Polar varieties and Euclidean distance degree

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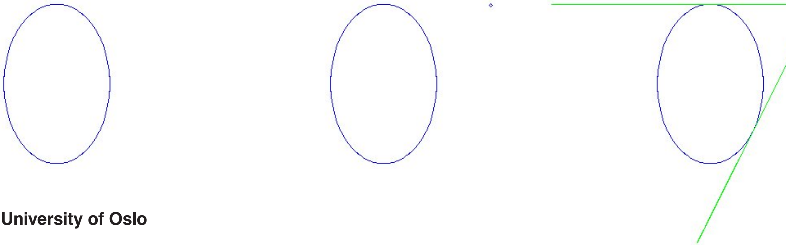
Pole and polars in the plane

Let Q be a conic section. Let P be a point in the plane. There are two tangents to Q passing through P. The polar of P is the line joining the two points of tangency. Conversely, if L is a line, it intersects the conic in two points. The pole of L is the intersection of the tangents to Q at these two points.



Polarity (reciprocation)

A quadric hypersurface Q in $\mathbb{P}^n$ given by a quadratic form q, sets up a polarity between points and hyperplanes:

$$P = (b_0 : \cdots : b_n) \mapsto P^\perp = H\colon \sum b_i \frac{\partial q}{\partial X_i} = 0$$

The polar hyperplane $P^\perp$ of P is the linear span of the points on Q such that the tangent hyperplane at that point contains P.

If $P \in Q$, then $P^\perp = T_P Q$.

If $L \subset \mathbb{P}^m$ is a linear space, then $L^\perp = \cap_{P \in Q \cap L} T_P Q$.

If H is a hyperplane, its pole $H^\perp$ is the intersection of the tangent hyperplanes to Q at the points of intersection with H.

Example

If the quadric is $q = \sum X_i^2$, then the polar of $P = (b_0 : \cdots : b_m)$ is the hyperplane $P^\perp : b_0X_0 + \ldots + b_nX_n = 0$.

The polar of the hyperplane $H : b_0X_0 + \ldots + b_nX_n = 0$ is the point $H^\perp = (b_0 : \cdots : b_m)$.

Polar varieties and Chern classes

Let $L_k \subset \mathbb{P}^n$ be a linear subspace of codimension $m - k + 2$.

The *kth polar variety* of of $X \subset \mathbb{P}^n$ with respect to L_k is

$$M_k := \{x \in X \mid \dim(T_{X,x} \cap L_k) \geq k - 1\}.$$

Its class is $[M_k] = c_k(\mathcal{P}^1_X(1)) \cap [X]$, hence we get the Todd–Eger relation

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} c_i(T_X) \cap [X], \qquad (1)$$

where $h = c_1(\mathcal{O}_X(1))$ is the class of a hyperplane.

Singular varieties and Nash transform

If X is singular, we take its Nash transform $\pi : \overline{X} \to X$ and replace Ω^1_X by the Nash bundle Ω on $\overline{X}$.

The Mather–Chern classes of X are

$$c_i^M(X) = \pi_*(c_i(\Omega^\vee) \cap [\overline{X}]),$$

and we get for the polar varieties:

$$[M_k] = \sum_{i=0}^{k} (-1)^i \binom{m-i+1}{m-k+1} h^{k-i} \cap c_i^M(X). \tag{2}$$

The Euclidean normal bundle (Catanese–Trifogli)

Consider $X \subset \mathbb{P}^n = \mathbb{P}(V)$, fix a hyperplane $H_\infty \subset \mathbb{P}^n$ at infinity and a smooth quadric Q in H_∞.

Use the polarity in $H_\infty \cong \mathbb{P}^{n-1} = \mathbb{P}(V')$ induced by Q to define *Euclidean normal spaces* at each smooth point $P \in X \setminus H_\infty$:

$$N_P X = \langle P, (T_P X \cap H_\infty)^\perp \rangle$$

Consider $0 \to V'' \to V \to V' \to 0$ $(\dim V'' = 1)$ and

$$0 \to \mathcal{N}_X(1) \to V_X \to \mathcal{P}^1_X(1) \to 0.$$

Assume (transversality of tangent spaces and H_∞) this induces

$$0 \to V'' \to \mathcal{P}^1_X(1) \to \mathcal{P} \to 0.$$

Then we get

$$0 \to \mathcal{N}_X(1) \to V'_X \to \mathcal{P} \to 0.$$

The polarity in H_∞ given by Q, gives $V'^\vee \cong V'$, so we have

$$V'_X \cong V'^\vee_X \to \mathcal{N}_X(1)^\vee$$

corresponding to the spaces perpendicular to the spaces $T_P X \cap H_\infty$, and combining with $V_X \to V'_X$ and $V_X \to \mathcal{O}_X(1)$, we get

$$V_X \to \mathcal{N}_X^\vee(-1) \oplus \mathcal{O}_X(1)$$

whose fibers correspond to the Euclidean normal spaces $N_P X$.

We call $\mathcal{E} := \mathcal{N}_X^\vee(-1) \oplus \mathcal{O}_X(1)$ the *Euclidean normal bundle* of X.

Reciprocal polar varieties

Instead of imposing conditions on the tangent spaces of a variety, one can similarly impose conditions on the Euclidean normal spaces.

Let $L \subset \mathbb{P}^n$ have codimension w, $n - m \leq w \leq n$. Set $k = w - (n - m)$ and define *reciprocal polar varieties*

$$M_k(L)^\perp = \{P \in X | N_P(X) \cap L \neq \emptyset\}.$$

Then (Porteous' formula) $M_k^\perp$ have classes

$$[M_k^\perp] = s_k(\mathcal{E}) \cap [X] = [s(\mathcal{N}_X^\vee(-1))s(\mathcal{O}_X(1))]_k \cap [X],$$

hence

$$[M_k^\perp] = \sum_{i=0}^{k} c_1(\mathcal{O}_X(1))^{k-i} \cap [M_i].$$

Applications

Polar and reciprocal polar varieties have been applied to study

- singularities (Lê–Teissier, Merle, . . .)
- the topology of real affine varieties (Bank, Giusti, Heinz et al., Safey El Din–Schost)
- real solutions of polynomial equations (Giusti, Heinz, et al.)
- complexity questions (Bürgisser–Lotz)
- foliations (Soares, . . .)
- Euclidean distance degree (Draisma et al.)
- finding nonsingular points on every component of a real affine plane curve (Banks et al. for smooth curves, Mork–P for compact curves with ordinary multiple points, counterexamples for curves with worse singularities).

The Euclidean endpoint map

Consider $\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n$.

Let $p\colon \mathbb{P}(\mathcal{E}) \to X$ and $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ denote the projections on the first and second factor. The map q is called the *endpoint map*.

Let $A \in \mathbb{P}^n \setminus H_\infty$. Then $p(q^{-1}(A))$ is a reciprocal polar variety:

$$p(q^{-1}(A)) = \{P \in X \mid A \in N_P X\} = M_m(A)^\perp.$$

Hence:

$$\deg q = \deg M_m^\perp = \sum_{k=0}^{m} \deg M_k.$$

Euclidean distance degree

The degree of the endpoint map $q\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$ is also called the *Euclidean distance degree*[1]:

$$\operatorname{E}\deg X = \deg q = \deg s_m(\mathcal{E}) = \sum_{k=0}^{m} \deg M_k.$$

The points in $M_m(A)^\perp$ are the points $P \in X$ where the line $\langle P, A\rangle$ is perpendicular to the tangent space $T_P X$. Hence they are max/min points for the "distance function" induced by the perpendicularity (defined by the quadric $Q \subset H_\infty$).

[1] Draisma–Horobet–Ottaviani–Sturmfels–Thomas.

Hypersurfaces with isolated singularities

If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d, then $\deg M_k = d(d-1)^k$.

If X has only isolated singularities, then only $\deg M_{n-1}$ is affected, and we get (from Teissier's formula and the Plücker formula for hypersurfaces with isolated singularities (Teissier, Laumon))

$$\mathrm{E}\deg X = \frac{d((d-1)^n - 1)}{d-2} - \sum_{P \in \mathrm{Sing}(X)} (\mu_P^{(n)} + \mu_P^{(n-1)}),$$

where $\mu_P^{(n)}$ is the Milnor number and $\mu_P^{(n-1)}$ is the sectional Milnor number of X at P.

Surface with ordinary singularities

Assume $X \subset \mathbb{P}^3$ is a generic projection of a smooth surface of degree d, so that X has *ordinary* singularities: a double curve of degree ϵ, t triple points, and ν_2 pinch points. Then (using known formulas for $\deg M_1$ and $\deg M_2$)

$$\mathrm{E}\deg X = \deg X + \deg M_1 + \deg M_2 = d^3 - d^2 + d - (3d-2)\epsilon - 3t - 2\nu_2.$$

Example

The Roman Steiner surface: $d = 4$, $\epsilon = 3$, $t = 1$, $\nu_2 = 6$

$$\mathrm{E}\deg X = 7$$

The focal locus

The focal locus Σ_X is the branch locus of the map q.

It is the image of the subscheme R_X given by the Fitting ideal $F^0(\Omega^1_{\mathbb{P}(\mathcal{E})/\mathbb{P}^n})$, so its class is

$$[\Sigma_X] = q_*((c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^*c_1(\Omega^1_{\mathbb{P}^n})\big) \cap [\mathbb{P}(\mathcal{E})]).$$

Example

$X \subset \mathbb{P}^2$ is a (general) plane curve of degree d. Then the focal locus is the *evolute* (or caustic) of X. Its degree is the degree of the class

$$q_*\big((c_1(\Omega^1_{\mathbb{P}(\mathcal{E})}) - q^*c_1(\Omega^1_{\mathbb{P}^2})\big)c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cap [\mathbb{P}(\mathcal{E})]\big)$$

which one can compute: $\deg \Sigma_X = 3d(d-1)$.

In the case that X is a "Plücker curve" of degree d having only δ nodes and κ ordinary cusps as singularities, then we obtain the classical formula due to Salmon

$$\deg \Sigma_X = 3d(d-1) - 6\delta - 8\kappa.$$

The focal locus of a hypersurface

Let $X \subset \mathbb{P}^n$ be a general hypersurface $(m = n-1)$ of degree d. It is known that in this case $R_X \to \Sigma_X$ is birational. We compute

$$\deg \Sigma_X = (n-1) \deg M_{n-1} + 2(d-1) \sum_{i=0}^{n-2} \deg M_i.$$

For a smooth hypersurface of degree d in $\mathbb{P}^n$, we have $\mu_i = d(d-1)^i$. Hence

$$\deg \Sigma_X = (n-1)d(d-1)^{n-1} + 2d(d-1)((d-1)^{n-1} - 1)(d-2)^{-1},$$

which checks with the formula found by Trifogli.

Toric varieties

The Chern–Mather class of a toric variety X is equal to

$$c^{\mathrm{M}}(X) = \sum_{\alpha} \mathrm{Eu}_X(X_\alpha)[\overline{X}_\alpha],$$

where the sum is taken over all orbits X_α of the torus action on X, and where $\mathrm{Eu}_X(X_\alpha)$ denotes the value of the local Euler obstruction of X at a point in the orbit X_α.

Hence the degrees of the polar and reciprocal classes can be expressed in terms of the Euler obstructions and the volumes of the torus orbits, cf. the talks of Martin Helmer and Bernt-Ivar U. Nødland.

Thanks for your attention!