

Inflectional loci of scrolls

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Abstract Let $X \subset \mathbb{P}^N$ be a scroll over a smooth curve C and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_X$ denote the hyperplane bundle. The special geometry of X implies that certain sheaves related to the principal part bundles of $\mathcal{L}$ are locally free. The inflectional loci of X can be expressed in terms of these sheaves, leading to explicit formulas for the cohomology classes of the loci. The formulas imply that the only uninflected scrolls are the balanced rational normal scrolls.

Keywords Scroll · Inflectional locus · Principal parts bundle

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1 Introduction

Let $X \subset \mathbb{P}^N$ be a smooth, nondegenerate complex projective variety of dimension n , let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_X$ be the hyperplane bundle on X , and let V be the vector subspace of $H^0(X, \mathcal{L})$ giving rise to the embedding. Set $V_X = V \otimes \mathcal{O}_X$. For every integer $k \geq 0$, let $\mathcal{P}_X^k(\mathcal{L})$ denote the k th principal part bundle (or k th jet bundle) of $\mathcal{L}$, and let $j_k : V_X \rightarrow \mathcal{P}_X^k(\mathcal{L})$ be the sheaf homomorphism sending a section $s \in V$ to its k th jet $j_{k,x}(s)$ evaluated at x , for every $x \in X$. Recall that $j_{k,x}(s)$ is represented in local coordinates by the Taylor expansion of s at x , truncated after the order k . The homomorphisms $j_{k,x}$ allow us to define the osculating

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spaces to X at x as follows. The k th osculating space to X at x is $\text{Osc}_x^k(X) := \mathbb{P}(\text{Im}(j_{k,x}))$. Identifying $\mathbb{P}^N$ with $\mathbb{P}(V)$ (the set of codimension 1 vector subspaces of V) we see that $\text{Osc}_x^k(X)$ is a linear subspace of $\mathbb{P}^N$. Since the rank of the k th jet bundle is $\binom{k+n}{n}$, we have $\dim \text{Osc}_x^k(X) \leq \binom{k+n}{n} - 1$.

In the case that X is a scroll over a smooth curve C , for $k \geq 2$ this inequality is strict, for every point $x \in X$. Indeed, let $\pi : X \rightarrow C$ denote the structure map. Then, around every point $x \in X$, there are local coordinates $u, v_2, \dots, v_n$ such that u is mapped isomorphically by π to a local coordinate on C , while $v_2, \dots, v_n$ are local coordinates on the fibre through x , and every $s \in V$ can be written locally as $s = a(u) + \sum_{j=2}^n v_j b_j(u)$, where $a, b_2, \dots, b_n$ are regular functions of u . Hence, starting with $h = 2$, all derivatives of order h of s vanish, except perhaps $s_{u,u}, s_{u,v_j}, j = 2, \dots, n$, for $h = 2$, and $s_{u,\dots,u,u}, s_{u,\dots,u,v_j}, j = 2, \dots, n$, for $3 \leq h \leq k$. This implies that the rank of $j_{k,x}$ cannot exceed $kn + 1$, hence $\dim \text{Osc}_x^k(X) \leq kn$, at every point $x \in X$. If equality holds at every point, we say that X is *uninflected*.

Let $X \subset \mathbb{P}^N$ be a scroll over C , and assume $kn \leq N$ and that the generic rank of j_k is $kn + 1$. A main result of this paper is that the dual $\mathcal{Q}_k^\vee$ of the sheaf $\mathcal{Q}_k := \text{Coker } j_k$, and the quotient sheaf $\mathcal{E}_k := \mathcal{P}_X^k(\mathcal{L})^\vee / \mathcal{Q}_k^\vee$, are locally free (Theorem 1). The k th inflectional locus Φ_k of X is the set of points $x \in X$ such that $\text{rk } j_{k,x} < kn + 1$. Now let k be the largest integer such that $kn \leq N$ and assume that Φ_k has the expected codimension $N + 1 - kn$. Then Φ_k has a natural structure as a Cohen–Macaulay scheme, and its cohomology class is a Segre class of $\mathcal{E}_k$ (Theorem 2). The Segre class can be computed explicitly (Theorem 3) in terms of Chern classes on the curve C and the hyperplane bundle class on X . In particular, when there are only finitely many inflection points, their weighted number is equal to

$$\deg \Phi_k = (k + 1)(d + nk(g - 1)).$$

This formula, and the corresponding formulas in the cases that the expected dimension of Φ_k is positive, allow us to conclude that the only uninflected scrolls are the balanced rational normal scrolls.

In a previous version of this work, we conjectured the degree formula in Corollary 1, but could only show it in special cases. We would like to express our deep gratitude to the referee, who showed us how Theorem 3, and thus Corollaries 1 and 2, follow from our exact sequences.

2 The theorems

We want to investigate the inflectional loci of scrolls, i.e., the loci where the rank of $j_{k,x}$ is smaller than expected. Suppose that at some point $x \in X$ (hence at a general point), we have $\text{rk}(j_{k,x}) = kn + 1$, and let $\mathcal{Q}_k$ denote the cokernel of the map j_k . Then we have an exact sequence of sheaves

$$0 \rightarrow \mathcal{P}_k \rightarrow \mathcal{P}_X^k(\mathcal{L}) \rightarrow \mathcal{Q}_k \rightarrow 0, \quad (1)$$

where $\mathcal{P}_k := \text{Im}(j_k)$.

The k th inflectional locus Φ_k of X is the locus where the map $j_k : V_X \rightarrow \mathcal{P}_X^k(\mathcal{L})$ does not have the maximal rank $kn + 1$, hence where the sheaf $\mathcal{P}_k$ is not a vector bundle. It is also the locus where the dual map $j_k^\vee : \mathcal{P}_X^k(\mathcal{L})^\vee \rightarrow V_X^\vee$ does not have maximal rank. Clearly, for $k \geq k'$ we have $\Phi_k \supseteq \Phi_{k'}$, because of the surjections $\mathcal{P}_k \rightarrow \mathcal{P}_{k'}$.

For simplicity, we shall denote by Ω_Y the cotangent sheaf Ω_Y^1 of a variety Y . We shall write T_Y for the tangent sheaf, equal to the dual sheaf $\Omega_Y^\vee$. If $\mathcal{N}$ is a line bundle, we write $\mathcal{N}^i$ instead of $\mathcal{N}^{\otimes i}$ and $\mathcal{N}^{-1}$ for $\mathcal{N}^\vee$.

Theorem 1 *Let $X \subset \mathbb{P}^N$ be a n -dimensional scroll over a smooth curve C , with hyperplane bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. For all $k \geq 1$ such that $kn \leq N$, assume the generic rank of j_k is $kn + 1$, and set $\mathcal{Q}_k = \text{Coker } j_k$ as above. Then, for such k ,*

- (i) $\mathcal{Q}_k^\vee$ is a locally free sheaf of rank $\binom{n+k}{n} - (kn + 1)$,
- (ii) there exist locally free sheaves $\mathcal{M}_k$ of rank $\binom{n+k-1}{n-1} - n$ and exact sequences

$$0 \rightarrow \mathcal{Q}_{k-1}^\vee \rightarrow \mathcal{Q}_k^\vee \rightarrow \mathcal{M}_k^\vee \rightarrow 0, \quad (2)$$

and

$$0 \rightarrow \pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L} \rightarrow S^k \Omega_X \otimes \mathcal{L} \rightarrow \mathcal{M}_k \rightarrow 0, \quad (3)$$

- (iii) the quotient sheaves $\mathcal{E}_k := \mathcal{P}_X^k(\mathcal{L})^\vee / \mathcal{Q}_k^\vee$ are locally free, of rank $kn + 1$, and there exist exact sequences

$$0 \rightarrow \mathcal{E}_{k-1} \rightarrow \mathcal{E}_k \rightarrow \pi^* T_C^{k-1} \otimes T_X \otimes \mathcal{L}^{-1} \rightarrow 0. \quad (4)$$

Proof There is an obvious inclusion

$$\pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L} \subset S^{k-1} \Omega_X \otimes \Omega_X \otimes \mathcal{L}.$$

By restricting the natural homomorphism $S^{k-1} \Omega_X \otimes \Omega_X \otimes \mathcal{L} \rightarrow S^k \Omega_X \otimes \mathcal{L}$ we get an injective and locally split homomorphism

$$\iota_k : \pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L} \rightarrow S^k \Omega_X \otimes \mathcal{L}. \quad (5)$$

To see this, let $x \in X$ and let u denote a local coordinate on the base curve C around $\pi(x)$ and $v_2, \dots, v_n$ local coordinates in the fibre of X through x , around x . Then $u, v_2, \dots, v_n$ are local coordinates on X around x . So, letting $A := \mathcal{O}_{X,x}$, we have the following isomorphisms:

$$(\pi^* \Omega_C)_x \cong A du, \quad \Omega_{X,x} \cong A du \oplus \bigoplus_{i=2}^n A dv_i,$$

and

$$(S^{k-1} \Omega_X)_x \cong \bigoplus_i A du^{i_1} dv_2^{i_2} \dots dv_n^{i_n}, \quad \text{with } \sum_j i_j = k-1.$$

The map $\iota_{k,x}$ on the stalks is clearly injective, since it acts on the differentials in the following way:

$$du^{k-1} \otimes du \mapsto du^k \quad \text{and} \quad du^{k-1} \otimes dv_i \mapsto du^{k-1} dv_i.$$

The same is true for the map $\iota_k(x)$ on the fibres. Hence ι_k is locally split. Set $\mathcal{M}_k := \text{Coker } \iota_k$. It follows that $\mathcal{M}_k$ is locally free, and we get the second exact sequence (3).

Let $\mathcal{E}_k := \mathcal{P}_X^k(\mathcal{L})^\vee / \mathcal{Q}_k^\vee$ denote the quotient sheaf, and consider the following diagram, all of whose horizontal sequences are exact:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{Q}_{k-1}^\vee & \rightarrow & \mathcal{P}_X^{k-1}(\mathcal{L})^\vee & \rightarrow & \mathcal{E}_{k-1} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{Q}_k^\vee & \rightarrow & \mathcal{P}_X^k(\mathcal{L})^\vee & \rightarrow & \mathcal{E}_k & \rightarrow 0 \\ & & & \downarrow & & & \\ 0 \rightarrow & \mathcal{M}_k^\vee & \rightarrow & (S^k \Omega_X \otimes \mathcal{L})^\vee & \xrightarrow{\iota_k^\vee} & (\pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L})^\vee & \rightarrow 0 \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

We will complete it to a commutative diagram in which all vertical sequences are also exact.

Let us first consider the composition α of the map ι_k (5) with the maps $S^k \Omega_X \otimes \mathcal{L} \rightarrow \mathcal{P}_X^k(\mathcal{L})$ and $\mathcal{P}_X^k(\mathcal{L}) \rightarrow \mathcal{Q}_k$,

$$\alpha: \pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L} \rightarrow \mathcal{Q}_k.$$

We want to show that this map is generically 0. Let $x \in X$ be a general point and use local coordinates as above. Then $(\mathcal{P}_X^k(\mathcal{L}))_x \cong \mathcal{P}_X^{k-1}(\mathcal{L})_x \oplus (S^k \Omega_X \otimes \mathcal{L})_x$ and α_x sends the generators $du^{k-1} \otimes du$ and $du^{k-1} \otimes dv_i$ to du^k and $du^{k-1} dv_i$ in the second summand. But these elements are in $\text{Im}(j_{k,x}) = (\mathcal{P}_k)_x$, hence they go to 0 in $(\mathcal{Q}_k)_x$, by (1).

Since α is generically 0, so is its dual,

$$\alpha^\vee: \mathcal{Q}_k^\vee \rightarrow (\pi^* \Omega_C^{k-1} \otimes \Omega_X \otimes \mathcal{L})^\vee = \pi^* T_C^{k-1} \otimes T_X \otimes \mathcal{L}^{-1}.$$

Since the target sheaf is locally free, it has no torsion, hence $\alpha^\vee$ is everywhere zero. Therefore we get induced maps ψ and β making the diagram commute:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{Q}_{k-1}^\vee & \rightarrow & \mathcal{P}_X^{k-1}(\mathcal{L})^\vee & \rightarrow & \mathcal{E}_{k-1} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \mathcal{Q}_k^\vee & \rightarrow & \mathcal{P}_X^k(\mathcal{L})^\vee & \rightarrow & \mathcal{E}_k & \rightarrow 0 \\ & \psi \downarrow & & \downarrow & & \beta \downarrow & \\ 0 \rightarrow & \mathcal{M}_k^\vee & \rightarrow & (S^k \Omega_X \otimes \mathcal{L})^\vee & \rightarrow & \pi^* T_C^{k-1} \otimes T_X \otimes \mathcal{L}^{-1} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}.$$

We show that: (a) ψ is surjective, and (b) $\text{Ker } \psi = \mathcal{Q}_{k-1}^\vee$. First note that $\mathcal{E}_{k-1} \rightarrow \mathcal{E}_k$ is injective, since both sheaves are subsheaves of $V_X^\vee$. Then fact (a) follows from the snake lemma: we have the exact sequence $\mathcal{P}_X^{k-1}(\mathcal{L})^\vee \rightarrow \mathcal{E}_{k-1} \rightarrow \text{Coker } \psi \rightarrow 0 \rightarrow 0$; the first map is surjective by the diagram, hence the second is zero; thus the third is injective, but since its image is zero, we conclude that $\text{Coker } \psi$ itself is zero. As for (b), clearly $\mathcal{Q}_{k-1}^\vee \subseteq \text{Ker } \psi$, by an easy diagram chase. The converse also follows by a diagram chase: Let $\xi \in (\text{Ker } \psi)_x$ for some $x \in X$. Since ξ goes to zero in $(\mathcal{M}_k^\vee)_x$, then its image, say y in $(\mathcal{P}_X^k(\mathcal{L})^\vee)_x$, goes to zero via both the horizontal and the vertical maps. Then y comes from an element $z \in (\mathcal{P}_X^{k-1}(\mathcal{L})^\vee)_x$, which must go to zero by the horizontal map, due to the commutativity of the right-upper square. Thus there exists a $w \in (\mathcal{Q}_{k-1}^\vee)_x$ mapping to z . Since the map $\xi \mapsto y$ is injective, we thus conclude that ξ is the image of w : this shows that $(\text{Ker } \psi)_x \subseteq (\mathcal{Q}_{k-1}^\vee)_x$.

To conclude, for every $k \geq 2$ the first vertical sequence gives the exact sequence (2). In particular, since $\mathcal{M}_k^\vee$ is locally free and $\mathcal{Q}_2^\vee = \mathcal{M}_2^\vee$ because $\mathcal{Q}_1 = 0$, this shows, by induction, that $\mathcal{Q}_k^\vee$ is locally free for every $k \geq 2$. The assertion on the rank of $\mathcal{Q}_k^\vee$ follows from the fact that it equals the generic rank of $\mathcal{Q}_k$, which is given by $\text{rk } \mathcal{P}_X^k(\mathcal{L}) - \text{rk } j_{k,x}$ for a general $x \in X$. This proves (i) and (ii).

To prove (iii), observe that since $\mathcal{Q}_1 = 0$, $\mathcal{E}_1 \cong \mathcal{P}_X^1(\mathcal{L})^\vee$ is locally free. Using the exactness of the rightmost vertical sequence in the commutative diagram with $k = 2$, we conclude that $\mathcal{E}_2$ must be locally free, since an extension of two locally free sheaves is locally free. Hence we deduce, recursively, that all $\mathcal{E}_k$ are locally free. $\square$

Theorem 2 *Let $X \subset \mathbb{P}^N$ be a n -dimensional scroll over a smooth curve C , with hyperplane bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Let k be the largest integer such that $kn \leq N$ and assume that the generic rank of j_k is $kn + 1$. If the k th inflectional locus Φ_k of X has codimension*

$\ell := N + 1 - kn$ or is empty, then it has a natural structure as a Cohen–Macaulay scheme, and its class is equal to the ℓ th term of the Segre class of $\mathcal{E}_k$,

$$[\Phi_k] = [c(\mathcal{E}_k)^{-1}]_\ell,$$

where $\mathcal{E}_k = \mathcal{P}_X^k(\mathcal{L})^\vee / \mathcal{Q}_k^\vee$.

Proof Note that the assumptions imply that the generic rank of $j_{k'}$ is $k'n + 1$ for $k' \leq k$, so that the assumptions of Theorem 1 are satisfied.

It follows from the definition that the k th inflectional locus is equal to the degeneracy locus of the map of locally free sheaves

$$0 \rightarrow \mathcal{E}_k = \mathcal{P}_X^k(\mathcal{L})^\vee / \mathcal{Q}_k^\vee \rightarrow V_X^\vee,$$

hence it has a natural structure as a Cohen–Macaulay scheme when it has the expected codimension $N + 1 - kn$. By Porteous' formula [1, Example 14.4.1, p. 255], the class of the k th inflectional locus is equal to the class

$$[c(V_X^\vee) c(\mathcal{E}_k)^{-1}]_\ell = [c(\mathcal{E}_k)^{-1}]_\ell$$

where $\ell = N + 1 - kn$. □

Note that, since k is the largest integer such that $kn \leq N$, we also have $N \leq (k + 1)n - 1$. This implies that

$$1 \leq \ell \leq n.$$

Theorem 3 *The j th term of $c(\mathcal{E}_k)^{-1}$, for $j = 1, \dots, n$, is equal to*

$$L^j + k(d + (n(k - 1) + 2j)(g - 1))L^{j-1}F,$$

where $L = c_1(\mathcal{L})$ denotes the class of a hyperplane section of X , F is the class of a fiber of the map $\pi : X \rightarrow C$, d is the degree of X , and g is the genus of C .

Proof We use the exact sequences (4) for $i = 1, \dots, k$ to get

$$c(\mathcal{E}_k) = c\left(\pi^*T_C^{k-1} \otimes T_X \otimes \mathcal{L}^{-1}\right) c(\mathcal{E}_{k-1}) = \prod_{i=0}^{k-1} c\left(\pi^*T_C^i \otimes T_X \otimes \mathcal{L}^{-1}\right) c(\mathcal{L}^{-1}).$$

The standard exact sequence

$$0 \rightarrow \pi^*\Omega_C \rightarrow \Omega_X \rightarrow \Omega_{X/C} \rightarrow 0$$

gives, by dualizing and tensoring with $\pi^*T_C^i$ and $\mathcal{L}^{-1}$, exact sequences

$$0 \rightarrow \pi^*T_C^i \otimes T_{X/C} \otimes \mathcal{L}^{-1} \rightarrow \pi^*T_C^i \otimes T_X \otimes \mathcal{L}^{-1} \rightarrow \pi^*T_C^{i+1} \otimes \mathcal{L}^{-1} \rightarrow 0.$$

The sheaf $\mathcal{F} := \pi_*\mathcal{L}$ is locally free, with rank n , and we have the standard exact sequence

$$0 \rightarrow \Omega_{X/C} \otimes \mathcal{L} \rightarrow \pi^*\mathcal{F} \rightarrow \mathcal{L} \rightarrow 0.$$

Dualizing and tensoring with $\pi^*T_C^i$, we obtain sequences

$$0 \rightarrow \pi^*T_C^i \otimes \mathcal{L}^{-1} \rightarrow \pi^*(T_C^i \otimes \mathcal{F}^\vee) \rightarrow \pi^*T_C^i \otimes T_{X/C} \otimes \mathcal{L}^{-1} \rightarrow 0.$$

Hence

$$c\left(\pi^*T_C^i \otimes T_X \otimes \mathcal{L}^{-1}\right) = c\left(\pi^*T_C^{i+1} \otimes \mathcal{L}^{-1}\right) c\left(\pi^*\left(T_C^i \otimes \mathcal{F}^\vee\right)\right) c\left(\pi^*T_C^i \otimes \mathcal{L}^{-1}\right)^{-1},$$

which gives, because of cancellations in the product, the expression

$$c(\mathcal{E}_k) = \prod_{i=0}^{k-1} \pi^* c(\mathcal{F}^\vee \otimes T_C^i) c(\pi^* T_C^k \otimes \mathcal{L}^{-1}).$$

The last Chern class in this product is the Chern class of a line bundle, so that $c(\pi^* T_C^k \otimes \mathcal{L}^{-1}) = 1 + k\pi^* c_1(T_C) - c_1(\mathcal{L}) = 1 - k(2g-2)F - L$, since $\pi^* c_1(T_C) = -\pi^* c_1(\mathcal{O}_C) = -(2g-2)F$. Since $\mathcal{F}^\vee \otimes T_C^i$ is a bundle on the curve C , its Chern class is just $1 + c_1(\mathcal{F}^\vee \otimes T_C^i) = 1 - c_1(\pi_* \mathcal{L}) + nic_1(T_C)$, and its inverse Chern class is $1 + c_1(\pi_* \mathcal{L}) + nic_1(\mathcal{O}_C)$. Because $\pi^* c_1(\pi_* \mathcal{L}) = dF$, we get $\pi^*(1 + c_1(\pi_* \mathcal{L}) + nic_1(\mathcal{O}_C)) = 1 + (d + 2in(g-1))F$, and thus

$$\begin{aligned} c(\mathcal{E}_k)^{-1} &= \prod_{i=0}^{k-1} (1 + (d + 2in(g-1))F)(1 - 2k(g-1)F - L)^{-1} \\ &= (1 + aF)(1 - bF - L)^{-1}, \end{aligned}$$

where we set $a := k(d + n(k-1)(g-1))$ and $b := 2k(g-1)$ and used the fact that $F^i = 0$ for $i > 1$. The j th term of this class is equal to

$$(bF + L)^j + aF(bF + L)^{j-1} = L^j + j b L^{j-1} F + a L^{j-1} F = L^j + (a + j b) L^{j-1} F,$$

which is what we wanted to prove. $\square$

Corollary 1 *Under the assumptions of Theorem 2, the class of the inflectional locus of X is equal to*

$$[\Phi_k] = L^{N+1-kn} + k(d + (n(k-1) + 2(N+1-kn))(g-1))L^{N-kn}F,$$

and its degree is equal to

$$\deg \Phi_k = (k+1)d + k(2(N+1) - (k+1)n)(g-1).$$

In particular, if $N = (k+1)n - 1$, then Φ_k is 0-dimensional, and its degree is equal to

$$\deg \Phi_k = (k+1)(d + nk(g-1)). \quad (6)$$

Corollary 2 *Let ℓ be an integer, $1 \leq \ell \leq n$. The only uninflected scroll $X \subset \mathbb{P}^{kn+\ell-1}$ of dimension n is the balanced rational normal scroll of degree kn in $\mathbb{P}^{(k+1)n-1}$.*

Proof If X is uninflected, then the assumptions of Theorem 2 are satisfied, since $\Phi_k = \emptyset$. By Corollary 1 the class

$$L^\ell + k(d + (n(k-1) + 2\ell)(g-1))L^{\ell-1}F$$

is 0. If $\ell < n$, we can intersect this class with $L^{n-\ell-1}F$ and obtain $L^{n-1}F = 0$, using the fact that $F^2 = 0$. But $L^{n-1}F = 1$ is the degree of the linear space F , thus we get a contradiction.

Hence we may assume $\ell = n$, so that $N = (k+1)n - 1$. Setting $\deg \Phi_k = 0$ in (6) implies $g = 0$ and $d = kn$. Therefore, X is a smooth, nondegenerate rational scroll of minimal degree, hence it is linearly normal. The explicit description of the maps j_k given in [7, p. 1050] shows that the only uninflected rational normal n -dimensional scrolls in $\mathbb{P}^{(k+1)n-1}$ are the balanced ones, i. e., the ones given by $X = \mathbb{P}(\pi_* \mathcal{L}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ on $C = \mathbb{P}^1$. $\square$

3 Examples

In this section we give geometric descriptions and details about inflectional loci in some particular, but relevant, cases.

In the situation of Theorem 2, when $n = 1$, we have $X = C$ and $N = k$, so that $X \subset \mathbb{P}^k$ is a nondegenerate curve. Corollary 1 gives the formula

$$\deg \Phi_k = (k+1)(d+k(g-1)) \quad (7)$$

for the total (weighted) number of inflection points. This classical formula, valid also when X is singular, goes back to Veronese and has been reproved many times (see, e.g. [6, Theorem 3.2]).

When $n = 2$ and $k = 2$, we have a surface scroll $X \subset \mathbb{P}^5$. In this case, Corollaries 1 and 2 were shown by Shifrin [9, Proposition 4.3, Theorem 4.3, p. 247]; in the more general case of a surface scroll $X \subset \mathbb{P}^{2k+1}$, Corollary 2 was shown by Pieni and Tai [8, p. 221].

Note that for $n = 2$ and $X \subset \mathbb{P}^{2k+2}$, we are outside the range of Corollary 2. In this case there are several examples of uninflected scrolls: for example, the scroll with $g = 1$, defined by an indecomposable rank 2 vector bundle of degree $2k+3$ [3, Theorem A], and scrolls with $g = 0$, both normal (the semibalanced scroll $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k+1))$ [8]) and non-normal [4, Theorem 3.4].

In the next example we consider at the same time the following cases:

- (i) $g = 0, k \geq 2$;
- (ii) $g = 1, k \geq 3$.

For $i = 1, \dots, n$, consider line bundles $\mathcal{L}_i \in \text{Pic}(C)$ such that $\deg \mathcal{L}_i = g + k - 1$ for $i = 1, \dots, n-1$ and $\deg \mathcal{L}_n = g + k$. Take $X = \mathbb{P}(\mathcal{F})$, where $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{L}_i$, and let $\mathcal{L}$ be the tautological line bundle on X . Clearly $\mathcal{L}$ is very ample; moreover, $h^0(\mathcal{L}) = h^0(\mathcal{F}) = (n-1)k + k + 1 = nk + 1$. So, X embedded by $|\mathcal{L}|$ is a linearly normal scroll in $\mathbb{P}^{kn}$ of degree $d = \deg \mathcal{F} = n(g+k) - (n-1)$ in both cases (i) and (ii). Let C_i be the generating section of the scroll X corresponding to the i th summand $\mathcal{L}_i$ of $\mathcal{F}$. Note that C_i is embedded in $\mathbb{P}^{k-1}$ as a rational (resp. elliptic) normal curve in case (i) [resp. (ii)] for $i = 1, \dots, n-1$. The same holds for C_n in $\mathbb{P}^k$. Thus $\Phi_{k-1}(C_i) = \emptyset$ and $\Phi_k(C_i) = C_i$ for $i = 1, \dots, n-1$ in case (i), while $\Phi_{k-1}(C_i)$ consists of k^2 points, according to (7), and $\Phi_k(C_i) = C_i$ for $i = 1, \dots, n-1$ in case (ii). Similarly, $\Phi_k(C_n)$ is either empty (case (i)) or consists of $(k+1)^2$ points (case (ii)). Let $Y := \mathbb{P}(\bigoplus_{i=1}^{n-1} \mathcal{L}_i)$ denote the $(n-1)$ -dimensional sub-scroll of X generated by the sections C_i , for $i = 1, \dots, n-1$. The above facts imply that the k th inflectional locus Φ_k of X is equal to Y in case (i), and to the union of Y and $(n-1)k^2 + (k+1)^2$ fibers in case (ii). This follows from [7, p. 1050] in case (i) and [5, p. 152] in case (ii) (see also [4, Proposition 2.6, Corollary 2.9]). We have $[Y] = L - (g+k)F$, since $\deg \mathcal{L}_n = g+k$. This gives $[\Phi_k] = L - kF$ in case (i), and $[\Phi_k] = L - (k+1)F + ((n-1)k^2 + (k+1)^2)F = L + k(nk+1)F$ in case (ii). This agrees with Corollary 1, which gives $[\Phi_k] = L - kF$ when $N = kn$ and $g = 0$, and $[\Phi_k] = L + kdF$ when $N = kn$ and $g = 1$.

Note that the only rational nondegenerate scroll in $\mathbb{P}^{2n}$ is the linearly normal rational scroll of degree $n+1$ considered in case (i) above, with $k = 2$. In fact, for any smooth n -dimensional scroll $X \subset \mathbb{P}^{2n}$, the well known double point (or self-intersection) formula becomes

$$(d-n)(d-n-1) = n(n+1)g, \quad (8)$$

so that for $g = 0$, we must have $d = n+1$ if X is nondegenerate.

It is in fact conjectured that any scroll $X \subset \mathbb{P}^{2n}$ of dimension n has $g = 0$ or 1 ; this conjecture holds for $n \leq 4$ [2, Corollary 5].

For $g = 1$, the case $k = 2$ is not covered by (ii) in the above example. Note that, by (8), such a scroll must have degree $2n + 1$. In fact, it is well known that the only such scrolls are the ones constructed as follows. Consider a smooth, elliptic curve C , and define inductively rank i , degree 1 sheaves $\mathcal{F}_i$ by starting with $\mathcal{F}_1 = \mathcal{O}_C(p)$, for some $p \in C$, and using the non-split exact sequences $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F}_{i+1} \rightarrow \mathcal{F}_i \rightarrow 0$. Taking points $p_1, p_2 \in C$, then $X = \mathbb{P}(\mathcal{F}_n(p_1 + p_2))$ can be embedded by the tautological line bundle, giving an indecomposable scroll of degree $2n + 1$ in $\mathbb{P}^{2n}$. If the assumptions of Theorem 2 are satisfied, then Corollary 1 gives $[\Phi_2] = L + 2(2n + 1)F$.

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