Return of the plane evolute

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Topology and real algebraic geometry
in low dimension
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Apollonius of Perga, 262–200 BC

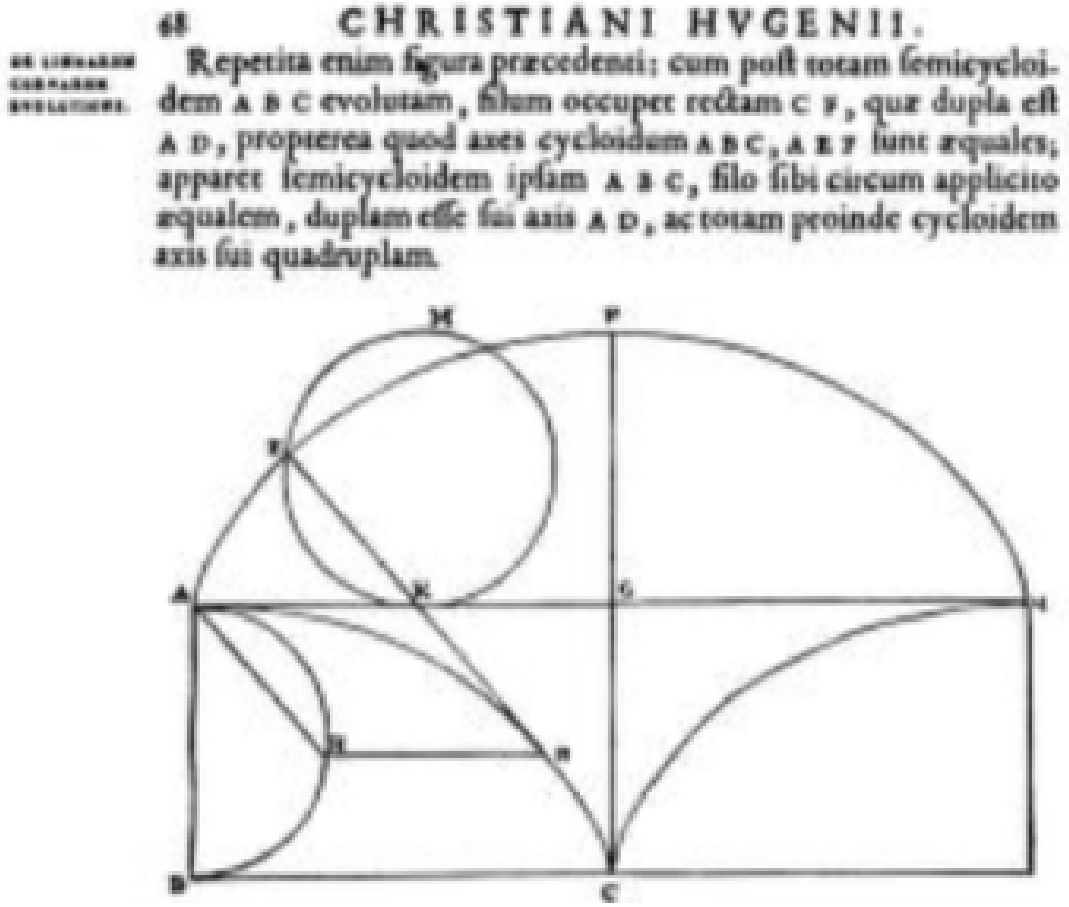
In his Treatise on Conic Sections, Book V, Apollonius constructs and studies *normals* to a conic – he finds these more interesting than the tangents:[1]

> *In the fifth book I have laid down propositions relating to maximum and minimum straight lines. You must know that our predecessors and contemporaries have only superficially touched upon this investigation of the shortest lines, and have only proved what straight lines touch the sections . . .*

He spends 16 pages finding the number of normals through a given point (the Euclidean distance degree!) and considers the points "where two normals fall together" – the evolute!

[1] T. L. Heath, Apollonius of Perga, Cambridge University 1961.

Christian Huygens: Horologium Oscillatorium, 1673

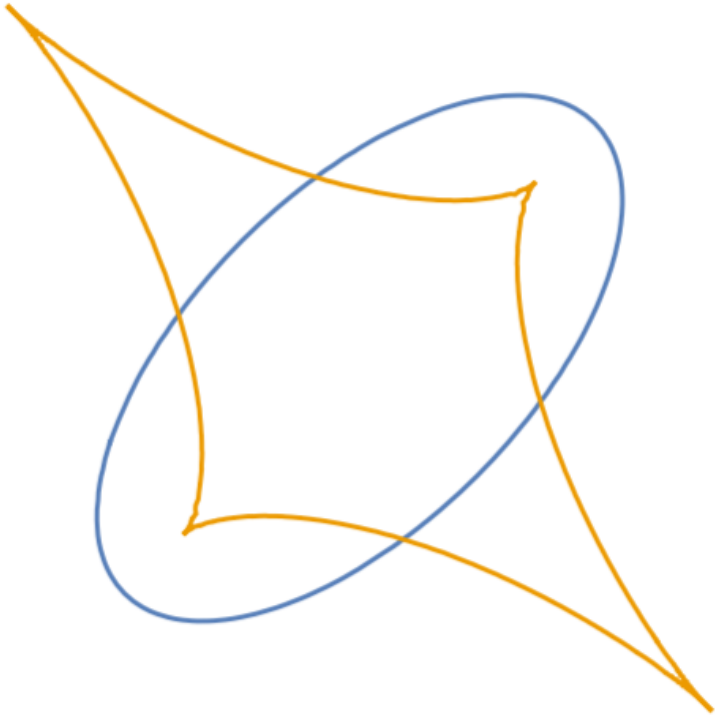


68 CHRISTIANI HVGENII.

Repetita enim figura præcedenti; cum post totam semicycloidem A B C evolutam, filum occupet rectam C F, quæ dupla est A D, propterea quod axes cycloidum A B C, A E F sunt æquales; apparet semicycloidem ipsam A B C, filo sibi circum applicito æqualem, duplam esse sui axis A D, ac totam proinde cycloidem axis sui quadruplam.

"The evolute of a cycloid is a cycloid."

An ellipse and its evolute (an astroid)



Envelopes of families of linear spaces

Consider a variety $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ of dimension r.

Given a $(n-r+1)$-quotient

$$V_X \to \mathcal{F},$$

we get a family of linear spaces of dimension $n-r$

$$\varphi_{\mathcal{F}} : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}(V) \to X.$$

The *envelope* of φ is the branch locus $E_{\mathcal{F}} \subset \mathbb{P}(V)$ of

$$\psi_{\mathcal{F}} : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}(V) \to \mathbb{P}(V).$$

A stratification of the envelope

The envelope $E_{\mathcal{F}}$ is given by $\Sigma^1(\psi_{\mathcal{F}})$.

Its "cuspidal edge" $C_{\mathcal{F}}$ is given by $\Sigma^{1,1}(\psi_{\mathcal{F}})$.

The cuspidal locus $\kappa_{\mathcal{F}}$ of $C_{\mathcal{F}}$ is given by $\Sigma^{1,1,1}(\psi_{\mathcal{F}})$, etc.

Set $\overline{c}_i := c_i(\psi_{\mathcal{F}}^* T_{\mathbb{P}(V)} - T_{\mathbb{P}(\mathcal{F})})$. Then

$$
\begin{aligned}
[E_{\mathcal{F}}] &= \psi_{\mathcal{F}*}(\overline{c}_1) \cap [\mathbb{P}(V)] \\
[C_{\mathcal{F}}] &= \psi_{\mathcal{F}*}(\overline{c}_1^2 + \overline{c}_2) \cap [\mathbb{P}(V)] \\
[\kappa_{\mathcal{F}}] &= \psi_{\mathcal{F}*}(\overline{c}_1^3 + 3\overline{c}_1\overline{c}_2 + 2\overline{c}_3) \cap [\mathbb{P}(V)] \\
[\lambda_{\mathcal{F}}] &= \psi_{\mathcal{F}*}(6\overline{c}_4 + 9\overline{c}_1\overline{c}_3 + 2\overline{c}_2^2 + 6\overline{c}_1^2\overline{c}_2 + \overline{c}_1^4) \cap [\mathbb{P}(V)] \ldots
\end{aligned}
$$

Euclidean geometry

Fix a hyperplane $H_\infty = \mathbb{P}(V') \subset \mathbb{P}(V)$ and a quadric $Q_\infty \subset H_\infty$. The sum of the maps

$$V_X \to V'_X \cong V'^\vee_X \to \mathcal{N}_{X/\mathbb{P}(V)}(-1) \text{ and } V_X \to \mathcal{O}_X(1)$$

whose fiber at a general point $p \in X$ is the *Euclidean normal*: set $L_p := T_p \cap H_\infty$ and let $L_p^\perp \subset H_\infty$ w.r.t Q_∞. Then $N_p := \langle p, L_p^\perp \rangle$ is the $(n-r)$-space perpendicular to T_p.

Example: $\mathbb{P}(V) = \mathbb{P}^2_{\mathbb{C}}$, $H_\infty = \{z = 0\}$, and

$Q_\infty = \{x^2 + y^2 = 0\} = \{(1 : i), (1 : -i)\} \subset H_\infty$ (the circular points)

gives Euclidean geometry in $\mathbb{R}^2 \subset \mathbb{C}^2 = \mathbb{P}^2_{\mathbb{C}} \setminus H_\infty$.

The Euclidean normal bundle

The *Euclidean normal bundle* $\mathcal{E}$ is the image of the map

$$V_X \to \mathcal{N}_{X/\mathbb{P}(V)}(-1) \oplus \mathcal{O}_X(1).$$

The *evolutes* of $X \subset \mathbb{P}(V)$ are

$E_X := E_{\mathcal{E}}$, $C_X := C_{\mathcal{E}}$, $\kappa_X := \kappa_{\mathcal{E}}$, ...

If $X \subset \mathbb{P}(V) = \mathbb{P}^2$ is a plane curve, its *curve of normals* $N_X \subset \mathbb{P}(V^\vee) = (\mathbb{P}^2)^\vee$ is given by the 1-quotient

$$V_X^\vee \cong \wedge^2 V_X \to \wedge^2 \mathcal{E}.$$

It is a classical fact that the curve of normals is equal to the dual curve of the evolute:

$$N_X = E_X^\vee \subset \mathbb{P}(V)^\vee.$$

Evolutes as centers of curvature

Choose coordinates in $\mathbb{P}(V) = \mathbb{P}^2$ such that $H_\infty = \{z = 0\}$ and $Q_\infty = \{x^2 + y^2 = 0\} \subset H_\infty$. Let $(x(t), y(t))$ be a local parameterization of $X \cap (\mathbb{P}^2 \setminus H_\infty) = X \cap \mathbb{C}^2$. The curve of normals $\{N_p\}_{p \in X} \subset (\mathbb{P}^2)^\vee$ is given by

$$V_X^\vee \cong \bigwedge^2 V_X \to \bigwedge^2 \begin{pmatrix} x(t) & y(t) & 1 \\ y'(t) & -x'(t) & 0 \end{pmatrix} = (x' : y' : -xx' - yy').$$

The evolute is obtained by taking the dual of the curve of normals:

$$V_X \cong \bigwedge^2 V_X^\vee \to \bigwedge^2 \begin{pmatrix} x' & y' & -xx' - yy' \\ x'' & y'' & -xx'' - x'^2 - yy'' - y'^2 \end{pmatrix},$$

which gives the *center of curvature*:

$$\left(x - \tfrac{y'(x'^2+y'^2)}{x'y''-x''y'} : y + \tfrac{x'(x'^2+y'^2)}{x'y''-x''y'} : 1\right).$$

Numerical characters

Let $X \to \mathbb{P}(V) \cong \mathbb{P}^2$ be a complex plane curve of degree d. Given a line H_∞ and a quadric $Q_\infty \subset H_\infty$. If $p \in X \cap Q_\infty$ but $T_p \neq H_\infty$, then the evolute has an inflectional tangent through p. If $T_q = H_\infty$ for some $q \in X \cap H_\infty$, then the evolute has an inflection point on H_∞.

Assume the intersections of X with H_∞ are transversal or tangential. Then (as in Salmon 1852)

$$\deg N_X = d + d^\vee - \iota_E \ \left(\leq d^2\right)$$

$$\deg E_X = 3d + \iota - 3\iota_E = 3d^\vee + \kappa - 3\iota_E \ \left(\leq 3d(d-1)\right)$$

$$\#C_X = 6d - 3d^\vee + 3\iota - 5\iota_E \ \left(\leq 3d(2d-3)\right)$$

where ι_E is the number of inflection points on E_X.
(Note that $\#C_X \cap H_\infty = d$. So $\#C_X \cap (\mathbb{P}^2 \setminus H_\infty) = 2d(3d-5)$ is the number of *vertices* of X.)

Nodes on the normal curve and on the evolute

The normal curve N_X has an ordinary d-multiple point (corresponding to the line at infinity).

The nodes of N_X are the *diameters* of the curve. For a general curve X the number of diameters of X is

$$\delta_N = \binom{d}{2}(d^2 + d - 4),$$

and the number of nodes on the evolute is

$$\delta_E = \frac{1}{2}d(3d - 5)(3d^2 - d - 6).$$

Evolutes of space curves

Let $X \to \mathbb{P}(V) \cong \mathbb{P}^3$ be a space curve of degree d and genus g. Fix a plane H_∞ and a quadric $Q_\infty \subset H_\infty$, and let $\mathcal{E}$ denote the Euclidean normal bundle to X. The envelope of the normal planes to X is a surface E_X, called the *polar developable* by Monge and the *polar surface* by Darboux. The *space evolute* of X is the cuspidal edge C_X of E_X, called the *evolute of the second type* by Blaschke and Lichtweiss and the *focal curve* by Uribe-Vargas.

The osculating planes of the space evolute are the normal planes to X. The space evolute of X is the locus of its centers of spherical curvature.

Enumerative formulas for a space curve

Assume $X \to \mathbb{P}^3$ is in general position w.r.t. Q_∞ and H_∞.

The degree of the polar developable is

$$\deg E_X = 6(d + g - 1) - 2k_0.$$

The degree of the space evolute is

$$\deg C_X = 3(3d + 4g - 4 - k_0).$$

The space evolute has

$$\#\kappa_X = 4(3d + 5g - 5 - k_0)$$

cusps.

The first two formulas agree with those found by Salmon (and with the formulas for a plane curve).

What about real curves?

Theorem (Klein)

If a real algebraic plane curve X of degree d and class $d^\vee$ has no other point or tangent singularities than nodes, cusps, bitangents, and inflectional tangents, then

$$d - 2\delta^{\mathrm{ac}} - \kappa_{\mathbb{R}} = d^\vee - 2\tau^{\mathrm{ac}} - \iota_{\mathbb{R}}.$$

For example, a nonsingular real curve can have no more than $d(d-2)$ real inflection points, whereas there are $3d(d-2)$ complex ones.

Some integral formulas based on Euler characteristic (1988)

6.D. (Generalized Klein formula). For any complex plane projective curve A (which is not necessarily real)

$$\deg A - \int_{A \cap \mathbb{R}P^2} m_A(x)\, d\chi(x) = \deg A^\vee - \int_{A^\vee \cap \mathbb{R}P^{2\vee}} m_{A^\vee}(x)\, d\chi(x).$$

Diameters and vertices of a plane curve

Notice that a crunode of N_Γ (i.e., the real node with two real branches) corresponds to the *diameter* of Γ which is a straight segment connecting pairs of points on Γ and which is perdendicular to the tangent lines to Γ at these endpoints.

Observe also that a real cusp of E_Γ (resp. an inflection point on N_Γ) corresponds to a *vertex* of Γ which is a critical point of Γ's curvature.

Vertices of plane appear, for example, in the classical 4-vertex theorem and its numerous generalizations.

$\mathbb{R}$-degree

To formulate our problems we need to introduce the following notion which deserves to be better known.

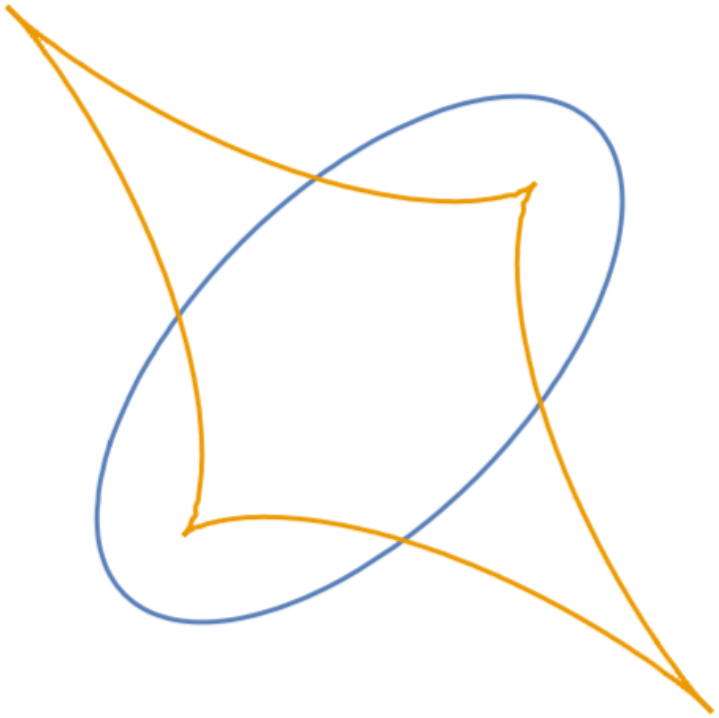
Definition

Given a closed semi-analytic hypersurface $H \subset \mathbb{R}^n$ without boundary, we define its $\mathbb{R}$-*degree* as the supremum of the cardinality of $H \cap L$ taken over all lines $L \subset \mathbb{R}^n$ such that L intersects H transversally. (Observe that we count points in $H \cap L$ without multiplicity.)

In what follows, we denote the $\mathbb{R}$-degree of H by $\mathbb{R}\deg(H)$. For a real-algebraic (or piecewise real-algebraic) hypersurface $H \subset \mathbb{R}^n$, one has $\mathbb{R}\deg(H) \leq \deg(H)$ where $\deg(H)$ is the usual degree of H (respectively the degree of the Zariski closure of H).

In particular, the $\mathbb{R}$-degree of a real-algebraic hypersurface is always finite which is in general not the case for real-analytic hypersurfaces.

$\mathbb{R}$-degree of an astroid is 4



Problems

Problem (1)

For a given positive integer d, what are the maximal possible $\mathbb{R}$-degrees of the evolute E_Γ and of the curve of normals N_Γ where Γ runs over the set of all real-algebraic curves of degree d?

Problem (2)

For a given positive integer d, what its the maximal possible number of real cusps on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of vertices a real-algebraic curve Γ of degree d might have?

To make Problem (2) well-defined we have to assume that Γ does not have a circle as its irreducible component.

Problems, cont.

Problem (3)

For a given positive integer d, what its the maximal possible number of crunodes on N_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal number of (real) diameters Γ might have?

Here we again have to assume that Γ does not have a circle as its irreducible component.

Problem (4)

For a given positive integer d, what its the maximal possible number of crunodes on E_Γ where Γ runs over the set of all real-algebraic curves of degree d? In other words, what is the maximal possible number of points in $\mathbb{R}^2$ which are the centers for at least two distinct (real) curvature circles of Γ?

Initial result

Proposition. For any $d \geq 3$, the maximal $\mathbb{R}$-degree among the evolutes of algebraic curves of degree d is not less than $d(d-2)$.

Remark. Complex answer $3d(d-1)$.

Proof. Recall that each real inflection point of a real curve corresponds to its evolute going to infinity. Notice that from Klein's theorem follows that a real-algebraic curve of degree d has at most one third of its inflection points real and this bound is achieved. The number of complex inflection points of a generic curve of degree d equals $3d(d-2)$. Thus there exists a smooth real-algebraic curve of degree d with $d(d-2)$ real inflection points. The evolute of such curve hits the line at infinity (transversally) at $d(d-2)$ real points. Thus its $\mathbb{R}$-degree is at least $d(d-2)$. $\square$

Comments

The above lower bound is apparently not sharp. For $d = 2$ the sharp bound is 4. For $d = 3$, taking a small deformation of three lines creating a compact oval one gets an example with $\mathbb{R}$-degree of the evolute greater than or equal to 6 while the number of real inflections is 3. The complex answer is $3d(d-1)$ which has leading coefficient 3 while our bound has leading coefficient 1. The correct leading coefficient at d^2 is unknown at the moment.

Second result

Proposition. There exists a real-algebraic curve Γ of degree d and a point $p \in \mathbb{R}^2$ such that all d^2 complex normals to Γ through p are, in fact, real. In other words, the maximal $\mathbb{R}$-degree of N_Γ equals d^2 which is the usual degree of N_Γ.

Proof. A crunode (which is a transversal intersection of two smooth real local branches) admits two types of real smoothing. By theorem of Brusotti any (possibly reducible) plane real-algebraic curve with only nodes as singularities admits a small real deformation which realizes independently prescribed smoothing types of all its crunodes.

Second result, proof

Given a crunode and a point z such that the line L through this point and through the crunode is not tangent to the real local branches at the crunode, there exists a smoothing type of the crunode such that, slightly rotating the line L around z, one obtains two real normals to this smoothing.

Now take an arrangement $\mathcal{A} \subset \mathbb{R}^2$ of d real lines in general position and a point z outside these lines. By Brusotti, smoothing all $d(d-1)/2$ nodes in an appropriate way we obtain $d(d-1)$ normals close to the lines joining z with the nodes of $\mathcal{A}$. Additional d normals are obtained by small deformations of the altitudes connecting z with each of the d given lines. Thus, there exist d^2 real normals through z implying the $\mathbb{R}$-degree of the curve of normals for the obtained curve is a least d^2. But its usual degree is d^2. The result follows. $\square$

Illustration

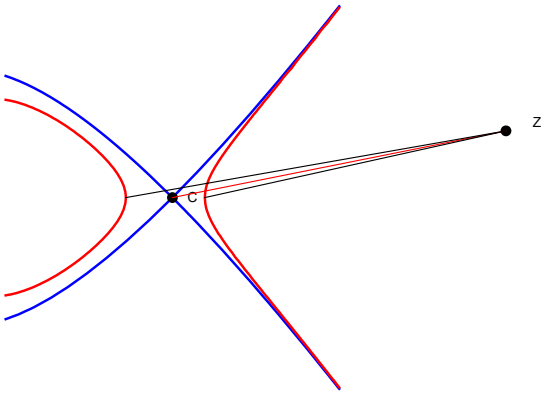


Figure: Good deformation of a crunode relative to a point z.

Advances in Problem 2

Theorem. The maximal total number of cusps on evolutes of real algebraic curves of degree d is at least $d(2d-3)$ which is exactly $1/3$ of $\#C_\Gamma = 3d(2d-3)$.

Proposition. The number of real cusps for the evolute of an arbitrary small deformation $\mathcal{R}$ of any generic line arrangement $\mathcal{A} \subset \mathbb{R}^2$ consisting of d lines equals $d(d-1)$ plus the number of bounded edges of $\mathcal{A}$ respected by $\mathcal{R}$. All bounded edges of $\mathcal{A}$ are respected by $\mathcal{R}$ if and only if the small deformation $\mathcal{R}(\mathcal{A})$ is a convex curve.

Advance in Problem 2, cont.

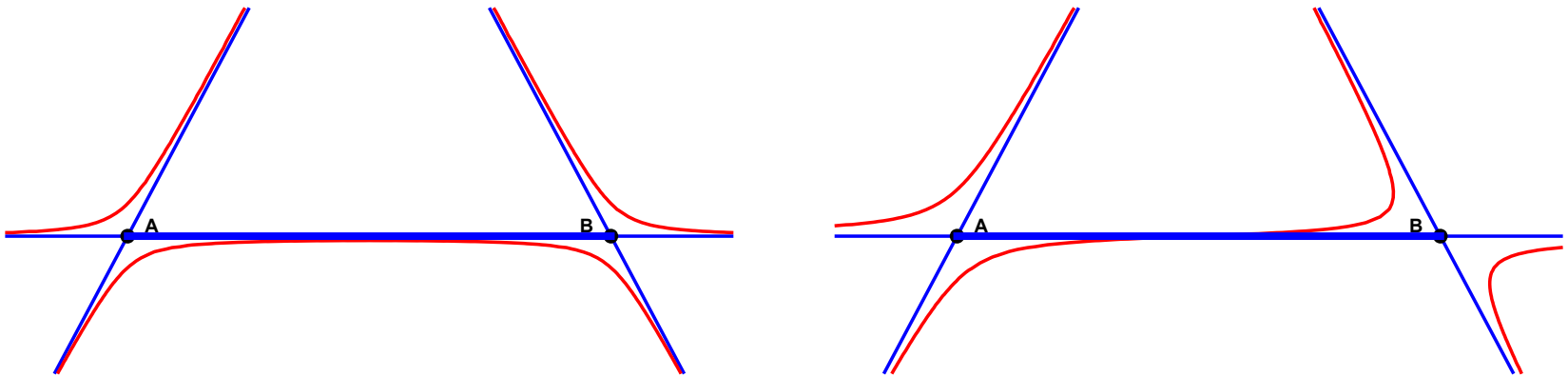


Figure: Two types of deformations of a bounded edge of a line arrangement. The left one respects the edge while the right twists it.

Its proof

Proof: Consider the complement $\mathbb{R}^2 \setminus \mathcal{A}$. It consists of $2d$ infinite convex polygons and $\binom{d-1}{2}$ bounded convex polygons. Now take any small deformation $\mathcal{R}$ of $\mathcal{A}$. Locally near any vertex v of $\mathcal{A}$ the smooth curve $\mathcal{R}(\mathcal{A})$ will consists of two convex branches for each of which the curvature has a local maximum near v. These local maxima will correspond to two cusps on the evolute of $\mathcal{R}(\mathcal{A})$ which gives totally $2\binom{d}{2} = d(d-1)$ cusps corresponding to local maxima of curvature. Let us now show that every bounded edge of $\mathcal{A}$ respected by $\mathcal{R}$ corresponds to the unique point on $\mathcal{R}(\mathcal{A})$ where the curvature attains its minimum.

Proof, cont.

Moreover all extremal points of the curvature belong either to the first or to the second types. On the other hand, every twisted edge corresponds to an inflection point on $\mathcal{R}(\mathcal{A})$ which means that the evolute goes to infinity. The total number of bounded edges of any generic arrangement with d lines equals $d(d-2)$. Easy to show that there exist exactly two small deformations for which all bounded edges will be respected and in such case we get $d(d-1)+d(d-2)=d(2d-3)$ extrema of curvature on $\mathcal{R}(\mathcal{A})$. $\square$

Comment

Remark. It is not clear that Klein's bound 1/3 is valid for evolutes which are highly singular curves. We tried to apply Klein's equation to the evolute, but have not got any definite conclusion yet. Thus it is unclear at the moment whether our lower bound is optimal.

Problem 3

Proposition. One has the following lower bound for $\mathbb{R}Diam(d)$,

$$\mathbb{R}Diam(d) \geq \frac{d^4}{2} - d^3 + \frac{d}{2}. \tag{1}$$

The complex answer is

$$\delta_N = \binom{d}{2}(d^2 + d - 4).$$

Some notions

Notation: - A line arrangement $\mathcal{A} \subset \mathbb{R}^2$ is called *strongly generic* if in addition to the requirements that no two lines are parallel and no three lines intersect at the same point we require that no two lines are *perpendicular*.

By an *altitude* of a given line arrangement $\mathcal{A} \subset \mathbb{R}^2$ we call a straight segment connecting a vertex of $\mathcal{A}$ with a point on a line belonging to $\mathcal{A}$ and which is perpendicular to this line. (Notice that if $\mathcal{A} \subset \mathbb{R}^2$ is strongly generic then no altitude of $\mathcal{A}$ connects its two vertices.)

Important notion

For a given strongly generic $\mathcal{A}$, its two vertices v_1 and v_2 and any resolution $\mathcal{R}$, we say that v_1 and v_2 *have each other in sight w.r.t.* $\mathcal{R}$ if $v_2 \in \mathcal{C}^{\perp}_{v_1}(\mathcal{R})$ and $v_1 \in \mathcal{C}^{\perp}_{v_2}(\mathcal{R})$.

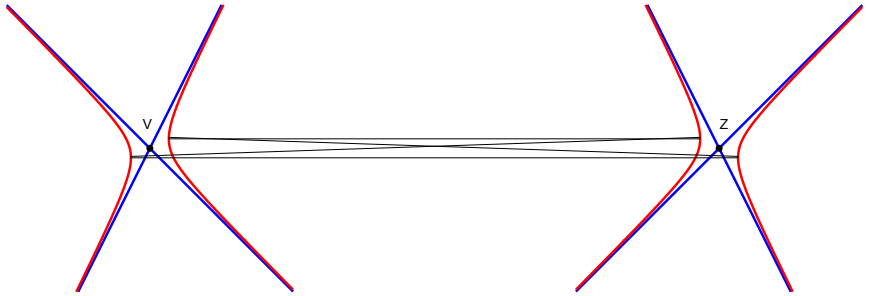


Figure: Two vertices having each other in sight w.r.t $\mathcal{R}$.

Illustration

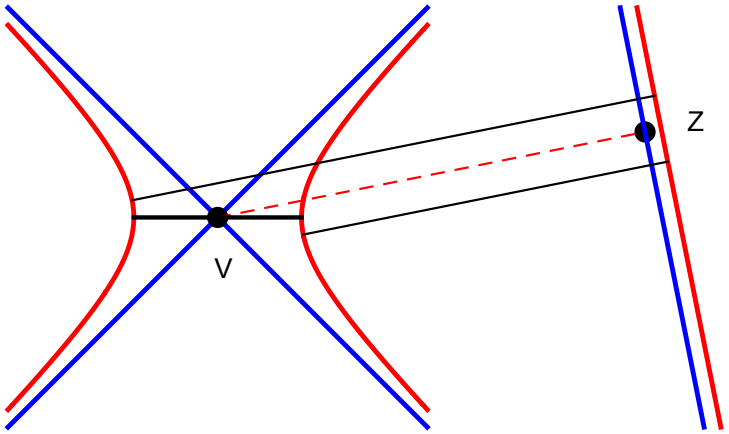


Figure: An altitude creating two diameters after an admissible deformation

Key technical result

Lemma. Given a strongly generic line arrangement $\mathcal{A}$, the following holds:

(i) Any small resolution of a vertex of $\mathcal{A}$ creates one diameter;

(ii) If an altitude al is admissible w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates two diameters close to al;

(iii) If v_1 and v_2 have each other in sight w.r.t. a small deformation $\mathcal{R}$ then $\mathcal{R}$ creates four diameters close to the segment (v_1, v_2).

Main statement

Proposition. Given any small resolution $\mathcal{R}$ of a strongly generic arrangement $\mathcal{A}$ consisting of d lines, the number of diameters of the obtained smooth curve $\mathcal{R}(\mathcal{A})$ equals

$$\#_{diam}(\mathcal{R}(\mathcal{A})) = \#_{ver} + 2\#_{adm.alt} + 4\#_{pairs\ of\ vert.\ s.\ each\ other},$$

where $\#_{ver} = \binom{d}{2}$ is the number of vertices of $\mathcal{A}$,
$\#_{adm.alt}$ is the number of admissible altitudes w.r.t. $\mathcal{R}$,
$\#_{pairs\ of\ vert.\ s.\ each\ other}$ is the number of vertices seeing each other.

Illustration

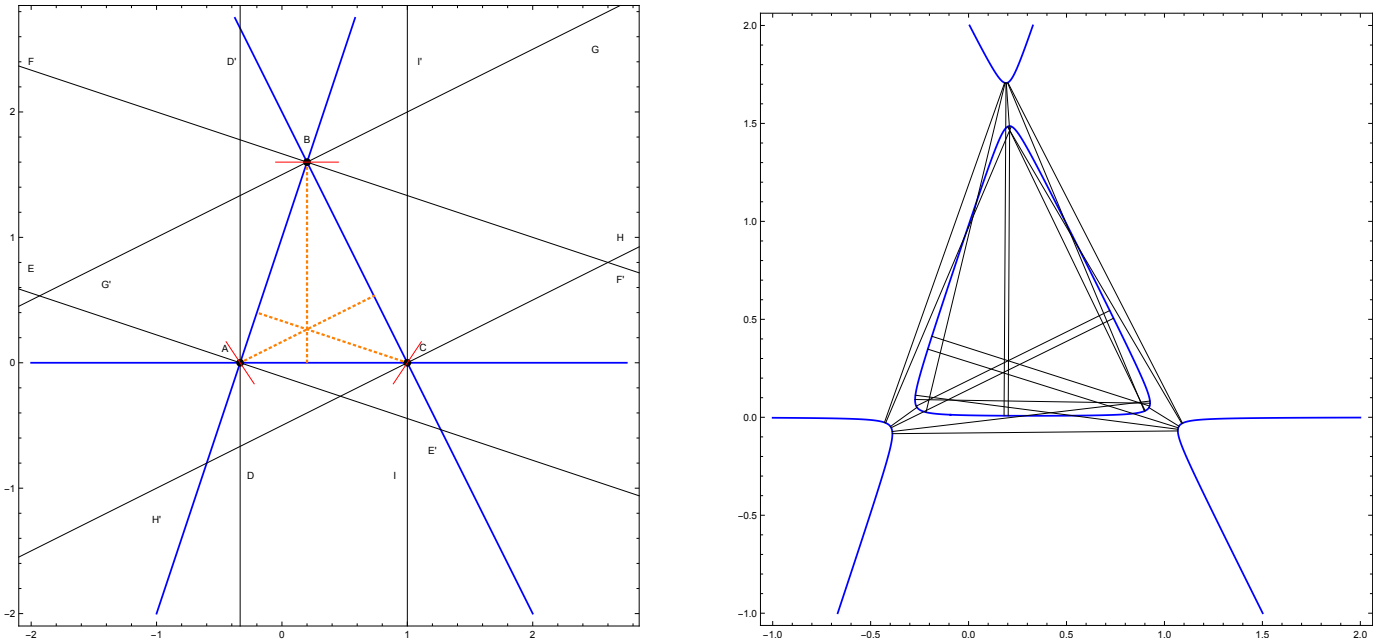
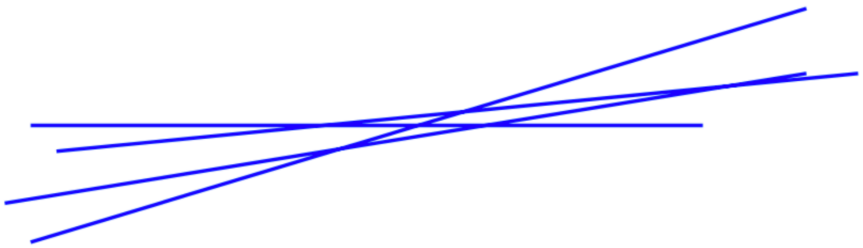


Figure: Simple line arrangement and its deformation creating 21 diameters.

The complex answer is 24.

Sketch of proof



To settle the Proposition we need to introduce some class of arrangements. We say that an arrangement $\mathcal{A}$ is *oblate* if the slopes of all lines in $\mathcal{A}$ are close to each other.

Take a small resolution of an oblate arrangement for which we make narrow cones at each vertex. Then every pair of vertices will see each other.On the other hand, all altitudes will be non-admissible. Thus we get $4\binom{\binom{d}{2}}{2} + \binom{d}{2} = \frac{d^4}{2} - d^3 + \frac{d}{2}$ diameters for this resolution. $\square$

Problem 4

Recall that the number of nodes of the evolute of a generic curve of degree d is given by

$$\delta_E(d) = \frac{d}{2}(3d-5)(3d^2-d-6).$$

Denote by $\delta_E^{cru}(d)$ the maximal number of crunodes for the evolutes of real-algebraic curves of degree d.
We have the following lower bound for this number of crunodes.

Proposition

$$\delta_E^{cru}(d) \geq \left(\left[\frac{d-1}{2}\right] + d - 2\right)^4 - \frac{1}{2}.$$

Improvements wanted...

There is apparently a lot of space for improvement of the suggested bounds (which are very naive) as well as for other real-algebraic problems related to the evolutes, curves of normals and their high-dimensional analogs!

Our small zoo, cage 1

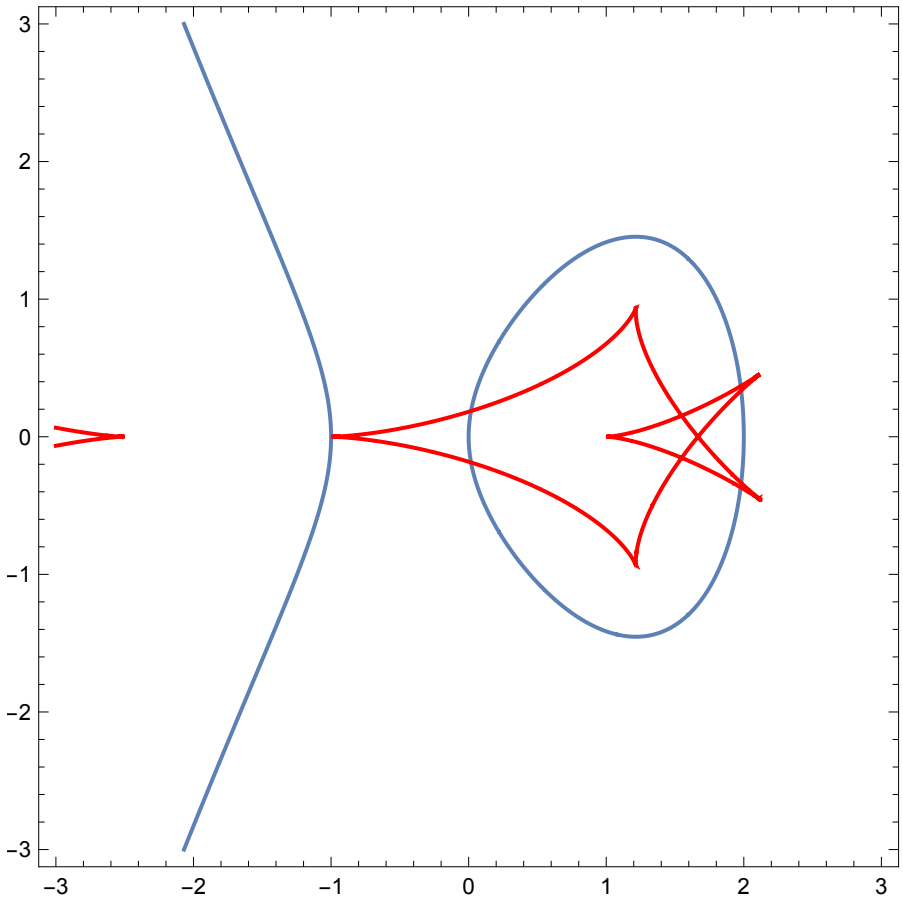


Figure: The Weierstrass cubic in blue and its evolute in red.

Our small zoo, cage 2

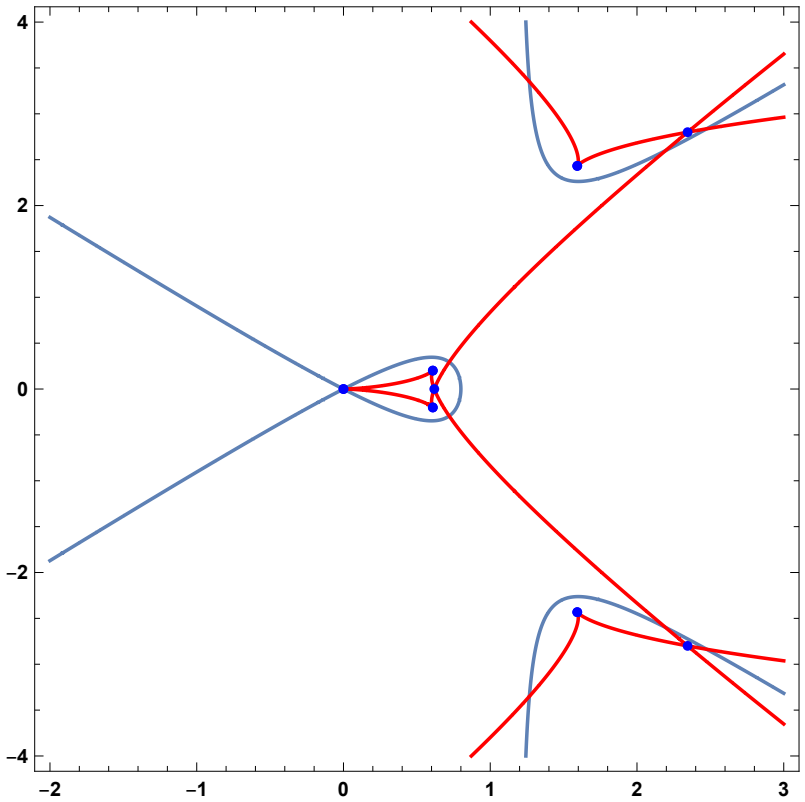


Figure: The nodal cubic $5(x^2 - y^2)(x - 1) + (x^2 + y^2) = 0$ in blue and its evolute in red.

Our small zoo, cage 3

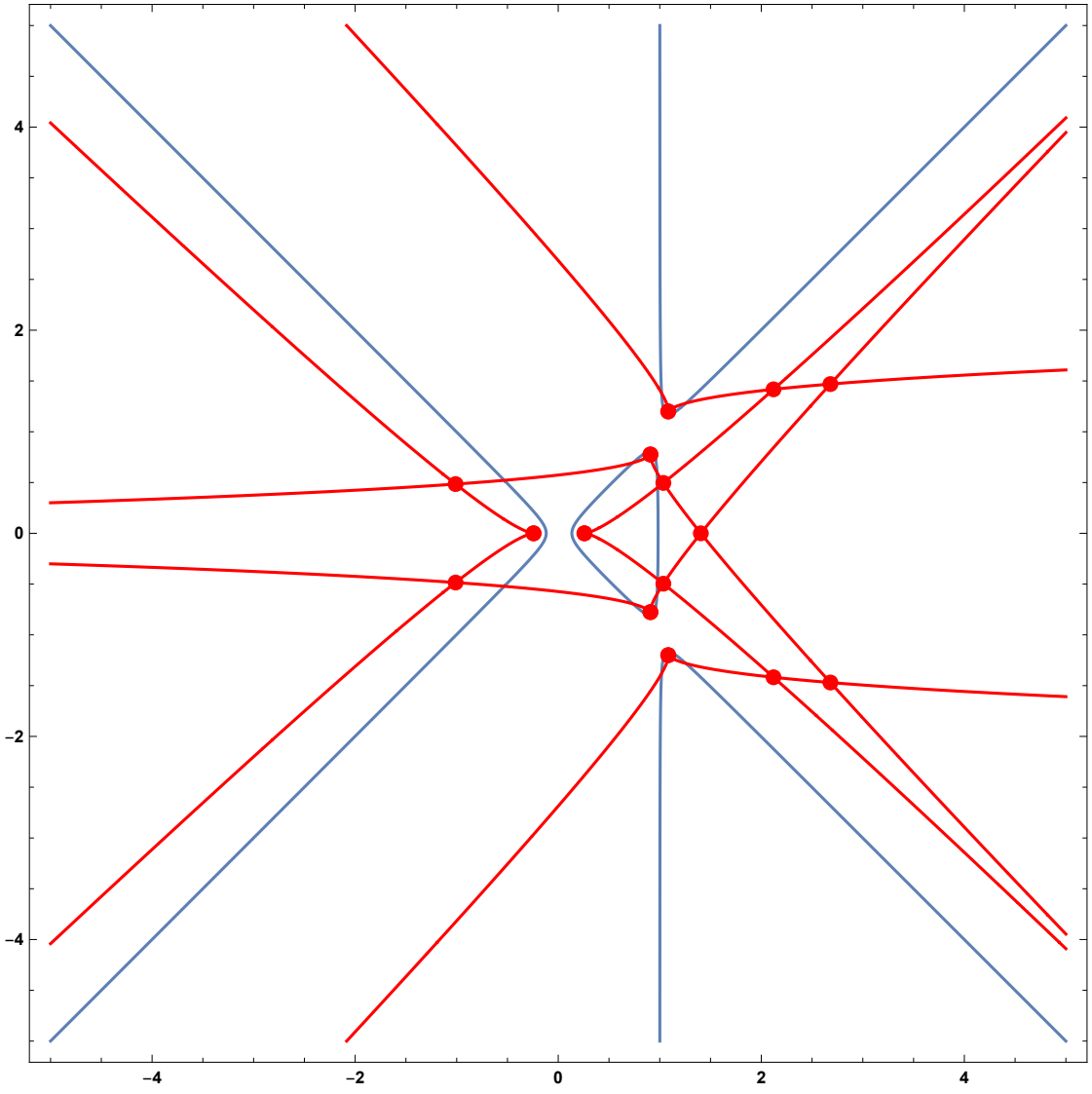


Figure: A non-singular cubic in blue and its evolute in red.

Our small zoo, cage 4

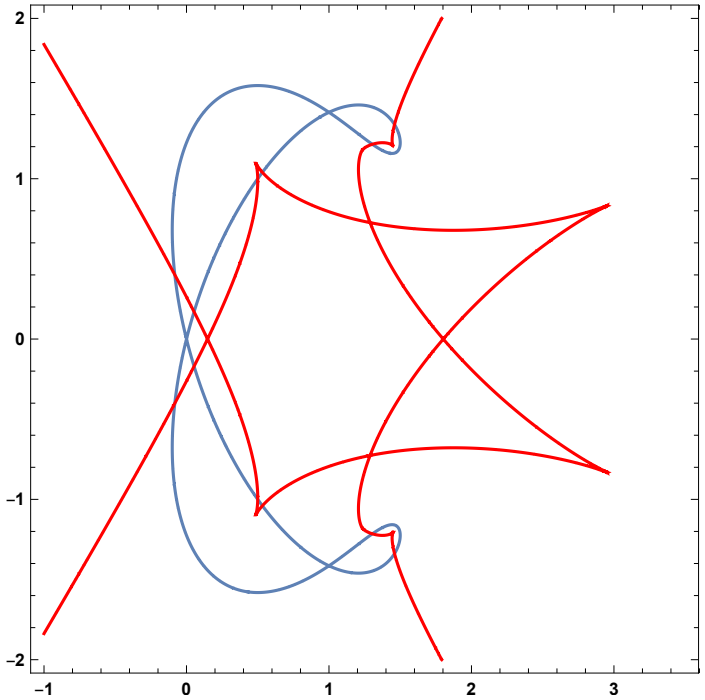


Figure: The ampersand curve in blue and its evolute in red

Grand finale

Figure: Many happy returns to the Wonderful Wizard of Math!