Node polynomials for curves on surfaces

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Introduction

Consider a smooth, projective surface S, and a line bundle $\mathcal{L}$.

Let $V \subset H^0(S, \mathcal{L})$ be a linear system on S of dimension $\dim |V| = r$.

Problem: *How many curves with r nodes are in $|V|$?*

The number N_r depends only on the Chern numbers $d = c_1(\mathcal{L})^2$, $k = c_1(\mathcal{L})c_1(\Omega_S)$, $s = c_1(\Omega_S)^2$, $x = c_2(\Omega_S)$ of S and $\mathcal{L}$.

$N_1 = 3d + 2k + x$
$N_2 = \frac{1}{2}(9d^2 + 12dk + 6dx + 4k^2 + x^2 + 4kx - 42d - 39k - 6s + 7x)$
$N_3 = \dots$

A very short history

For $S = \mathbb{P}^2$: $d = m^2$, $k = -3m$, $s = 9$, $x = 3$.

$N_1(m) = 3(m-1)^2$ (Steiner 1848)
$N_2(m) = \frac{3}{2}(m-1)(m-2)(3m^2 - 3m - 11)$ (Cayley 1863)
$N_3(m) = \frac{9}{2}m^6 - 27m^5 + \frac{9}{2}m^4 + \frac{423}{2}m^3 - 229m^2 - \frac{829}{2}m + 525$
(Roberts 1875)

$N_r(m)$ for $r = 4, 5, 6$ Vainsencher 1995.

Recursive formula for all r by Caporaso–Harris 1998.

For *arbitrary* S: Vainsencher $r \leq 6(7)$, Kleiman–Piene $r \leq 8$.

It was natural to conjecture:
N_r is given by a universal polynomial in d, k, s, and x.

Göttsche's conjecture

$$\sum_r N_r t^r = A_1^d A_2^k A_3^s A_4^x$$

where the $A_j \in \mathbb{Q}[[t]]$ are universal power series.

Proved in 2010 by Tzeng and by Kool–Shende–Thomas.

Equivalent formulation:

$$\sum_r N_r t^r = \exp(\sum_i a_i t^i / i!)$$

where the $a_i = a_i(d, k, s, x)$ are linear forms defined by $\log(A_1^d A_2^k A_3^s A_4^x) = d \log A_1 + \ldots = \sum a_i(d, k, s, x) t^i$.

Bell polynomials

E.T. Bell defined recursively polynomials by $P_0 = 1$ and

$$P_{r+1}(a_1, \ldots, a_{r+1}) = \sum_{j=0}^{r} \binom{r}{j} P_{r-j}(a_1, \ldots, a_{r-j}) a_{j+1}.$$

Equivalently, by the formal identity

$$\sum_{r\geq 0} P_r(a_1, \ldots, a_r) t^r / r! = \exp\left(\sum_{i\geq 1} a_i t^i / i!\right).$$

or by

$$P_r(a_1, \ldots, a_r) = \sum_{k_1+2k_2 \cdots + rk_r = r} \frac{r!}{k_1! \cdots k_r!} \left(\frac{a_1}{1!}\right)^{k_1} \cdots \left(\frac{a_r}{r!}\right)^{k_r}$$

Hence we have:

$$N_r = P_r(a_1, \ldots, a_r)/r!$$

where the $a_i = a_i(d, k, s, x)$ are (universal) linear forms. Their coefficients are integers (Kleiman–P for $i \leq 8$, Qviller for all i).

Why Bell polynomials?

- A recursive formula for N_r fits with the Bell polynomials (reminiscent of "derivations" like in the Faà di Bruno formula).
- An intersection theoretic approach on the configuration space S^r, shows that each a_i comes from an intersection class supported on a diagonal, and each product of a_i's to a class on a corresponding polydiagonal (Qviller).

The advantage of knowing the Bell form of the node polynomials N_r is that in order to compute N_{r+1} from the previous N_i's one only needs to compute one new term, a_{r+1}:

$$N_{r+1} = \frac{1}{(r+1)!} \sum_{i=0}^{r-1} \binom{r}{i} (r-i)! N_{r-i}\, a_{i+1} + \frac{1}{(r+1)!} a_{r+1}$$

Faà di Bruno's formula

Let $h(t) = f(g(t))$ be a composed function.

Differentiate once: $h'(t) = f'(g(t))g'(t)$

and twice: $h''(t) = f''(g(t))g'(t)^2 + f'(t)g''(t)$.

Set $h_i = h^{(i)}(t)$, $f_i(t) = f^{(i)}(g(t))$, $g_i = g^{(i)}(t)$. Then

$h_1 = f_1\, g_1$, $h_2 = f_2\, g_1^2 + f_1\, g_2$, $h_3 = f_3\, g_1^3 + 3 f_2\, g_1\, g_2 + f_1\, g_3$,

$$\text{and } h_n = \sum_{k=1}^{n} f_k\, P_{n,k}(g_1, \ldots, g_{n-k+1}),$$

where the $P_{n,k}$ are the *partial* Bell polynomials

$$P_{n,k}(a_1, \ldots, a_{n-k+1}) = \sum \binom{n}{j_1 \cdots j_{n-k+1}} \left(\tfrac{a_1}{1!}\right)^{j_1} \cdots \left(\tfrac{a_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

summing over $\sum i j_i = n$, $\sum j_i = k$.

From $r-1$ to r nodes

Consider the family of curves $D \subset F := S \times Y \to Y$, and a new family

$$\pi' \colon F' \to F = S \times Y,$$

obtained by blowing up the diagonal in $F \times_Y F$.

The new family is a family of surfaces S_x, where S_x is the blow up of S in the point x. Let $X \subset F$ be the set of singular points of the fibers of D; then the r-nodal fibers of $D \to Y$ correspond to the $(r-1)$-nodal fibres $(\pi'^* D - 2E)|_X \to X$.

We get the r-nodal formula u_r by pushing down the $(r-1)$-nodal formula u'_{r-1}.

Derivations

We express $u'_{r-1} = \pi'*u_{r-1} + z_{r-1}$, z_{r-1} a correction class.

Then $\pi'_*(u'_{r-1} \cdot [X]) = u_{r-1}u_1 + \pi'_* z_{r-1}$, since $\pi'_*[X] = u_1$.

This creates a "derivation formula" of the form

$$r u_r = u_{r-1}u_1 + \partial(u_{r-1})$$

Pretend ∂ behaves like a derivation, and set $a_i = \partial^{i-1}(u_1)$. Then

$$2u_2 = u_1^2 + \partial(u_1) = a_1^2 + a_2,$$
$$3!u_3 = (u_1^2 + \partial(u_1))u_1 + \partial(u_1^2 + \partial(u_1)) = a_1^3 + 3a_1a_2 + a_3$$

$$r!u_3 = P_r(u_1, \ldots, \partial^{r-1}(u_1)) = r!P(a_1, \ldots, a_r).$$

Polydiagonals

Let X be a space, and consider

$$X^n = X \times \cdots \times X = \{(x_1, \ldots, x_n) \,|\, x_i \in X\}.$$

By a polydiagonal of type $\mathbf{k} = (k_1, \ldots k_r)$ we mean the subset where k_2 pairs of points of $(x_1, \ldots, x_r)$ are equal, k_3 triples of points of $(x_1, \ldots, x_r)$ are equal, and so on, with $k_1 + 2k_2 + \cdots + rk_r = r$. There are precisely

$$\frac{r!}{k_1! \cdots k_r!} \left(\frac{1}{1!}\right)^{k_1} \cdots \left(\frac{1}{r!}\right)^{k_r}$$

polydiagonals of type $(k_1, \ldots, k_r)$.

This is precisely the coefficient of $a_1^{k_1} \cdots a_r^{k_r}$ in the Bell polynomial.

Intersections on configuration spaces

Let $X \subset D \times Y \subset F = S \times Y$ denote the set of points that are singular on the fibres, and set $\xi = [X]$. Want to compute $p_1^*\xi \cdots p_r^*\xi$ on $F \times_Y \ldots \times_Y F$ *modulo* the equivalences of all polydiagonals (which represent excess intersection).

Let $B^{(i)}$ denote the equivalence of all distinguished varieties whose support is contained in the (small) diagonal $\Delta^{(i)}$.

Then Nikolay Qviller proved:

$$a_i = (-1)^{i-1}(i-1)!p_*B^{(i)},$$

where $p : F \times_Y \ldots \times_Y F \to Y$.

Products of a_i's "correspond" to polydiagonals, again making the Bell polynomials natural in this context.

Plane curves

Di Francesco–Itzykson conjectured in 1994 that the node polynomials $N_r(m)$, in the case of plane curves of degree m, had a particular shape.

The conjecture was refined by Göttsche, and proved by Nikolay Qviller in his 2012 Ph.D. Thesis:

$$N_r(m) = \frac{3^r}{r!} \sum_{\mu=0}^{2r} \frac{1}{\mu! 3^{\lfloor \mu/2 \rfloor}} \frac{r!}{(r - \lceil \mu/2 \rceil)!} Q_\mu(r) m^{2r-\mu},$$

where Q_μ is a polynomial with integer coefficients and degree $\lfloor \mu/2 \rfloor$.

Thank you for your attention!