Envelopes of plane curves: return of the evolute

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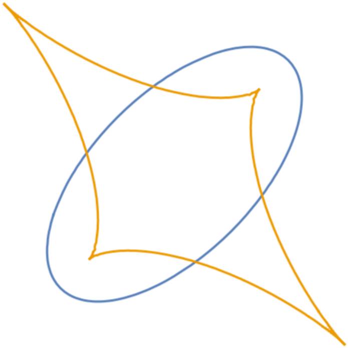
UiO : University of Oslo

An ellipse and its evolute (an astroid)



Apollonius of Perga, 262–200 BC

In his Treatise on Conic Sections, Book V, Apollonius constructs and studies *normals* to a conic – he finds these more interesting than the tangents:

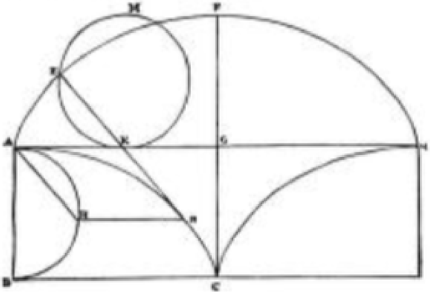
> *In the fifth book I have laid down propositions relating to maximum and minimum straight lines. You must know that our predecessors and contemporaries have only superficially touched upon this investigation of the shortest lines, and have only proved what straight lines touch the sections . . .*

He spends 16 pages finding the number of normals through a given point (the Euclidean distance degree!) and considers the points "where two normals fall together" – the evolute!

Christian Huygens: Horologium Oscillatorium, 1673

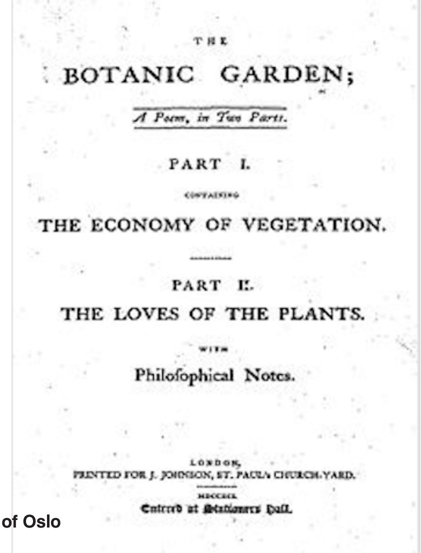
68 CHRISTIANI HVGENII.

Repetita enim figura præcedenti; cum poſt totam ſemicycloidem A B C evolutam, filum occupet rectam C F, quæ dupla eſt A D, propterea quod axes cycloidum A B C, A E F ſunt æquales; apparet ſemicycloidem ipſam A B C, filo ſibi circum applicito æqualem, duplam eſſe ſui axis A D, ac totam proinde cycloidem axis ſui quadruplam.



“The evolute of a cycloid is a cycloid.”

Erasmus Darwin 1791: The Loves of the Plants


THE

BOTANIC GARDEN;

A Poem, in Two Parts.

PART I.

CONTAINING

THE ECONOMY OF VEGETATION.

PART II.

THE LOVES OF THE PLANTS.

WITH

Philoſophical Notes.

LONDON,
PRINTED FOR J. JOHNSON, ST. PAUL'S CHURCH-YARD.

MDCCXCI.

Entered at Stationers Hall.

George Canning, the Anti-Jacobin 1798

THE LOVES OF THE TRIANGLES.

A MATHEMATICAL AND PHILOSOPHICAL POEM.

INSCRIBED TO DR. DARWIN.

Debased, corrupted, groveling, and confined,
No DEFINITIONS touch *your* senseless mind;
To *you* no POSTULATES prefer their claim,
No ardent AXIOMS *your* dull souls inflame;
For *you* no TANGENTS touch, no ANGLES meet,
No CIRCLES join in osculation sweet!

For *me*, ye CISSOIDS, round my temples bend
Your wandering Curves; ye CONCHOIDS extend;
Let playful PENDULES quick vibration feel,
While silent CYCLOIS rests upon her wheel;

Footnotes

Ver. 9. *Tangents*-- So called from touching, because they touch Circles, and never cut them.

Ver. 10.—*Circles*—See Chambers's Dictionary, Article Circle.

Ditto. *Osculation*—For the *Osculation*, or kissing of Circles and other Curves, see Huygens, who has veiled this delicate and inflammatory subject in the decent obscurity of a learned language.

Ver. 11. *Cissois*--A Curve supposed to resemble the sprig of ivy, from which it has its name, and therefore peculiarly adapted to poetry.

Envelopes of families of lines

Consider a plane curve $C_0 \subset \mathbb{P}(V) \cong \mathbb{P}^2$ of degree d, with normalization $\nu : C \to C_0$. Given a 2-quotient

$$V_C \to \mathcal{F},$$

we get a family of lines

$$\varphi : \mathbb{P}(\mathcal{F}) \subset C \times \mathbb{P}(V) \to C.$$

The *envelope* of φ is the branch locus (discriminant) of

$$\psi : \mathbb{P}(\mathcal{F}) \subset C \times \mathbb{P}(V) \to \mathbb{P}(V).$$

The envelope of the tangents

The set of tangents to C is a family of lines, parameterized by the curve, with total space

$$Z = \mathbb{P}(\mathcal{P}) \subset C \times \mathbb{P}^2,$$

where $V_C \to \nu^* \mathcal{P}^1_{C_0}(1) \to \mathcal{P}$ is the Nash quotient.

Let $\varphi : Z \to C$ and $\psi : Z \to \mathbb{P}^2$ denote the projections.
Then $\psi(\varphi^{-1}(p))$ is the tangent to C at $p \in C$.
The map ψ is a finite cover of $\mathbb{P}^2$ of degree

$$d^\vee = \#\psi^{-1}(q) = \#\{p \in C \,|\, q \in T_{C,p}\}.$$

If $q \in C$, then $\#\psi^{-1}(q) = d^\vee - 1$.

Hence C is (part of) the branch locus of the map ψ.

The Euclidean normal bundle

Fix a line $L_\infty = \mathbb{P}(V') \subset \mathbb{P}(V)$ and a quadric $Q_\infty \subset L_\infty$.

Set $\mathcal{K} := \operatorname{Ker}(V_C \to \mathcal{P})$. The sum of the maps

$$V_C \to V'_C \cong V'^{\vee}_C \to \mathcal{K}^\vee \text{ and } V_C \to \mathcal{O}_C(1)$$

whose fiber at a general point $p \in C$ is the *Euclidean normal*: let $T_p \cap L_\infty = \{q\}$ and let $q^\perp \in L_\infty$ w.r.t Q_∞. Then (by definition), $N_p := \langle p, q^\perp \rangle$ is the line perpendicular to T_p.

The *Euclidean normal bundle* is the image of the map

$$V_C \to \mathcal{E} \subseteq \mathcal{K}^\vee \oplus \mathcal{O}_C(1).$$

Example: $\mathbb{P}(V) = \mathbb{P}^2_{\mathbb{C}}$, $L_\infty = \{z = 0\}$, and

$$Q_\infty = \{(1 : i : 0), (1 : -i : 0)\} \text{ (the circular points)}$$

gives Euclidean geometry in $\mathbb{R}^2 \subset \mathbb{C}^2 = \mathbb{P}^2_{\mathbb{C}} \setminus L_\infty$.

Evolutes

The *evolute* E_C of C is the envelope of the family of normals $\mathbb{P}(\mathcal{E}) \to C$ (cf. Apollonius).

The evolute is a *caustic*.

The *curve of normals* $N_C \subset \mathbb{P}(V)^\vee$ is the image of the morphism $C \to \mathbb{P}(V)^\vee$ given by the one-quotient

$$V_C^\vee \cong \wedge^2 V_C \to \wedge^2 \mathcal{E}.$$

The normals to C are the tangents to E_C:

Proposition

The curve of normals is equal to the dual curve of the evolute:

$$N_C = E_C^\vee \subset \mathbb{P}(V)^\vee.$$

Evolutes as centers of curvature

Choose coordinates in $\mathbb{P}(V)$ such that $L_\infty = \{z = 0\}$ and $Q_\infty = \{x^2 + y^2 = 0\}$. Let $(x(t), y(t))$ be a local parameterization of $C_0 \cap (\mathbb{P}(V) \setminus L_\infty = \mathbb{C}^2$. The curve of normals:

$$V_C^\vee \cong \bigwedge^2 V_C \to \bigwedge^2 \begin{pmatrix} x(t) & y(t) & 1 \\ y'(t) & -x'(t) & 0 \end{pmatrix} = (x' : y' : -xx' - yy').$$

The evolute is obtained by taking the dual of the curve of normals:

$$V_C \cong \bigwedge^2 V_C^\vee \to \bigwedge^2 \begin{pmatrix} x' & y' & -xx' - yy' \\ x'' & y'' & -xx'' - x'^2 - yy'' - y'^2 \end{pmatrix},$$

which gives the *center of curvature*:

$$\left(x - \frac{y'(x'^2+y'^2)}{x'y''-x''y'} : y + \frac{x'(x'^2+y'^2)}{x'y''-x''y'} : 1\right).$$

Numerical characters

Let $C \to C_0 \subset \mathbb{P}(V)$ be a complex plane curve. Given a line L_∞ and a quadric $Q_\infty \subset L_\infty$. If $p \in C_0 \cap Q_\infty$ but $T_p \neq L_\infty$, then the evolute has an inflectional tangent through p. If $T_q = L_\infty$ for some $q \in C \cap L_\infty$, then the evolute has an inflection point on L_∞. Assume the intersections of C with L_∞ are transversal or tangential. Then (as in Salmon 1852)

$$\deg N_C = d + d^\vee - \iota_E$$

$$\deg E_C = 3d + \iota - 3\iota_E = 3d^\vee + \kappa - 3\iota_E$$

$$\kappa_E = 6d - 3d^\vee + 3\iota - 5\iota_E$$

where ι_E is the number of inflection points on E_C. The formulas can be adjusted to more special situations.

Brusotti's theorem (1921)

Let us now turn to the discussion of real algebraic curves and their evolutes and curves of normals. We will need some tools.

Theorem

Any real-algebraic curve $\Gamma \subset \mathbb{R}^2$ with only nodes as singularities admits a small real deformation of the same degree which realizes any independently prescribed smoothing types of all its crunodes.

Klein and Klein–Schuh

Theorem (Klein)

If a real algebraic plane curve C of degree d and class $d^\vee$ has no other point or tangent singularities than nodes, cusps, bitangents, and inflectional tangents, then

$$d + 2\tau^{\mathrm{ac}} + i_{\mathbb{R}} = d^\vee + 2\delta^{\mathrm{ac}} + \kappa_{\mathbb{R}}.$$

Theorem (Klein–Schuh)

Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a curve of degree d and class $d^\vee$. Then

$$d - d^\vee = \sum_{p \in C(\mathbb{R})} (m_p(C) - r_p(C)) - \sum_{q \in C^\vee(\mathbb{R})} (m_q(C^\vee) - r_q(C^\vee)),$$

where m_p is the multiplicity and r_p the number of real branches.

The $\mathbb{R}$-degree of a curve

The $\mathbb{R}$-degree of a curve $D \subset \mathbb{R}^2$ is

$$\mathbb{R}\deg(D) := \sup_L \#\,(D \cap L)$$

where $L \subset \mathbb{R}^2$ are all lines intersecting D transversally.

Problem 1: What are

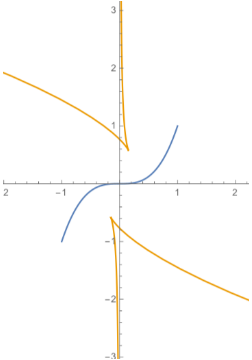
$$e(d) := \max_\Gamma \mathbb{R}\deg(E_\Gamma) \text{ and } n(d) := \max_\Gamma \mathbb{R}\deg(N_\Gamma),$$

where the maximum is taken over all curves $\Gamma \subset \mathbb{R}^2$ of degree d.

Answer: $e(d) \geq d(d-2)$ and $n(d) = d^2$

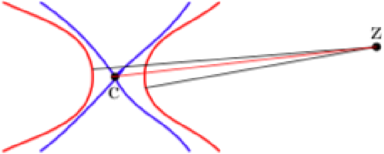
Proof for the $\mathbb{R}$-degree of the evolute

By Klein's theorem, a (smooth) curve of degree d has $\leq d(d-2)$ real inflection points, and the bound is sharp. Take Γ of degree d with $d(d-2)$ real inflection points. Each inflection point gives a point of the evolute on the line at infinity. By a small deformation, we get a real line in $\mathbb{R}^2$ intersecting the evolute transversally in $d(d-2)$ points.



Proof for the $\mathbb{R}$-degree of the curve of normals

Consider an arrangement $\mathcal{A} \subset \mathbb{R}^2$ of d lines in general position and a point $z \notin \mathcal{A}$. By Brusotti, we can smoothen all $d(d-1)/2$ nodes of $\mathcal{A}$ as in the figure. For each node, we get two normals (in black) through z. Additionally, d normals are obtained by deforming each altitude from z to a line in $\mathcal{A}$. This give d^2 normals through z to the smooth curve.



Vertices

A *vertex* of a plane curve plane curve $\Gamma \subset \mathbb{R}^2$ is a critical point of the curvature function, i.e., a cusp of the evolute $E_\Gamma \cap \mathbb{R}^2$.

Problem 2: What is

$$v(d) := \max_\Gamma \kappa_{\mathbb{R}}(E_\Gamma),$$

where the maximum is taken over all curves $\Gamma \subset \mathbb{R}^2$ of degree d.

Answer: $v(d) \geq d(2d-3)$

For complex cusps: $2d(3d-5)$.

Diameters

A *diameter* of a plane curve $\Gamma \subset \mathbb{R}^2$ is a line L which is the normal to Γ at two distinct points. The diameters are double points of the curve of normals N_Γ.

Problem 3: What is

$$\delta_N(d) := \max_\Gamma \delta^{\mathrm{cru}}(N_\Gamma),$$

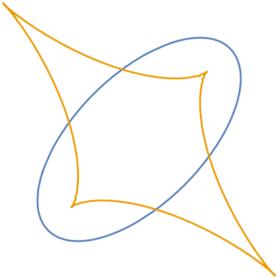
where the maximum is taken over all curves $\Gamma \subset \mathbb{R}^2$ of degree d?

Answer: $\delta_N(d) \geq \frac{1}{2}d^4 - d^3 + \frac{1}{2}d$ and conjecturally (for $d \geq 3$),

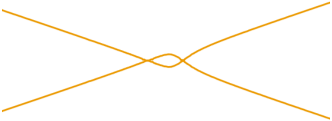
$$\delta_N(d) \leq \frac{1}{2}d^4 - 3d^2 + \frac{5}{2}d.$$

For complex nodes: $\frac{1}{2}d^4 - \frac{5}{2}d^2 + 2d$ – the number of bottle necks!

An ellipse, its evolute and its curve of normals



Vertices: $\kappa_{\mathbb{R}}(E_\Gamma) = 4$



Diameters: $\delta^{\mathrm{cru}}(N_\Gamma) = 2$

Crunodes of the evolute

For a given curve $\Gamma \subset \mathbb{R}^2$, how many points in the plane are the centers of more than one circle of curvature, i.e., how many crunodes does the evolute have?

Problem 4: What is

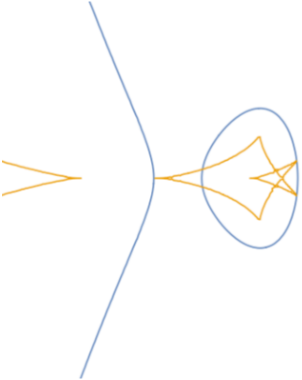
$$c(d) := \max_\Gamma \delta^{\text{cru}}(E_\Gamma),$$

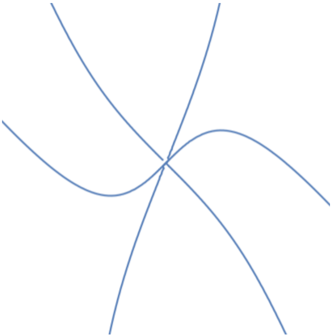
where the maximum is taken over all curves $\Gamma \subset \mathbb{R}^2$ of degree d?

Answer: $c(d) \geq \binom{d(d-3)+1}{2} = \frac{1}{2}d^4 - 3d^3 + 5d^2 - \frac{3}{2}d$

For complex nodes: $\frac{9}{2}d^4 - 9d^3 - \frac{13}{2}d^2 + 15d$

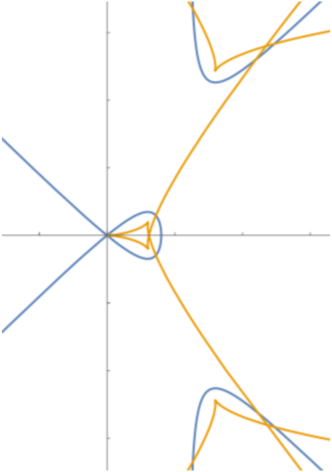
The cubic $y^2 + x(x-2)(x+1) = 0$

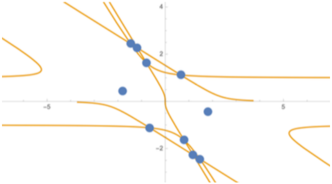




$\deg \Gamma = 3$, $\deg \Gamma^\vee = 6$, $\deg E_\Gamma = 3 \cdot 6 - 3 \cdot 2 = 12$,
$\deg N_\Gamma = 3 + 6 - 2 = 7$

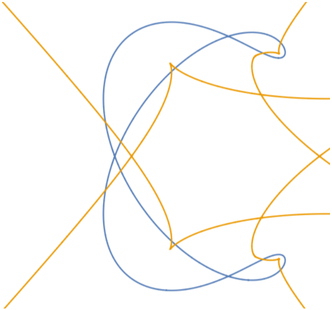
The nodal cubic $5(x^2 - y^2)(x-1) + (x^2+y^2) = 0$

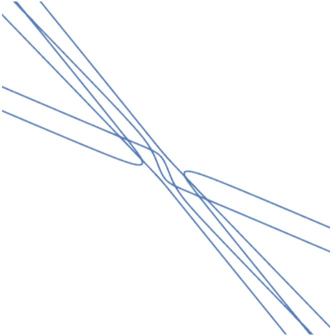




$$\deg \Gamma = 3,\ \deg \Gamma^\vee = 4,$$
$$\deg E_\Gamma = 3 \cdot 4 = 12,$$
$$\deg N_\Gamma = 3 + 4 = 7$$

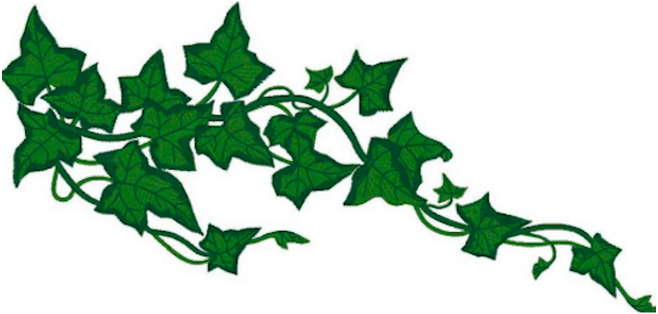
The ampersand, its evolute & its curve of normals





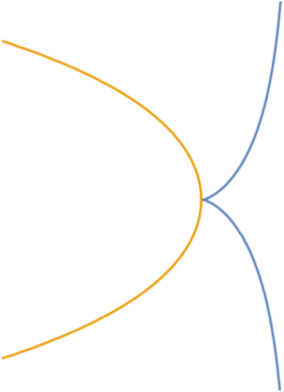
$$\deg \Gamma = 4,\ \deg \Gamma^\vee = 6,\ \deg E_\Gamma = 3 \cdot 6 = 18,\ \deg N_\Gamma = 4 + 6 = 10$$

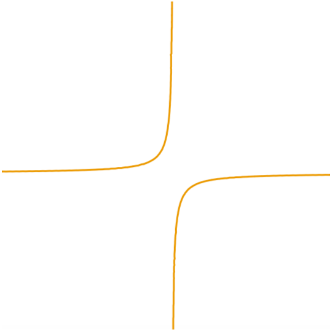
Ivy ($\kappa\iota\sigma\sigma\acute{o}\varsigma$)



Ver. 11. *Cissois*——A Curve supposed to resemble the sprig of ivy, from which it has its name, and therefore peculiarly adapted to poetry.

The cissoid of Diocles $(x^2 + y^2)x - 4y^2 = 0$

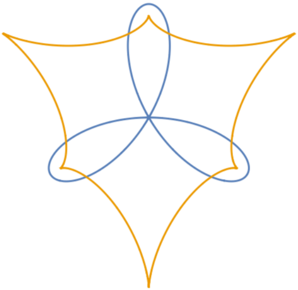


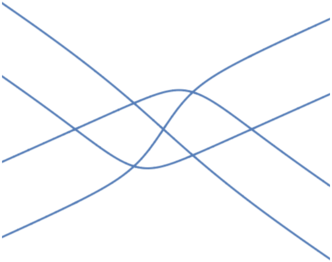


$\deg \Gamma = 3$, $\deg \Gamma^\vee = 3$, $\deg E_\Gamma = 3 \cdot 3 + 1 - 3 \cdot 2 = 4$,
$\deg N_\Gamma = 3 + 3 - 2 = 4$

Trifolium pratense

The trifolium $(x^2+y^2)^2 - x^3 + 3xy^2 = 0$

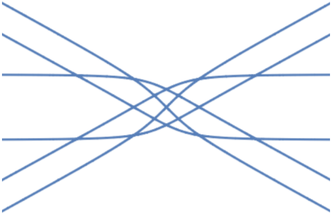




$\deg \Gamma = 4$, $\deg \Gamma^\vee = 6$, $\deg E_\Gamma = 10$, $\deg N_\Gamma = 6$

The fourleafed clover

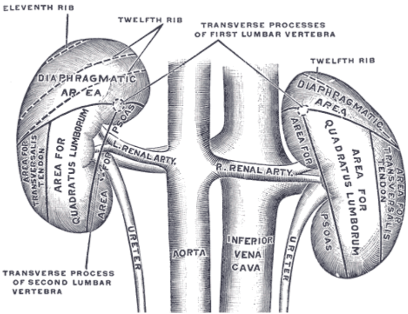
The quadrifolium $(x^2 + y^2)^3 - 4x^2y^2 = 0$



$\deg \Gamma = 6$, $\deg E_\Gamma = 14$, $\deg N_\Gamma = 8$

Nephrology

"The nephron is the minute or microscopic structural and functional unit of the kidney."



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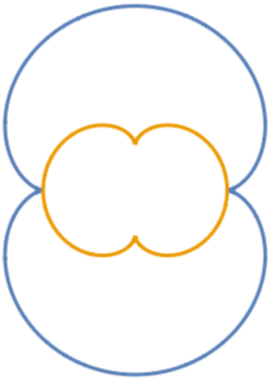
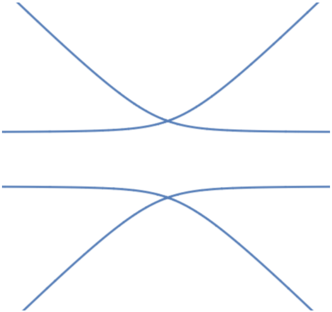
The nephroid $4(x^2 + y^2 - 1)^3 - 27y^2 = 0$

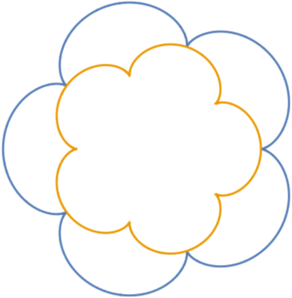


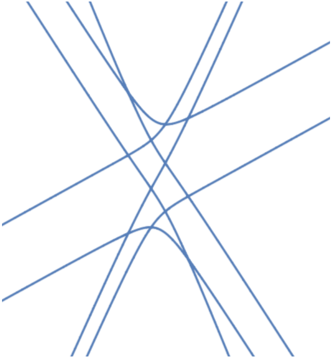


$\deg \Gamma = 6$, $\deg E_\Gamma = 6$, $\deg N_\Gamma = 4$

Ranunculus

The ranunculoid $x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + \cdots = 0$





$\deg \Gamma = 12,\ \deg E_\Gamma = 12,\ \deg N_\Gamma = 7$

THANK YOU FOR YOUR ATTENTION!