

Enriques diagrams and equisingular strata of families of curves

(Joint work with Steve Kleiman)

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Motivation

- ▶ Count the number of curves in a given family which have certain types of singularities, or that have a given genus (e.g., are rational), and which satisfy some other conditions (like going through given points, or are tangent to a given curve).

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- ▶ More generally, the number N_r can be replaced with a cycle class on the parameter variety of the family.

Examples.

- ▶ 1. Plane curves of degree d .
Determine the variety $V_r \subset \mathbf{P}^N$, $N = d(d+3)/2$, of curves with r nodes (the Severi varieties).
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- ▶ 2. Lefschetz pencil.
Let $S \subset \mathbf{P}^3$ be a smooth surface, $L \subset \mathbf{P}^3$ a general line. Then # singular curves in the pencil

$$\{S \cap H \mid L \subset H\}$$

is the degree of the dual surface $S^\vee$.

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- ▶ An *Enriques diagram* is a finite forest $\mathbf{T}$ where each vertex V is assigned a weight $m_V \geq 1$. The forest is equipped with a binary relation, denoted ' $\succ$ ' and called *proximity*. A vertex V is called a *satellite* if it is proximate to a remote predecessor; if not, V is said to be *free*. A root is free, and a leaf can be either free or a satellite.

- ▶ **Law of Proximity.** A root is proximate to no vertex. If a vertex is not a root, then it is proximate to its immediate predecessor and to at most one other vertex; the latter must be a remote predecessor. If one vertex is proximate to a second, and if a distinct third lies between the two, then it too is proximate to the second.

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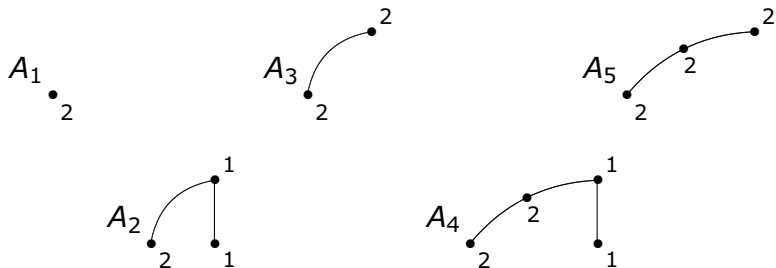
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- ▶ **Proximity Inequality** For each vertex V ,

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- ▶ **Law of Succession** A vertex V may have any number of free immediate successors, but at most two may be satellites, and they must be satellites of different vertices.

Examples of diagrams

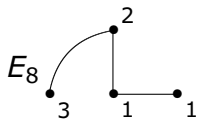
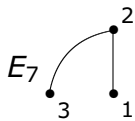
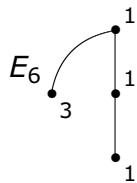


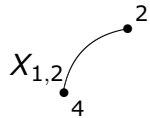
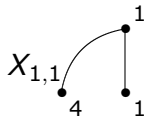
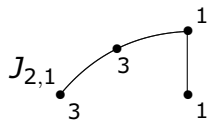
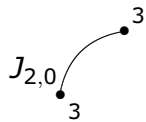
$$D_4 \bullet_3$$

$$D_5 \begin{array}{c} \bullet_1 \\ \curvearrowleft \\ \bullet_3 \end{array} \begin{array}{c} \bullet_1 \\ \downarrow \\ \bullet_1 \end{array}$$

$$D_6 \begin{array}{c} \bullet_2 \\ \curvearrowleft \\ \bullet_3 \end{array}$$

$$D_7 \bullet$$





Numerical characters.

- Fix an Enriques diagram $\mathbf{T}$ and introduce some numerical characters:

$\text{rts}(\mathbf{T}) :=$ the number of roots

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- In terms of the preceding characters, define the following ones:

$\text{dim}(\mathbf{T}) := \text{rts}(\mathbf{T}) + \text{frs}(\mathbf{T})$

$\text{deg}(\mathbf{T}) := \sum_{V \in \mathbf{T}} \binom{m_V + 1}{2}$

$\text{cod}(\mathbf{T}) := \text{deg}(\mathbf{T}) - \text{dim}(\mathbf{T})$

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- ▶ $\text{mod}(\mathbf{T}) := \mu(\mathbf{T}) - \text{cod}(\mathbf{T})$

Lemma. Let $\mathbf{T}$ be a minimal Enriques diagram with one root R , and let $\mathbf{S}$ be the set of satellites. Then:

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Fibonacci diagrams

Let φ_k denote the k th Fibonacci number:

$$\varphi_0 = 0, \varphi_1 = 1, \varphi_2 = 1, \varphi_3 = 2, \varphi_4 = 3, \varphi_5 = 5, \dots$$

The k th Fibonacci diagram, for $k \geq 3$, is the weighted tree with one leaf, starting with a root of weight φ_k , followed by a free vertex of weight φ_{k-1} , followed by a succession of satellites with weights $\varphi_{k-2}, \dots, 1, 1$. It has the following numerical characters:

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Zariski clusters

- ▶ Assume $x \in C \subset S$ is a singular point on a reduced curve on a smooth surface. Let $\beta : S' \rightarrow S$ be a good embedded resolution — it determines a weighted diagram with a proximity relation, $\mathbf{T} = \{(V, m_V), \succ\}$, the *Enriques diagram* of the singularity. The Enriques diagram determines the topological type of the singularity.

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- ▶ The *Zariski cluster* of the singularity is the subscheme defined by the complete ideal

$$\mathcal{I} := \beta_* \mathcal{O}_{S'}(-\sum_V m_V E_V),$$

where E_V is the full pullback to S' of the exceptional divisor coming from blowing up the point V .

Geometric interpretation

Let $x \in C \subset S$ be a singular point of a reduced curve on a smooth projective surface and $\mathbf{T}$ its Enriques diagram. There are the following interpretations:



$$\text{rts}(\mathbf{T}) = 1$$

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$$\text{deg}(\mathbf{T}) = \dim \mathcal{O}_C/\mathcal{I}$$

$\text{cod}(\mathbf{T}) = \text{cod}(\text{equisingular stratum in versal def. space})$

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- ▶ $\text{mod}(\mathbf{T})$ = the modality

Fibonacci curves

The rational curve $C_k: y^{\varphi_k} z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0$ has precisely two cusps and no other singularities. In addition to the Fibonacci cusp, there is a “semi-Fibonacci” cusp: $z^{\varphi_{k-1}} - x^{\varphi_{k+1}} = 0$. The strict transform of this cusp is a $(k-1)$ th Fibonacci cusp.

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$$\deg C_k^* = 2\varphi_{k+1} - 2 - (\varphi_{k-1} - 1 + \varphi_k - 1) = \varphi_{k+1}$$

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- ▶ The sum of the $\bar{M}$ -numbers is

$$\varphi_{k+2} - 4 + \varphi_{k+1} - 4 + \varphi_{k-1} - 1 = 3(\varphi_{k+1} - 3)$$

(cf. a conjecture by Orevkov).

Families of curves

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- ▶ The candidate for such an enumerating cycle is the pushdown $U(\mathbf{T})$ of the fundamental cycle of the closure of $Z(\mathbf{T})$, where

$$Z(\mathbf{T}) := H(\mathbf{T}) \cap \mathrm{Hilb}_{D/Y}^d \subset \mathrm{Hilb}_{F/Y}^d,$$

and where $H(\mathbf{T})$ is defined as follows.

Hilbert schemes

- ▶ Let $\mathbf{T}$ be an Enriques diagram, and set $d := \deg \mathbf{T}$. The set of Zariski clusters in the fibers of π defined by complete ideals $\mathcal{I}$ with diagram $\mathbf{T}$, determines a locally closed subset $H(\mathbf{T}) \subset \mathrm{Hilb}_{F/Y}^d$.

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- ▶ **Proposition.** $H(\mathbf{T})$ is smooth, irreducible, and of dimension $\dim \mathbf{T} := \mathrm{rts} \mathbf{T} + \mathrm{frs} \mathbf{T}$.
- ▶ Sketch of proof: We construct, by induction on the number of vertices of $\mathbf{T}$, a smooth, irreducible variety P parameterizing ordered configurations with (unweighted) diagram $\mathbf{T}$. Forgetting the ordering gives a map to $H(\mathbf{T})$, which is therefore irreducible and of the right dimension. We use Zariski's Main Theorem to show that $H(\mathbf{T})$ is the quotient of P by the automorphisms of the orderings.

Theorem 1. Assume the fibers of D are reduced and that for each $\mathbf{T}$ such that $Y(\mathbf{T}) \neq \emptyset$, we have

$$\mathrm{cod} Y(\mathbf{T}) \geq \min(r + 1, \mathrm{cod} \mathbf{T}).$$

Then either $Y(r\mathbf{A}_1)$ is empty, or it has pure codimension r ; in either case, its closure is the support of the natural nonnegative cycle $U(r\mathbf{A}_1)$.

Node polynomials

Write $U(r) := U(r\mathbf{A}_1)$. The key to computing a formula for $U(r)$ is the following proposition.

- **Proposition.** Assume the family of curves D on $\pi : F \rightarrow Y$ is reduced in codimension $\leq r$ and that $\text{cod } Y(\mathbf{T}) \geq \min(r + 1, \text{cod } \mathbf{T})$. Set $r_i := r - \binom{i+1}{2} + 2$. Then the following formula holds:

$$ru(D, r) = \pi_* \sum_{i \geq 2} (-1)^i u(D_i, r_i).$$

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- Here D_i , $\pi_i : F_i \rightarrow Y_i$ are families derived by blowing up i -multiple points of the fibers of D . The new base Y_i is a modification of the scheme of zeros of the section of $\mathcal{P}_{F/Y}^{i-1}(D)$ induced by D .

Theorem 2. With the above assumptions, if $r \leq 8$, then the rational equivalence class $[U(r)]$ is given by the formula

$$[U(r)] = P_r(a_1, \dots, a_r)/r!$$

where $a_q := \pi_* b_q$ and b_q is a polynomial, output by a certain algorithm, in the Chern classes $c_1(\mathcal{O}_F(D))$, $c_1(\Omega_{F/Y}^1)$, $c_2(\Omega_{F/Y}^1)$, and the P_j are Bell polynomials.

Bell polynomials

The *Bell* polynomials $P_i(a_1, \dots, a_i)$ are defined via the formal identity in t :

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- ▶ If we assign a_j weight j , then $P_i(a_1, \dots, a_i)$ is weighted homogeneous of degree i .