

Some counting problems and their generating functions

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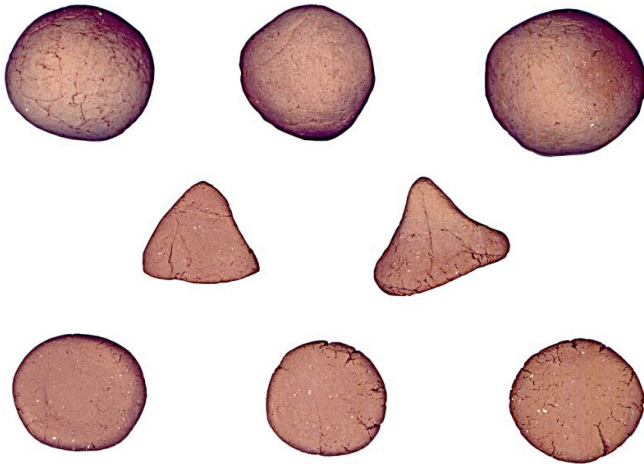


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Counting



MS 5067/1-8

Neolithic plain counting tokens. Near East, ca. 8000--3500 BC



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A counting problem

Let n be an integer. Count the way you can write n as a sum of two positive integers. Or as a sum of three? ... How many ways altogether?

$$1$$

$$2 = 1 + 1$$

$$3 = 2 + 1 = 1 + 1 + 1$$

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 \\ = 1 + 1 + 1 + 1 + 1$$

Get a sequence of integers:

$$p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, \dots$$



$$7 = 4 + 3$$



MS 4647

Numbers 3+4. Sumer, ca. 3500-3200 BC



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Partitions of an integer

Let $p(n)$ denote the number of partitions of the integer n , i.e., $p(n)$ is the number of ways we can write $n = n_1 + n_2 + \dots$, where $n_1 \geq n_2 \geq \dots > 0$.

Then $p(n)$ is clearly the coefficient of x^n in the power series expansion of the product

$$(1+x+x^2+\dots)(1+x^2+(x^2)^2+\dots)\dots(1+x^m+(x^m)^2+\dots)\dots$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \dots$$

The *generating function* of integer partitions is

$$P(x) = \sum_{n \geq 0} p(n) x^n = \prod_{m \geq 1} (1 - x^m)^{-1}.$$



“A generating function is a clothesline on which we hang up a sequence of numbers for display.”

H.S.Wilf
generatingfunctionology
Academic Press Inc., 1990



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Plane partitions

If we instead consider

$$(1+x+x^2+\dots)(1+x^2+(x^2)^2+\dots)^2 \dots (1+x^m+(x^m)^2+\dots)^m \dots$$

it turns out that the coefficient of x^n in the expansion is the number $\pi(n)$ of plane partitions (or 3D Young diagrams) of n .

The generating function for this problem is the **MacMahon** function

$$\sum \pi(n)x^n = \prod_{m \geq 1} (1 - x^m)^{-m} =: M(x),$$

$$M(x) = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \dots$$

Modulo x^4 :

$$\begin{aligned}M(x) &\equiv (1 + x + x^2 + x^3)(1 + x^2)^2(1 + x^3)^3 \\&= 1 + x + x^2 + 2x^2 + x^3 + x \cdot 2x^2 + 3x^3\end{aligned}$$

$$x \longleftrightarrow 1$$

$$x^2 \longleftrightarrow 2$$

$$2x^2 \longleftrightarrow 1 \mid 1 \text{ and } \frac{1}{1}$$

$$x^3 \longleftrightarrow 3$$

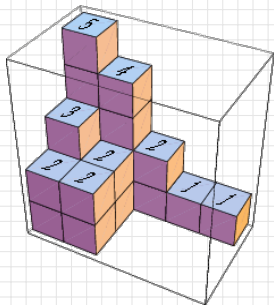
$$x \cdot 2x^2 \longleftrightarrow 2 \mid 1 \text{ and } \frac{2}{1}$$

$$3x^3 \longleftrightarrow 1 \mid 1 \mid 1 \text{ and } \frac{1}{1} \mid \frac{1}{1} \text{ and } \frac{1}{1}$$



A plane partition of 22

5	4	2	1	1
3	2			
2	2			



Enumerative geometry

Unlike the case of enumerative combinatorics, generating functions did not use to play a prominent role in enumerative geometry. One reason is that there often are not natural sequences that need to be displayed, rather one looks for specific numbers, or general formulas.



Storm P. on H.G. Zeuthen

- What is it about?
- It is about the theory of algebraic curves and surfaces, from which enumerative geometry originated!



— Hvad er det angaaende?
— Det er angaaende de algebraiske
Kurvers og Fladers Teori, hvorved
Antalgeometrien er opstaaet!



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Enumerative geometry

Unlike the case of enumerative combinatorics, generating functions did not use to play a prominent role in enumerative geometry. One reason is that there often are not natural sequences that need to be displayed, rather one looks for specific numbers, or general formulas.

Schubert calculus: Enumerate linear spaces in a given projective space satisfying a certain number of incidence conditions with other linear spaces.

Example

How many lines intersect four given lines in $\mathbb{P}^3$?

Grassmann varieties

More generally: how many lines meet $2n$ general $(n - 1)$ -planes in projective $(n + 1)$ -space?

Schubert calculus gives the answer

$$\frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \binom{2n}{n}.$$

How can we display the sequence

1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ...?

Catalan numbers

The n th *Catalan number* is

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

They satisfy the convolution formula

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

From this one deduces the equation $xC(x)^2 - C(x) + 1 = 0$ for the generating function for the line counting problem:

$$C(x) := \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

The Catalan numbers also count:

- ▶ The number of ways to insert n pairs of parentheses in a word of $n + 1$ letters.

For $n = 3$ there are 5 ways:

$((ab)(cd)), (((ab)c)d), ((a(bc))d), (a((bc)d)), (a(b(cd)))$.

- ▶ The ways of joining $2n$ points on a circle to form n nonintersecting chords.





MS 3049
Properties of chords of circles, in the Sumerian sexagesimal system.
Babylonia, ca. 17th c. BC



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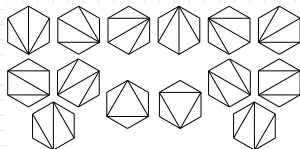
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- ▶ The ways of joining $2n$ points on a circle to form n nonintersecting chords.
- ▶ The number of ways to triangulate a regular $(n + 2)$ -gon.
For $n = 4$:



Recursions

If a sequence of numbers is defined recursively, we might be able to find the generating functions, and maybe even a closed formula for the numbers.

Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Then

$$F(x) := \sum_{n \geq 1} F_n x^n = x + \sum_{n \geq 2} F_{n-1} x^n + \sum_{n \geq 2} F_{n-2} x^n$$

Hence the generating function for the Fibonacci numbers is

$$F(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - (1 - \varphi)x} \right)$$

and $F_n = \frac{1}{\sqrt{5}} (\varphi^n - (1 - \varphi)^n)$, where $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio.

Of course, it is easier to just solve the second order homogeneous difference equation

$$F_n - F_{n-1} - F_{n-2} = 0$$

Its characteristic equation is

$$r^2 - r - 1 = 0$$

which has roots φ and $1 - \varphi$.

The initial conditions $F_0 = 0$ and $F_1 = 1$ give the desired solution

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (1 - \varphi)^n)$$

Bell numbers

Consider the numbers defined recursively by

$$B_0 = 1 \text{ and } B_{n+1} = \sum_{i=0}^n \binom{n}{i} B_i$$

Consider the *exponential* generating function

$$B(x) := \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

Differentiating gives

$$B'(x) = \sum_{n=1}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} B_i \frac{x^n}{n!}$$



...and their generating function

Hence $B'(x)$ is a product

$$B'(x) = \sum_{n,i} B_i \frac{x^i}{i!} \frac{x^{n-i}}{(n-i)!} = B(x) \sum_j \frac{x^j}{j!} = B(x) e^x.$$

To find $B(x)$ we solve the differential equation

$$\frac{B'(x)}{B(x)} = e^x,$$

with initial condition $B(0) = 1$, and we get

$$B(x) = e^{e^x - 1}.$$



What do the Bell numbers count?

THE NUMBER OF PARTITIONS OF A SET

GIAN-CARLO ROTA, Massachusetts Institute of Technology

Let S be a finite nonempty set with n elements. A *partition* of S is a family of disjoint subsets of S called “blocks” whose union is S . The number B_n of distinct partitions of S has been the object of several arithmetical and combinatorial investigations. The earliest occurrence in print of these numbers has never been traced; as expected, the numbers have been attributed to Euler, but an explicit reference to Euler has not been given, and Bell [7] doubts that it can

1964]

THE NUMBER OF PARTITIONS OF A SET

499

be found in Euler’s work. The properties of these numbers are periodically being rediscovered, as recently as 1962 (cf. [13]). Following Eric Temple Bell, we shall call them the *exponential numbers*. Bell [4, 5, 6, 7], used the notation e_n ; on the other hand, Jacques Touchard [29 and 30] used a_n to celebrate the birth of his daughter Ann; Becker and Riordan [3] used B_n in honor of Bell. We shall follow their choice.

A great many problems of enumeration can be interpreted as counting the number of partitions of a finite set; for example, the number of rhyme schemes of length n , the number of ways of distributing n distinct objects into k boxes (empty boxes permitted), the number of equivalence re-



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Polydiagonals

Let X be a space, and consider the set of ordered n -tuples of elements of X , denoted $X \times \cdots \times X = X^n$. For each $k \leq n$ and each division $\mathbf{i}_k$ of $\{1, 2, \dots, n\}$ into k disjoint subsets $\{i_1^j, i_2^j, \dots, i_{l(j)}^j\}$, $j = 1, \dots, k$, $\sum l(j) = n$, the corresponding *polydiagonal* is defined as

$$\Delta(\mathbf{i}_k) = \{(x_1, \dots, x_n) \mid x_{i_r^j} = x_{i_s^j} \text{ for all } r, s, j\}$$

Then

$$B_n = \# \cup_{k=1}^n \{\Delta(\mathbf{i}_k)\}$$

is the number of all polydiagonals in X^n .

Plane rational curves

Fix $3d - 1$ points in the plane and count how many ways, N_d , there are to embed a line as a curve of degree d through the given points. Clearly $N_1 = 1$, and we have

Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

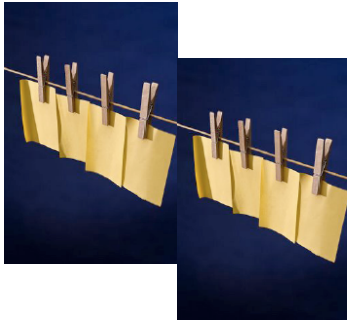
What can we say about the generating function

$$N(x) = \sum N_d x^d ?$$

or the *exponential* generating function

$$N^e(x) = \sum N_d x^d / d! ?$$

To display on the clothesline?



... the sequence of numbers:

$$N_1 = 1$$

$$N_2 = 1$$

$$N_3 = 12$$

$$N_4 = 620$$

$$N_5 = 87304$$

$$N_6 = 26312976$$

$$N_7 = 14616808192$$

$$N_8 = 13525751027392$$

$$N_9 = 19385778269260800$$

$$N_{10} = 40739017561997799680$$

$$N_{11} = 120278021410937387514880$$

$$N_{12} = 482113680618029292368686080$$



Gromov–Witten and Donaldson–Thomas invariants

Naively speaking, the **Gromov–Witten invariants** counts maps from a curve to a manifold, whereas the **Donaldson–Thomas invariants** count curves on a manifold.

In how many ways can you map a line to a quartic surface in projective 3-space, in such a way that the image curve has n self crossings?

The answer is that the generating function of this problem is

$$GW(x) = P(x)^{24} = \prod_{m \geq 1} (1 - x^m)^{-24}$$

There is a geometric reason: the surface can be degenerated to a fibration in elliptic curves, with precisely 24 singular (nodal) fibers, and the partitions appearing are ways to map a bunch of lines to each nodal fiber.



Degree zero DT function

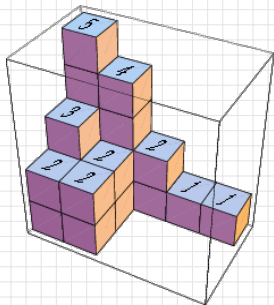
The *degree zero* Donaldson–Thomas invariants “count” *collapsed* curves, i.e., “fat” points on the manifold.

When the manifold is $\mathbb{C}^3$, this virtual count is equivalent, up to a sign, to counting monomial ideals in $\mathbb{C}[y_1, y_2, y_3]$ of given finite colength. Such an ideal is determined by the monomials $y_1^i y_2^j y_3^k$ that are *not* in the ideal, so if the colength is n , it corresponds to a plane partition of a set of n elements.



A plane partition of 22

5	4	2	1	1
3	2			
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Therefore, the n th degree zero DT invariant is equal to $(-1)^n \pi(n)$ and the generating function is given by the MacMahon-function:

$$DT_0(x) = \sum (-1)^n \pi(n) x^n = M(-x) = \prod_{m \geq 1} (1 - (-x)^m)^{-m}$$

Bell polynomials

Define polynomials $B_n(a_1, \dots, a_n)$ recursively by $B_0 = 1$ and

$$B_{n+1}(a_1, \dots, a_{n+1}) = \sum_{i=0}^n \binom{n}{i} B_{n-i}(a_1, \dots, a_{n-i}) a_{i+1}$$

Then $B_0 = 1$, $B_1(a_1) = a_1$, $B_2(a_1, a_2) = a_1^2 + a_2$, ...

and $B_n(1, \dots, 1) = B_n$ is the n th Bell number.

Another way to define $B_n(a_1, \dots, a_n)$ is by the formal identity

$$\sum_{i \geq 0} B_i(a_1, \dots, a_i) t^i / i! = \exp\left(\sum_{j \geq 1} a_j t^j / j!\right).$$

Plane nodal curves

A plane curve is defined as the zero set in $\mathbb{C}^2$ (or in $\mathbb{P}_{\mathbb{C}}^2$) of a polynomial in two variables (or a homogeneous polynomial in three variables).

Let $N_r(d)$ denote the number of plane curves of degree d with r ordinary double points (nodes) that pass through $\binom{d+2}{2} - 1 - r$ points.

From the proof of Göttsche's conjecture (Tzeng and Kool–Shende–Thomas) it follows (Kleiman–P, Qviller) that the generating function can be written as

$$\sum N_r(d) x^r = \sum B_r(a_1, \dots, a_r) \frac{x^r}{r!},$$

where the a_j are quadratic polynomials in d .



Why Bell polynomials?

- ▶ A recursive formula for $N_r(d)$ – proved for $r \leq 8$, conjectured in general (Kleiman–P) – fits with the Bell polynomials (reminiscent of “derivations” like in the Faà di Bruno formula).
- ▶ An intersection theoretic approach on the configuration space $(\mathbb{P}^2)^r$, shows that each a_j comes from an intersection class on a “polydiagonal” (Qviller).

The advantage of knowing the Bell form of the node polynomials $N_r(d)$ is that in order to compute $N_{r+1}(d)$ from the previous $N_i(d)$ one only needs to compute one new term, $a_{r+1}(d)$:

$$N_{r+1}(d) = \frac{1}{(r+1)!} \sum_{i=0}^r \binom{r}{i} (r-i)! N_{r-i}(d) a_{r+1}(d)$$



THANK YOU FOR YOUR ATTENTION!

A Place for Women

Massachusetts Institute of Technology



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The Hunting of the Snark (Lewis Carroll)

The Beaver had counted with scrupulous care,
Attending to every word:
But it fairly lost heart, and outgrabe in despair,
When the third repetition occurred.

It felt that, in spite of all possible pains,
It had somehow contrived to lose count,
And the only thing now was to rack its poor brains
By reckoning up the amount.

“Two added to one — if that could but be done,”
It said, “with one’s fingers and thumbs!”
Recollecting with tears how, in earlier years,
It had taken no pains with its sums.

“The thing can be done,” said the Butcher, “I think.
The thing must be done, I am sure.
The thing shall be done! Bring me paper and ink,
The best there is time to procure.”