Higher order tangent spaces – the toric case

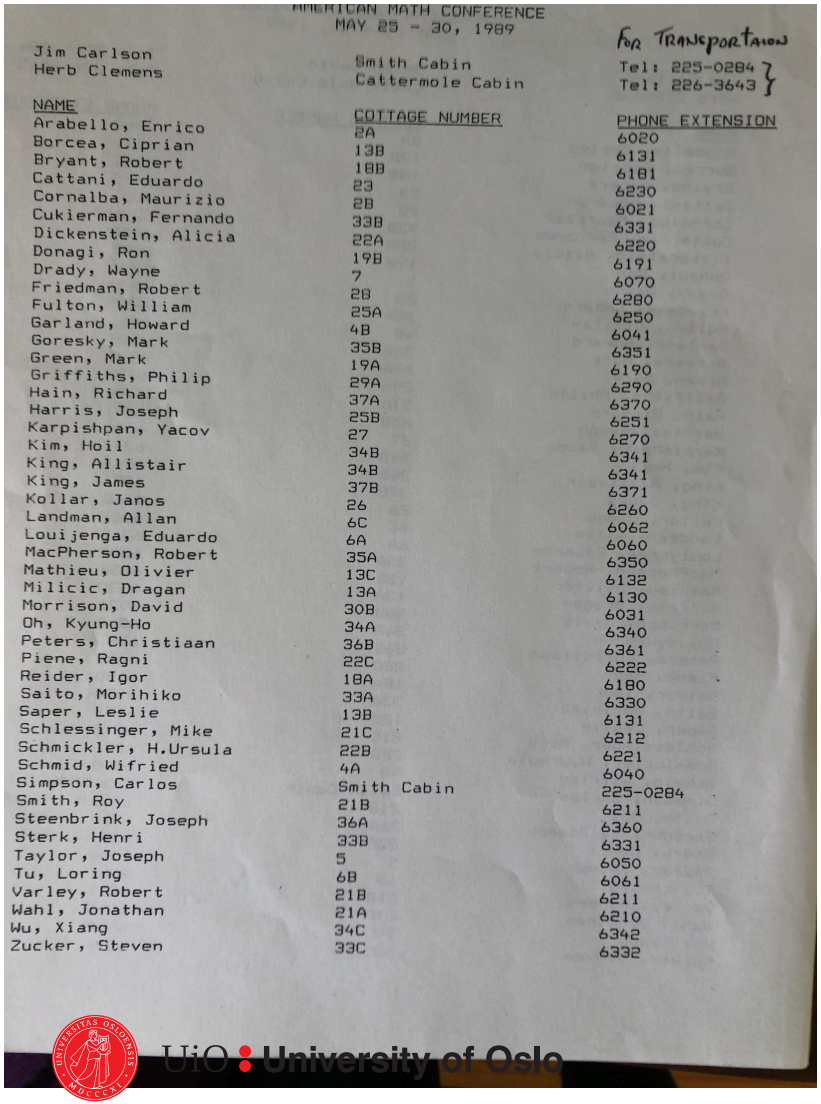
Ragni Piene

MateMax – Alicia 70
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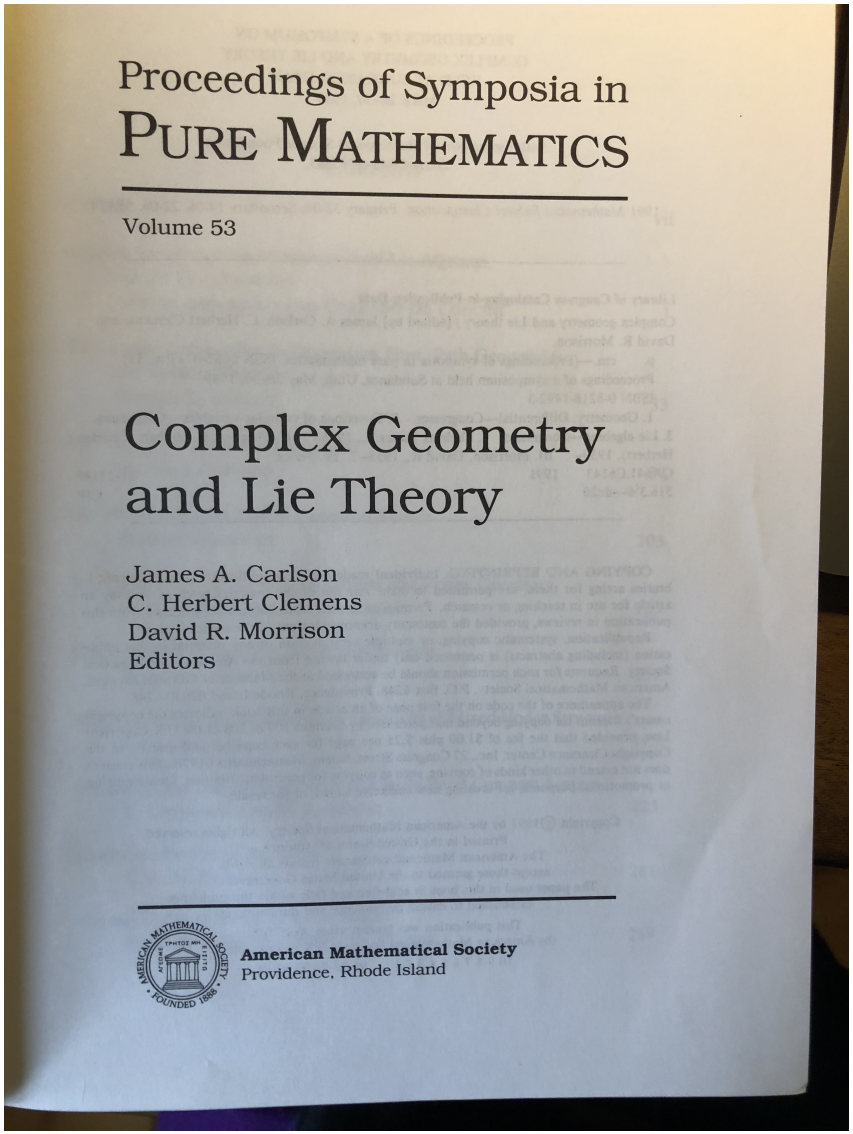


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AN INTEGRAL CRITERION FOR THE EQUIVALENCE OF PLANE CURVES

Alicia DICKENSTEIN and Carmen SESSA

We give in this paper a necessary and sufficient condition for the Zariski equivalence of algebroid plane irreducible curves $C \equiv (f=0)$ over an algebraically closed field of characteristic zero and, in the analytic case, an integral expression for the characteristic numbers. We express these numbers as the intersection multiplicities of $(f=0)$ with a family of cycles associated to f, and also as residues of meromorphic differentials on C.

Definitions and notation

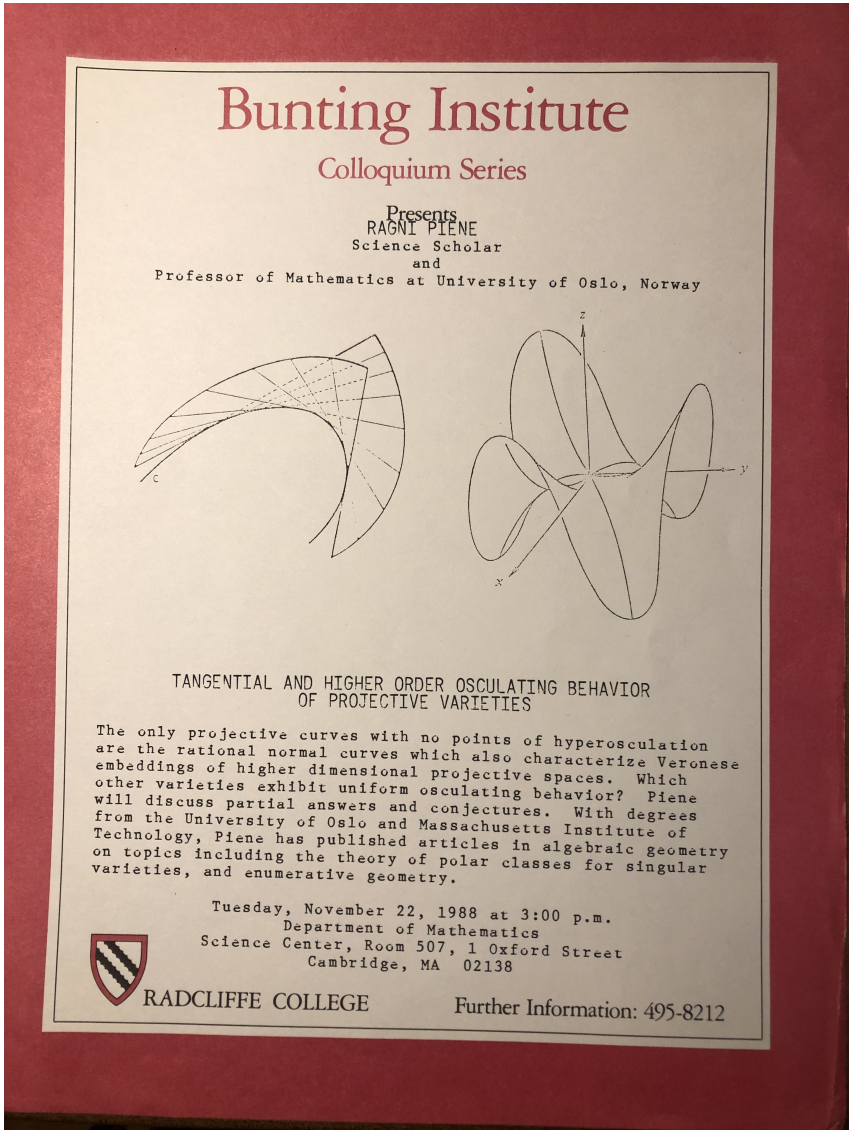
Let $C \equiv (f(x,y) = 0)$, $f \in k[[x,y]]$ irreducible and suppose $(x=0)$ is not a tangent. Let $x=t^n$ $y = \phi(t) = \sum_{i \geq n} a_i t^i$ be a parametrization of C. We will assume C singular $(n > 1)$ and G.C.D. $\{n,i : a_i \neq 0\} = 1$. The characteristic of C is the g+1 - tuple $(n;\beta_1,\beta_2,\ldots,\beta_g)$, where $n = \text{mult } C$; $e_o = n$; $\beta_{j+1} = \min \{i \in N : e_j \nmid i,\ a_i \neq 0\}$ and $e_j = \text{G.C.D } \{n,\beta_1,\ldots,\beta_j\}$ $(1 \leq j \leq g)$, $e_g = 1$.

Let us define $G_i = \{\tau \in k : \tau^{e_i} = 1\}$ $0 \leq i \leq g$. It is clear that $G_o \supseteq G_1 \supseteq \ldots \supseteq G_i \supsetneq G_{i+1} \ldots \supseteq G_g = \{1\}$ and $|G_i - G_{i+1}| = e_i - e_{i+1}$, where $|A|$ denotes the cardinal of the set A.

LEMMA 1: Let $x=t^n$ $y = \phi(t) = \sum_{i \geq n} a_i t^i$ be a parametrization of C ; then

$$\text{ord}_t(\phi(t) - \phi(\tau t)) = \beta_{i+1} \quad \text{if} \quad \tau \in G_i \setminus G_{i+1}.$$

0025-2611/82/0037/0001/$01.80



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Lattice points and toric varieties

Let $\mathbf{a}_0, \ldots, \mathbf{a}_n \in \mathbb{Z}^m$ be lattice points and set

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix} \in \mathbb{Z}^{(m+1)\times(n+1)}.$$



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The lattice points define a convex lattice polytope

$$P_A = \mathrm{Conv}\{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^m,$$

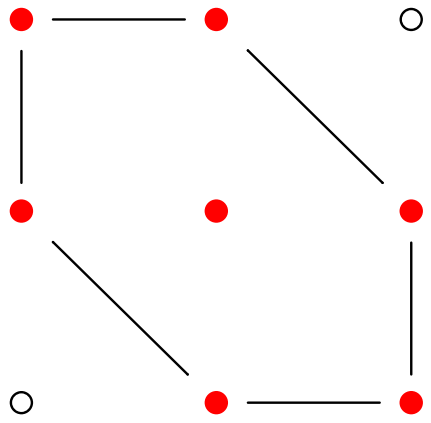
and a toric variety $X_A = \overline{\mathrm{im}\,\varphi} \subset \mathbb{P}^n$ where $\varphi\colon (\mathbb{C}^*)^m \to \mathbb{P}^n$

$$\mathbf{t} := (t_1, \ldots, t_m) \mapsto (\mathbf{t}^{\mathbf{a}_0} : \cdots : \mathbf{t}^{\mathbf{a}_n}).$$



Example: Del Pezzo's surface

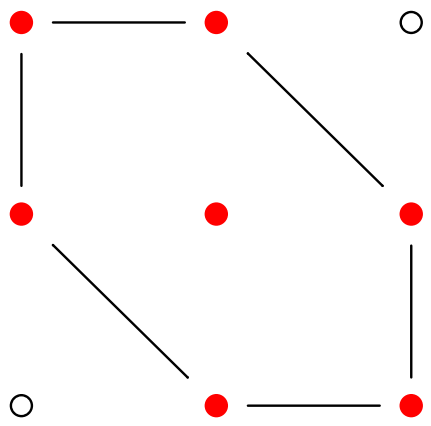
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}$$



P_A

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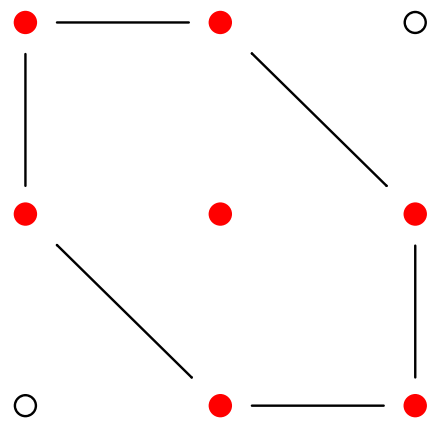
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$X_A \subset \mathbb{P}^6$ is defined by the parameterization

$$\mathbf{t} = (t_1, \ldots, t_m) \mapsto (t_1 : t_2 : t_1^2 : t_1 t_2 : t_2^2 : t_1^2 t_2 : t_1 t_2^2).$$

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The $\binom{m+k}{k} \times (n+1)$-matrix $A^{(k)}(\mathbf{t})$

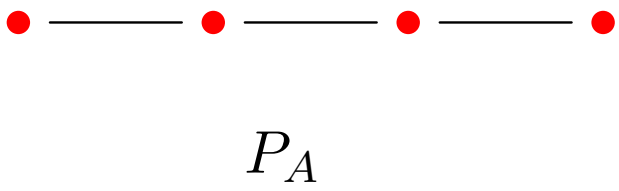
$$A^{(k)}(\mathbf{t}) := \begin{pmatrix} \mathbf{t}^{\mathbf{a}_0} & \mathbf{t}^{\mathbf{a}_1} & \cdots & \mathbf{t}^{\mathbf{a}_n} \\ \frac{\partial \mathbf{t}^{\mathbf{a}_0}}{\partial t_1} & \frac{\partial \mathbf{t}^{\mathbf{a}_1}}{\partial t_1} & \cdots & \frac{\partial \mathbf{t}^{\mathbf{a}_n}}{\partial t_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \mathbf{t}^{\mathbf{a}_0}}{\partial t_m} & \frac{\partial \mathbf{t}^{\mathbf{a}_1}}{\partial t_m} & \cdots & \frac{\partial \mathbf{t}^{\mathbf{a}_n}}{\partial t_m} \\ \frac{\partial^2 \mathbf{t}^{\mathbf{a}_0}}{\partial t_1^2} & \frac{\partial^2 \mathbf{t}^{\mathbf{a}_1}}{\partial t_1^2} & \cdots & \frac{\partial^2 \mathbf{t}^{\mathbf{a}_n}}{\partial t_1^2} \\ \frac{\partial^2 \mathbf{t}^{\mathbf{a}_0}}{\partial t_1 \partial t_2} & \frac{\partial^2 \mathbf{t}^{\mathbf{a}_1}}{\partial t_1 \partial t_2} & \cdots & \frac{\partial^2 \mathbf{t}^{\mathbf{a}_n}}{\partial t_1 \partial t_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^k \mathbf{t}^{\mathbf{a}_0}}{\partial t_m^k} & \frac{\partial^k \mathbf{t}^{\mathbf{a}_1}}{\partial t_m^k} & \cdots & \frac{\partial^k \mathbf{t}^{\mathbf{a}_n}}{\partial t_m^k} \end{pmatrix},$$

where $\frac{\partial^j}{\partial t_i^j}$ denotes the Hasse derivative, that is, $\frac{1}{j!}$ times the standard derivative. Note that $A^{(1)}(\mathbf{1}) = A$.



The twisted cubic

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$



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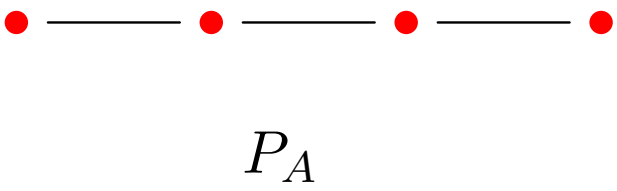
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$$A^{(2)}(\mathbf{t}) = \begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \end{pmatrix}$$

Higher order tangent (osculating) spaces

The kth osculating space to X_A at $\varphi(\mathbf{t})$ is the row span in $\mathbb{P}^n$ of the matrix $A^{(k)}(\mathbf{t})$. We have

$$\{\varphi(\mathbf{t})\} \subseteq T_{\varphi(\mathbf{t})} = \mathrm{Osc}^1_{\varphi(\mathbf{t})} \subseteq \mathrm{Osc}^2_{\varphi(\mathbf{t})} \subseteq \mathrm{Osc}^3_{\varphi(\mathbf{t})} \subseteq \cdots \subseteq \mathbb{P}^n.$$



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Set $m_k := \operatorname{rk} A^{(k)}(\mathbf{t}) - 1 = \dim \mathrm{Osc}^k_{\varphi(\mathbf{t})}$ for $\mathbf{t} \in (\mathbb{C}^*)^m$.

$A^{(k)}(\mathbf{t})$ is the matrix of the kth jet map $\mathcal{O}^{n+1}_{X_A} \to \mathcal{P}^k_{X_A}(1)$.



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On a Nash modification $\widetilde{X}^k_A$ the jet map induces a quotient

$$\mathcal{O}^{n+1}_{\widetilde{X}^k_A} \to \mathcal{P}^k, \text{ with } \operatorname{rk} \mathcal{P}^k = m_k + 1.$$

We call $\mathcal{P}^k$ the kth *osculating bundle*.



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Veronese embeddings

Example (The rational normal curve)

Consider $\mathbb{P}^1 \cong X_A \subset \mathbb{P}^n$, $P_A = n\Delta_1$,

$$\nu_n : \mathbf{t} = (t) \mapsto (1 : t : t^2 : \cdots : t^n).$$

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Example (The dth Veronese embedding)

Consider $\mathbb{P}^m \cong X_A \subset \mathbb{P}^n$, $n = \binom{m+d}{d} - 1$, $P_A = d\Delta_m$,

$$\nu_d \colon \mathbf{t} = (t_1, \ldots, t_m) \mapsto (1 : t_1 : \cdots : t_m : t_1^2 : \cdots : t_m^d).$$

By (Perkinson 1996): $\mathcal{P}^k = \mathcal{P}^k_{X_A}(1) \cong \mathcal{O}_{\mathbb{P}^m}(d-k)^{\oplus \binom{m+k}{k}}$.



Higher order dual varieties (P. 1983)

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Example (P.–Sacchiero 1984)

Let $\mathbb{P}^1 \cong X_A \subset \mathbb{P}^n$ be the rational normal curve of degree n.
Then, for $k < n$,

$$\mathcal{P}^k = \mathcal{P}^k_{X_A}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n-k)^{\oplus k+1} \text{ and } \deg X^{(k)} = (k+1)(n-k).$$



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Toric surfaces (Dickenstein–Di Rocco–P. 2014)

Let $X_A \subset \mathbb{P}^n$ be a k-regular toric surface. Then

$$\deg X_A^{(k)} = \tbinom{k+3}{4}\big(3\,\mathrm{Vol} - 2kE - \tfrac{1}{3}(k^2-4)V + 4(k^2-1)\big),$$

where Vol, E, V denote the (lattice) volume, length of edges, number of vertices of P_A.

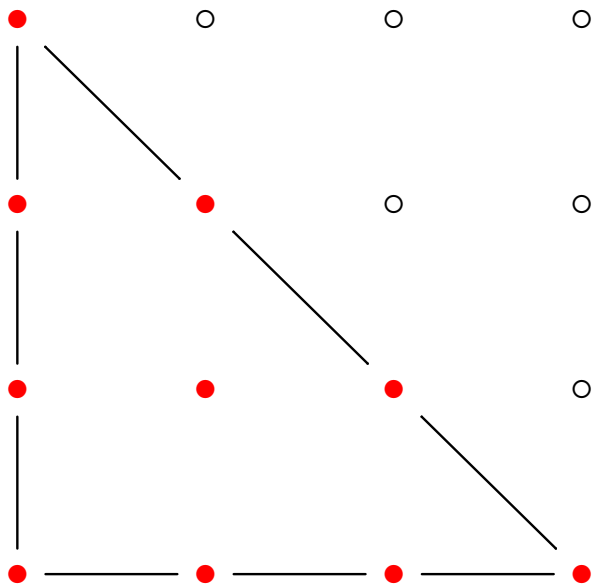


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$P_A = 3\Delta_2$

$\nu_3 \colon \mathbb{P}^2 \to \mathbb{P}^9$

$\mathrm{Vol} = 9,$
$E = 9,$
$V = 3,$

$\deg X_A^{(2)} =$
$5(27 - 36 + 12) = 15.$



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Toric threefolds (D–DR–P 2014)

Let $X_A \subset \mathbb{P}^n$ be a 2-regular toric threefold. Then

$$\deg X_A^{(2)} = 62\,\mathrm{Vol} - 57F + 28E - 8V + 58\,\mathrm{Vol}_0 + 51F_0 + 20E_0,$$

where Vol, F, E (resp. Vol_0, F_0, E_0) denote the (lattice) volume, area of facets, length of edges of P_A (resp. of the convex hull of the interior lattice points of P_A), and V is the number of vertices of P_A.

Toric threefolds (D–DR–P 2014)

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Example

Take P_A to be a cube with edge lengths 2. We have $\mathrm{Vol} = 6 \cdot 8 = 48$, $F = 6 \cdot 2 \cdot 4 = 48$, $E = 12 \cdot 2 = 24$, $V = 8$. Since P_A has only one interior lattice point, $\mathrm{Vol}_0 = F_0 = E_0 = 0$.

$$\deg X_A^{(2)} = 62 \cdot 48 - 57 \cdot 48 + 28 \cdot 24 - 8 \cdot 8 = 848.$$



Higher order selfdual toric varieties

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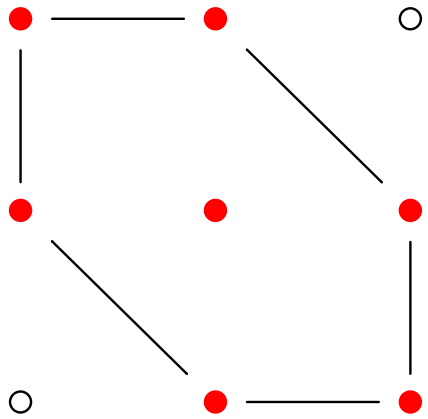
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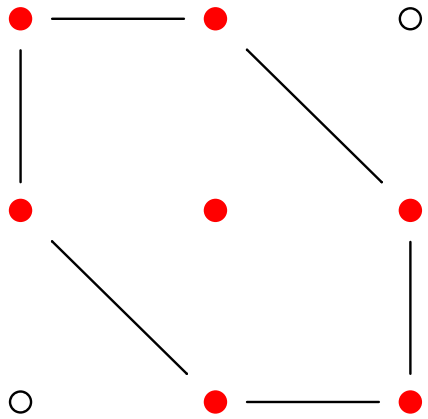
2. Any choice of $\binom{m+k}{k} + 1$ *general* points in $\mathbb{Z}^m$ gives a matrix A such that X_A is k-selfdual.



Del Pezzo and Togliatti

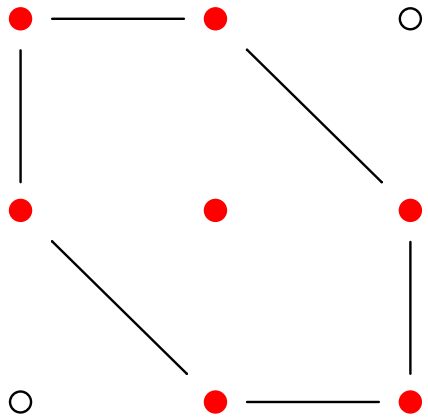


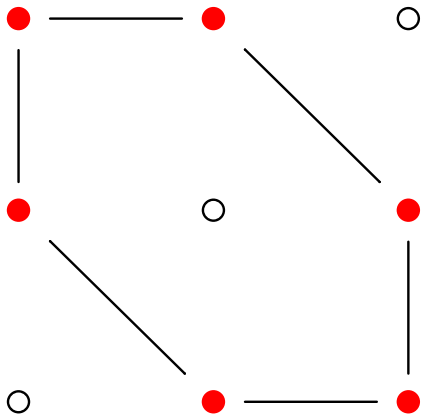
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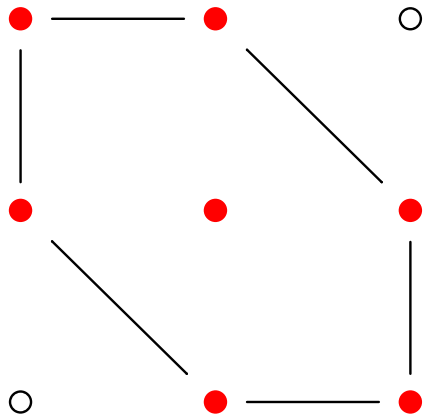
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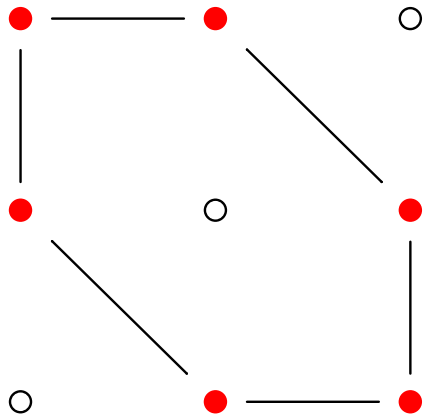


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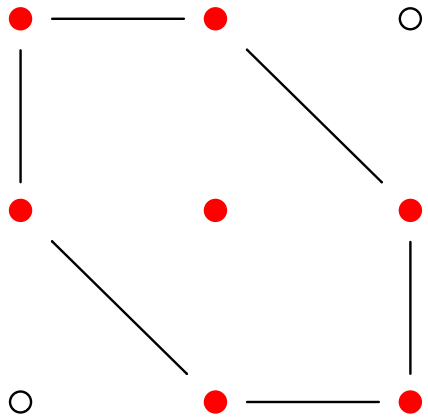


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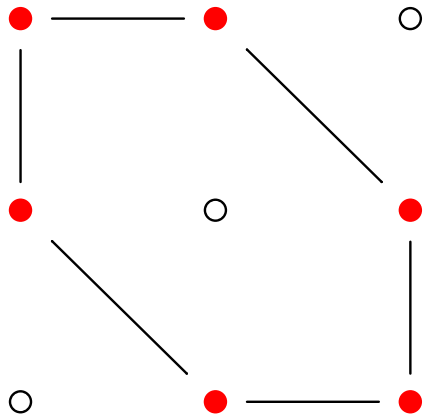


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Note that $P_A = P_{\overline{A}}$.

Higher order polar varieties (P. 2022)

Assume $m_k < n$. Let $L_{k,i} \subset \mathbb{P}^n$ be a linear subspace of codimension $m_k - i + 2$.



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The class $[M_{k,i}(L_{k,i})]$ is independent of the choice of a *general* $L_{k,i}$. We have

$$\deg[M_{k,i}] = \deg c_i(\mathcal{P}^k).$$



Projective invariance

The classes $[M_{k,i}]$ are projective invariants of X, like the usual polar classes:



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Let $X \subset \mathbb{P}^n$ and $f\colon X \to \mathbb{P}^r$ be a generic projection, with $m_k + 1 \leq r \leq n$. Then $[M_{k,i}] = [M^f_{k,i}]$.



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There are two ingredients: the commutativity of the jet maps and a version of Kleiman's Transversality Lemma. □

Corollary

If $X \subset \mathbb{P}^n$ is smooth and k-regular, i.e., $\mathcal{P}^k = \mathcal{P}^k_X(1)$, then a generic projection $f\colon X \to \mathbb{P}^r$ has $[M^f_{k,i}] = c_k(\mathcal{P}^k_X(1))$.



Reciprocal polar varieties

Bank, Giusti, Heintz, Mbakop, Pardo (1997–2005), Safey El Din, Schost (2003–2005) introduced *reciprocal* polar varieties in order to find real points on each connected component of a non-singular real algebraic variety. These were defined using Schubert conditions on the (Euclidean) normal spaces instead of the tangent spaces.



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We studied reciprocal polar varieties for *singular* real plane curves (Mork–P. 2008) and *singular* real surfaces (Mork 2011).



Orthogonality

Fix $H_\infty \subset \mathbb{P}^n$, the hyperplane at infinity. A nonsingular quadric $Q_\infty \subset H_\infty$ defines a polarity: $L_\infty \subset H_\infty \mapsto L_\infty^\perp \subset H_\infty$.



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Given a linear space $L \subset \mathbb{P}^n$, $L \not\subseteq H_\infty$, and $p \in L$. The *orthogonal space to L at p* is

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Then $L \cap H_\infty = (2:1:0)$ and $(L \cap H_\infty)^\perp = (1:-2:0)$.

Take $p = (2:-1:1)$. Then $L_p^\perp : 2x + y - 3z = 0$.



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We call $\mathcal{N}^k$ the *kth order Euclidean normal bundle.*



Higher order reciprocal polar varieties (P. 2022)

For $i = 0, \ldots, m$, let $L_i \subset \mathbb{P}^n$, $L_i \not\subseteq H_\infty$, have codimension $n - m_k + i$. Define *kth order reciprocal polar varieties*

$$M_{k,i}(L_i)^\perp := \overline{\{p \in X_{k-\text{cst}} \setminus H_\infty | N_p^k \cap L_i \neq \emptyset\}}.$$

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hence, since $s((\mathcal{K}^k)^\vee) = c(\mathcal{P}^k)$ and

$$s(\mathcal{O}_{\widetilde{X}^k}(1)) = 1 + c_1(\mathcal{O}_{\widetilde{X}^k}(1)) + c_1(\mathcal{O}_{\widetilde{X}^k}(1))^2 + \cdots,$$

$$[M_{k,i}^\perp] = \sum_{j=0}^{i} c_1(\mathcal{O}_X(1))^{i-j} \cap [M_{k,j}].$$



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Replacing the Euclidean normal spaces by the higher order normal spaces one defines higher order Euclidean distance degrees as the degree of $[M_{k,m}^{\perp}]$ (Di Rocco–Rose –Sodomaco), which gives that

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Since the higher polar classes are projective invariants, it follows that the Euclidean distance degrees are projective invariants.



The rational normal curve

Let $\mathbb{P}^1 \cong X_A \subset \mathbb{P}^n$ be the rational normal curve. Then

$$\mathcal{P}^k = \mathcal{P}^k_{X_A}(1) \cong \mathcal{O}_{\mathbb{P}^m}(n-k)^{\oplus\binom{m+k}{k}},$$



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We have already observed that X_A is $(n-1)$-self dual:

$X_A^{(n-1)} \subset (\mathbb{P}^n)^\vee$ is a rational normal curve.



Toric surfaces and threefolds

From the formulas of (D–DR–P 2014) it follows:

1. $X_A \subset \mathbb{P}^n$ a k-regular toric surface. Then

$$\deg[M_{k,2}^{\perp}] = (1 + \tbinom{k+2}{2} + 3\tbinom{k+3}{4})\,\mathrm{Vol} - ((\tbinom{k+2}{3} + 2k(\tbinom{k+3}{4}))E \\ - \tfrac{1}{3}(k^2-4)\tbinom{k+3}{4}V + 4(k^2-1)\tbinom{k+3}{4}.$$



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2. $X_A \subset \mathbb{P}^n$ a 2-regular toric threefold. Then

$$\deg[M^{\perp}_{2,3}] = 103\,\mathrm{Vol} - 88F + 34E - 8V + 76\,\mathrm{Vol}_0 + 60F_0 + 20E_0.$$



Veronese varieties

Let $\nu_d \colon \mathbb{P}^m \to \mathbb{P}^n$ be the dth Veronese embedding and set $X := \nu_d(\mathbb{P}^m)$. Then $\mathcal{P}^k = \mathcal{P}^k_X(1)$ and (Perkinson 1996)

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Hence

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The projective invariance gives $\deg[M^{f,\perp}_{k,m}] = \deg[M^\perp_{k,m}]$ for a generic projection $f\colon X \to \mathbb{P}^r$.



Toric interpolants

Interpolation problem: Given $X \subset \mathbb{P}^n$, find a variety $Y \supset X$ such that the tangent space to Y at (almost) every point $p \in X$ is equal to the kth tangent space to X at p. Such Y is called a kth interpolant.



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Theorem (Dickenstein–Di Rocco–P. 2024)

If $X_A \subset \mathbb{P}^n$ is toric, then there exists a unique toric kth interpolant Y.

Proof.

The row span of the matrix $A^{(k)}$ is the kth osculating space to X at the point $\mathbf{1}$. Then $Y := X_{A^{(k)}}$ is a kth toric interpolant of X, and we show that it is unique. □

The twisted cubic

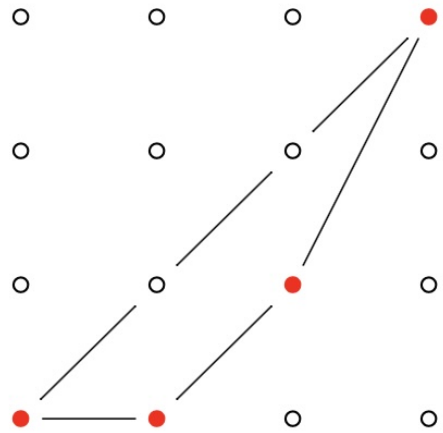
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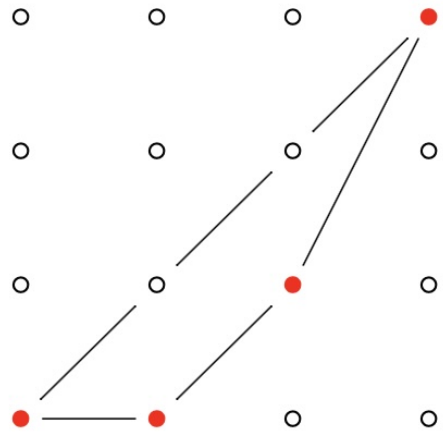
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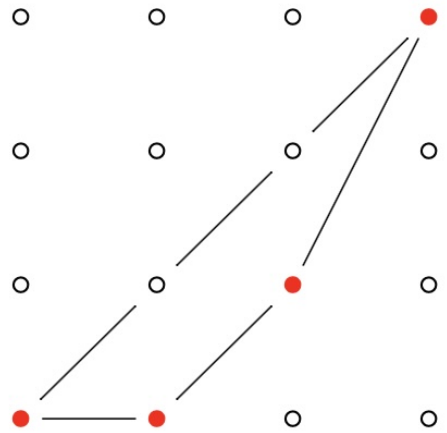


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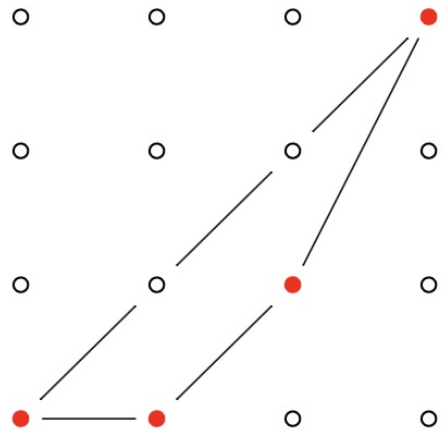
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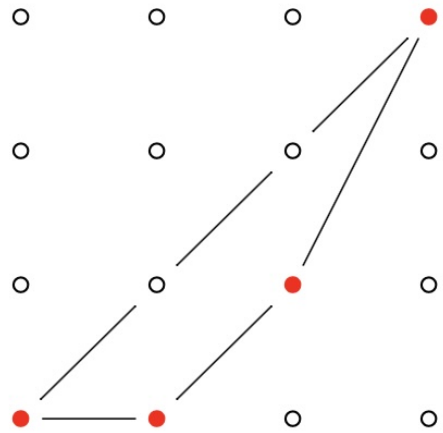
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$(X_A^{(2)})^\vee \subset \mathbb{P}^3$ is the (non toric) tangent developable of X_A.

The twisted cubic

Let $X \subset \mathbb{P}^3$ be the twisted cubic $t \mapsto (1 : t : t^2 : t^3)$.

$$A^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$



$X_{A^{(2)}} \subset \mathbb{P}^3$ is the selfdual toric surface $X_0X_2^3 - X_1^3X_3 = 0$.

$X_A^{(2)} \subset (\mathbb{P}^3)^\vee$ is a twisted cubic, so X_A is 2-selfdual.

$(X_A^{(2)})^\vee \subset \mathbb{P}^3$ is the (non toric) tangent developable of X_A.

$X_{A^{(2)}} \cap (X_A^{(2)})^\vee = 3X_A \cup 2T_{X,(1:0:0:0)} \cup 2T_{X,(0:0:0:1)} \cup S$, where the "shadow curve" S is a twisted cubic (K. Moe 2009).



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Alicia on Guldberg & Waage

Two Norwegians in the XIX-th century

The Law of Mass Action was proposed by Cato Guldberg (1836–1902) and Peter Waage (1833-1900).



Waage was a chemist and Guldberg was a mathematician. They were close friends and brothers in law.

Their work was published in Norwegian in 1862, then in French in 1867, and again in German around 1880, until it was recognized (in the meantime, it was rediscovered by the Dutch physicist van't Hoff.)

Happy birthday, Alicia!























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