

Generating functions and enumerative geometry

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Generating functions

Enumerative geometry

Bell polynomials

Curves on surfaces



Partitions

Let p denote the partition function: $p(n)$ is the number of ways of writing $n = n_1 + \dots + n_k$, where $n_1 \geq \dots \geq n_k \geq 1$ (= the number of Young–Ferrers diagrams of size n).

The generating function for p ,

$$\varphi(q) := \sum_{n \geq 0} p(n)q^n,$$

is equal to the formal power series

$$\prod_{m \geq 1} (1 - q^m)^{-1} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \dots$$

Fibonacci numbers

A recursive relation sometimes gives a closed formula for the corresponding generating function:

$$F_0 = 0, F_1 = 1 \text{ and}$$

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

This gives

$$\sum_{n \geq 2} F_n q^n = \sum_{n \geq 2} F_{n-1} q^n + \sum_{n \geq 2} F_{n-2} q^n$$

hence

$$f(q) := \sum_{n \geq 0} F_n q^n = \frac{q}{1 - q - q^2}$$



Enumerative geometry: count the number of certain geometric objects that satisfies an appropriate number of conditions.

- ▶ There are two lines in space meeting four given lines.
- ▶ There are eight circles tangent to three given circles in the real plane.

Two common methods to solve a problem in enumerative geometry:

- ▶ Degenerate the objects and/or the conditions so that the problem has a combinatorial solution.
- ▶ Represent the objects as points and the conditions as cycles, in some parameter space, and apply intersection theory.

Plane rational curves

Let N_d denote the number of plane rational curves of degree d passing through $3d - 1$ given points.

Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

If we set $n_d := \frac{N_d}{(3d-1)!}$, then

$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} \frac{d_1 d_2 ((3d_1-2)(3d_2-2)(d+2) + 8(d-1))}{6(3d-1)(3d-2)(3d-3)}$$

Kontsevich's original proof used a degeneration argument: mapping the one-dimensional family of curves through $3d - 2$ points to $\mathbb{P}^1$ using the cross-ratio, and counting degrees above 0 and 1.



For example, for $d = 3$:

$$N_3 = 4 \cdot 5 + 4 \cdot 5 - 2 \cdot 10 - 8 \cdot 1 = 12$$

The naive generating function of the problem is

$$\sum_d N_d q^d$$

However, the “correct” generating function turns out to be

$$\Gamma(q, t) := \sum_d N_d q^d t^{3d-1} / (3d-1)! = \sum_d n_d q^d t^{3d-1}$$

This function satisfies the differential equation

$$\Gamma_{222} = (\Gamma_{112})^2 - \Gamma_{111}\Gamma_{122}$$

which gives Kontsevich's formula (associativity of the quantum product).



Define the **Bell polynomials** $P_r(y_1, \dots, y_r)$ by $P_0 = 1$ and

$$P_{i+1}(y_1, \dots, y_{i+1}) = \sum_{k=0}^i \binom{i}{k} P_{i-k}(y_1, \dots, y_{i-k}) y_{k+1}$$

The generating function $\phi(q) := \sum P_i q^i / i!$ satisfies

$$\phi'(q) / \phi(q) = \sum y_k q^{k-1} / (k-1)!$$

hence is given by the formal identity

$$\phi(q) = \exp\left(\sum y_j q^j / j!\right)$$

The coefficient of $y_1^{k_1} \dots y_r^{k_r}$ in the polynomial P_r is equal to

$$\frac{r!}{k_1! \dots k_r!} \left(\frac{1}{1!}\right)^{k_1} \dots \left(\frac{1}{r!}\right)^{k_r}$$

(Link to Faà di Brunos formula!)



Then

$$[(e^{-st} D_t^s f) \xi_n(-x, t; r)]_{t=\pm\infty} = 0,$$

and hence, by repeated partial integration of the left of (3.4),

$$(3.4) \quad \int_{-\infty}^{\infty} f(t) (D_t^s e^{-st}) dt = (-1)^s \int_{-\infty}^{\infty} e^{-st} (D_t^s f) dt.$$

Thus if we choose $f(t) = P(t)$, where P is a polynomial of degree $< n$ in t , we have

$$(3.5) \quad \int_{-\infty}^{\infty} e^{-st} P(t) \xi_n(-x, t; r) dt = 0.$$

The conditions (3.3) with $f = P$ are satisfied when $x > 0$ and r is an even integer > 0 . Hence (3.2) follows from (3.5) when we take $m < n$.

II. The Polynomials ϕ_n , Y

4. The ϕ_n are a generalization of the ξ_n ; see (4.51). Let $\alpha_0, \alpha_1, \dots, \alpha_n, \dots$ be an infinite sequence of independent variables. The case when some or all of the α are absolute constants is included; in particular, α_n may be zero for all $n > m$, where m is a constant integer. The polynomials

$$\phi_n = \phi_n(\alpha) = \phi_n(\alpha_0, \dots, \alpha_n) \quad (n = 0, 1, \dots)$$

are defined by

$$(4.1) \quad \phi_0 = 1, \quad \phi_{-k-1} = 0, \quad \phi^{n+1} = \alpha(\phi + \alpha)^n, \quad n \geq 0.$$

In ordinary notation the last is

$$(4.2) \quad \phi_{n+1} = \sum_{i=0}^n \binom{n}{i} \alpha_{n+1} \phi_{n-i}.$$

For verifications of formulas we give the first 6 ϕ 's

$$\begin{aligned} \phi_0 &= 1, & \phi_1 &= \alpha_0, & \phi_2 &= \alpha_1^2 + \alpha_0, & \phi_3 &= \alpha_1^3 + 3\alpha_1\alpha_0 + \alpha_2, \\ \phi_4 &= \alpha_1^4 + 6\alpha_1^2\alpha_0 + 4\alpha_1\alpha_2 + 3\alpha_0^2 + \alpha_4, \\ \phi_5 &= \alpha_1^5 + 10\alpha_1^3\alpha_0 + 10\alpha_1^2\alpha_2 + 15\alpha_1\alpha_0^2 + 5\alpha_1\alpha_4 + 10\alpha_2\alpha_3 + \alpha_5. \end{aligned}$$

Let t be a parameter. Then, writing

$$f(t) = \exp \langle t\alpha \rangle = \alpha_0 + \alpha_1 t + \alpha_2 t^2 / 2! + \dots,$$

we have the generating identity of the ϕ ,

$$(4.3) \quad \exp [f(t) - \alpha_0] = \exp \langle t\phi \rangle$$

For, operation with D_t on (4.3) gives

$$\alpha \exp [(\alpha + \phi)t] = \phi \exp \langle t\phi \rangle,$$



Diagonals

Consider the r -fold product $X^r = X \times \dots \times X$.

Problem: *how many diagonals of each type exists?*

Use the following shorthand notation

$$d_i = [i \text{ points are equal}]$$

For example, $d_1^{r-5} d_2 d_3$ means a diagonal where, for fixed distinct integers i, j, k, l, m , $x_i = x_j$ and $x_k = x_l = x_m$.

The number of diagonals of type $d_1^{k_1} \dots d_r^{k_r}$ is equal to

$$\frac{r!}{k_1! \dots k_r!} \left(\frac{1}{1!}\right)^{k_1} \dots \left(\frac{1}{r!}\right)^{k_r},$$

the coefficient appearing in the Bell polynomial.



Configuration spaces

The configuration space of r points in X is the complement of the diagonals in $X^r = X \times \dots \times X$.

Fulton and MacPherson constructed a compactification $X[r]$ of this space by adding nonsingular divisors D_S for $S \subset \{1, \dots, n\}$, intersecting transversally (and non-emptily only if nested).

The Chow ring of $X[r]$ is generated over that of X^r by the $\sum_{k=2}^r \binom{r}{k} = 2^r - r - 1$ classes $[D_S]$, with explicit relations.

Manin computed the generating function for the Euler–Poincaré characteristics $\chi(X[r])$ in terms of $\chi(X)$.

The strata and nests are enumerated by means of trees.



Counting curves on surfaces

Let $\pi: F \rightarrow Y$ be a family of smooth projective surfaces and $D \subset F$ a relative divisor (a family of curves).

Set $Y_r := \{y \in Y \mid D_y \text{ has } r \text{ nodes}\}$.

The expected codimension of Y_r in Y is r .

Problem: *find the class $[\overline{Y}_r]$ in the Chow group $A^r Y$.*

For example, $\overline{Y}_1$ = the discriminant of $D \rightarrow Y$, and $[\overline{Y}_1] = \pi_*(\text{polynomial in the Chern classes of the family})$.

To find a formula for general r , proceed by recursion: resolve one node at a time, and reduce the family of r -nodal curves to a family of $(r-1)$ -nodal curves with “known” Chern classes.

Node polynomials

The solution can be expressed via Bell polynomials:

Theorem (Kleiman–P.) *If the family of curves $D \subset F \rightarrow Y$ is dimensionally general, then $\overline{Y}_r$ is the support of a natural nonnegative cycle U_r of codimension r .*

Moreover, for $r \leq 8$, the class $u_r := [U_r]$ is given by

$$u_r = P_r(a_1, \dots, a_r)/r!$$

where $a_i = \pi_* b_i$, and the b_i are polynomials in the Chern classes of D and F/Y and are output by a certain algorithm.

Benefit: To go from u_r to u_{r+1} , need only to determine a_{r+1} .



The proof of the Theorem is based on the recursion formula

$$ru(D)_r = \pi_* \sum_{i \geq 2} (-1)^i u(D_i)_{r_i},$$

where $D_i \subset F_i \rightarrow X_i$ are “derived” families with fewer nodes.

The real difficulties start for $r \geq 8$. Then the family $F_2 \rightarrow X_2$ has non-reduced fibres (coming from points of multiplicity ≥ 4 in the fibers of D) in “too high” dimension. Here we need to use the theory of residual, or excess, intersection.

The recursive relation is the key to the appearance of the Bell polynomials.

The other main point is the way the Chern classes of the derived families depend on the Chern classes of the original family.



Proof for $r \leq 7$

Let $X_i \subset F$ denote the set of i -multiple points on fibers of D .

The class $x_i := [X_i]$ is a polynomial in the Chern classes of D and F/Y .

Set $b_1 = x_2$, and define b_s by a certain algorithm.

Then

$$u_1(D) = \pi_*[X_2] = \pi_*x_2 =: a_1 = P_1(a_1).$$

Assume $r \geq 2$ and the theorem holds for all families verifying the conditions with r replaced by $r - 1$. Then

$$r!u_r(D) = \pi_*\left(P_{r-1}(a_1(D_2), \dots) - \frac{(r-1)!}{(r-4)!}P_{r-4}(a_1(D_3), \dots)\right)$$

By definition, $a_s(D_i) = \pi_{i*}b_s(D_i)$ and we have

$$P_s(a_1(D_i), \dots, a_s(D_i)) = P_s(\pi^*a_1(D) + Q(i, b_1), \dots) \cdot x_i$$

The Bell polynomials satisfy a “binomial” identity:

$$\begin{aligned} &P_s(\pi^*a_1(D, \pi) + Q(i, b_s), \dots, \pi^*a_s(D, \pi) + Q(i, b_s)) \\ &= \sum_{k=0}^s \binom{s}{k} P_{s-k}(\pi^*a_1, \dots, \pi^*a_{s-k}) P_k(Q(i, b_1), \dots, Q(i, b_k)). \end{aligned}$$

Using the definition of b_s and $a_s = \pi_*b_s$, we get

$$r!u_r(D) = \sum_{k=0}^{r-1} \binom{r-1}{k} P_{r-k}(a_1, \dots, a_{r-k}) a_{k+1} = P_r(a_1, \dots, a_r).$$

Curves on K3 surfaces

Let S be a K3 surface, $C \subset S$ a curve, r, g integers such that $C^2/2 = g + r - 1$, and set

$N_{g,r} := \#\{r\text{-nodal curves in } |C| \text{ of geometric genus } g \text{ through } g \text{ given points of } S\}$

Bryan and Leung proved: if S is general and $[C]$ is primitive, then

$$f_g(q) := \sum N_{g,r} q^r = \frac{(DG_2)^g}{\Delta} q^{1-g}$$

Here $\Delta = q \prod_{m \geq 1} (1 - q^m)^{24}$ is the discriminant (a modular form of weight 12) and DG_2 is the quasimodular form equal to the derivative of the Eisenstein series $G_2 = -\frac{1}{24} + \sum_{k \geq 1} \sigma(k) q^k$, where $\sigma(k) = \sum_{d|k} d$.



For $g = 0$, $f_0(q) = \Pi(1 - q^m)^{-24} = \varphi(q)^{24}$, where φ is the generating function of the partition function.

Proof ($g = 0$): Degenerate (in symplectic geometry) S to an elliptic fibration $S' \rightarrow \mathbb{P}^1$ with 24 nodal fibers, and C to $C' = s(\mathbb{P}^1) + r\phi$, where s is a section and ϕ is a fiber.

The number of stable genus 0 maps for a given r can be identified with the sum $\sum M_{a_1} \cdots M_{a_{24}}$, where M_{a_i} is the number of such curves mapping $a_i: 1$ to the i th nodal fibre, and the sum is taken over 24-tuples $(a_1, \dots, a_{24})$ with $\sum a_i = r$.

One can see that M_{a_i} is equal to $p(a_i)$, and hence the generating function is the 24th power of the generating function $\varphi(q)$.

Conjectures

Göttsche's conjecture: If $F = S \times Y$ and $Y = |C|$ is ample enough, then the generating function $\sum N_r q^r$, with N_r the number of r -nodal curves in the system, can be expressed in terms of two universal (but unknown and un-understood) formal power series and three quasimodular forms. If the canonical bundle of S is trivial, then only quasimodular forms appear.

Proved by Bryan and Leung if S is a general K3 or Abelian surface and the class of C is primitive.

Itzykson and Di Francesco conjectured that for plane curves of degree d :

$$N_r = \frac{3^r}{r!} d^{2r} - \frac{2 \cdot 3^r r}{r!} d^{2r-1} + \dots$$

Corresponds to the recursive formula giving $r!u_r = u_1^r + \dots$, where $u_1 = 3d^2 + \dots$

Thom polynomials

An alternative approach to determining the classes u_r : apply multiple point theory to the map $\pi|_{X_2}: X_2 \rightarrow Y$.

The problem with this approach is that the map is not generic (and X_2 is not smooth).

However, *Kazarian* observed that it is (generically) a Legendre map, and that therefore it is possible to find formulas based on the knowledge of the Thom polynomials of all relevant singularities (so this can only be done explicitly only “up to” the codimension for which all singularities are classified and their Thom polynomials known).

Kazarian's formula for $r!u_r$ is given as a linear combination of terms of the form

$$S_1^{k_1} \cdots S_r^{k_r}$$

with $k_1 + 2k_2 + \cdots + rk_r = r$.

The class $S_i = S_{A_1^i}$, for $i \geq 2$, corresponds to an “excess” or “residual” intersection coming from a diagonal of type d_i . The coefficients in the linear combination are the same as the ones appearing in the Bell polynomials.



Kazarian's formula translates to

$$r!u_r = P_r(S_1, -S_2, \dots, (-1)^r S_r)$$

but, unfortunately, this “diagonal approach” does not make the computations any easier.

For example, S_2 is equal to the “equivalence” of the diagonal in the product $S_1 S_1$ plus twice the class of fibers with A_2 singularities.

In general, to compute the contribution from d_i , *all* singularities of codimension i will have to be taken into account.



Tropical geometry

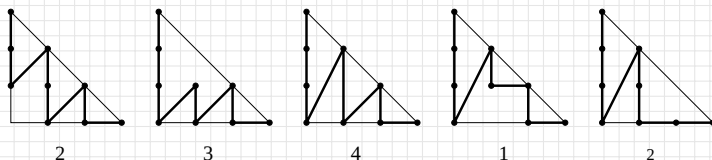
A new approach to the enumeration of (complex) curves on toric surfaces was given by Mikhalkin, using tropical geometry. The complex curves are counted by counting the corresponding tropical (real) curves, with appropriate multiplicities.

The number of curves is given by a count of certain lattice paths in the Newton polygon defining the tropical curves.

In the case of plane curves, the number $N_{g,d}$ is equal to the (weighted) number of so-called λ -increasing lattice paths of length $3d - 1 + g$ in the triangle with corners $(0, 0)$, $(d, 0)$, $(0, d)$.



Counting rational cubic curves passing through 8 points:



$$N_3 = 2 + 3 + 4 + 1 + 2 = 12$$

Hence the original enumerative geometry problem is reduced to a combinatorial one!

Jeopardy

What question has answer 3280?

Which magic number appears in the formula for u_8 ?

What question has answer 5895?

