. The second agrees with Salmon for $d=2$.



Evolute of a real surface



Umbilics

The local study of the geometry of the evolute a real surface is well known. Since the evolute is the locus of the focal points, i.e., the centers of spherical curvature, each normal to the surface will intersect the evolute in two points. When these two points come together, the curvature of the surface is the same in all directions. These points are the *umbilical points* of the surface.

According to Porteous, the umbilical points are the D_4 singularities of the real map $\psi_{\mathbb{R}} : X \times \mathbb{R} \rightarrow \mathbb{R}^3$.



Lagrange maps

The map $\psi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^3$ is supposedly Lagrangian.

According to Mikosz–Pragacz–Weber the Thom polynomial for D_4 for a Lagrangian map is

$$\tilde{Q}_{2,1} = a_1 a_2 - 2a_3,$$

for Chern classes a_i .

Question: Which bundles are the a_i the Chern classes of?

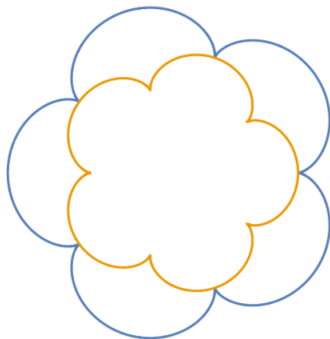
We know what to expect for $X \subset \mathbb{P}^3$ a surface of degree d , namely Salmon's 1882 formula:

$$\#D_4 = 2d(5d^2 - 14d + 11)$$



Ranunculus

The ranunculoid $x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + \dots = 0$



HAPPY BIRTHDAY, FERNANDO!



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