Counting problems and generating functions

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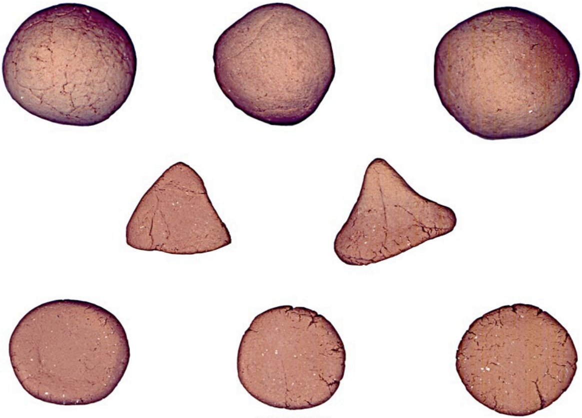
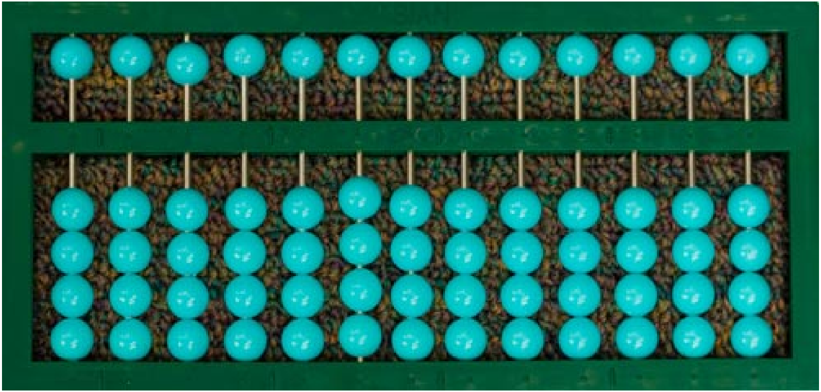
Ways of counting 1



MS 5067/1-8
Neolithic plain counting tokens. Near East, ca. 8000--3500 BC

Ways of counting 2



Ways of counting 3

Khipo in the Museo Machu Picchu (Pi3.124, CC BY-SA 4.0)

Partitions

Let n be a positive integer.

In how many ways can we write n as a sum of positive integers?

$1 = 1$
$2 = 2 = 1 + 1$
$3 = 3 = 2 + 1 = 1 + 1 + 1$
$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$

A partition of 7

MS 4647
Numbers 3+4. Sumer, ca. 3500-3200 BC

Partitions

Let n be a positive integer.

In how many ways $p(n)$ can we write n as a sum of positive integers?

$1 = 1$
$2 = 2 = 1 + 1$
$3 = 3 = 2 + 1 = 1 + 1 + 1$
$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$

$p(1) = 1$
$p(2) = 2$
$p(3) = 3$
$p(4) = 5$

A closed formula for $p(n)$?

The partition function (Euler)

Expand the infinite product

$$(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\cdots(1+q^m+q^{2m}+\cdots)\cdots$$

The coefficient of q^n is $p(n)$, the number of ways we can write $n = n_1 + n_2 + \ldots$, where $n_1 \geq n_2 \geq \cdots > 0$.

The *generating function*: $P(q) := \sum p(n)q^n$ is

$$P(q) = \prod_{m\geq 1}(1+q^m+q^{2m}+q^{3m}+\cdots) = \prod_{m\geq 1}(1-q^m)^{-1}.$$

The function $P(q)$ displays the integer sequence

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, 77, 101, \ldots$$

A closed formula: $p(n) = \frac{1}{2\pi i}\int_C \frac{P(q)}{q^{n+1}}dq$.

Generating functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

H.S.Wilf
generatingfunctionology
Academic Press Inc., 1990

Plane partitions

Consider next the number of *plane partitions* $\pi(n)$:

$$1 = 1 \Rightarrow \pi(1) = 1$$

$$2 = 2,\ \begin{array}{c|c} 1 & 1 \end{array},\ \begin{array}{c} 1 \\ \hline 1 \end{array} \Rightarrow \pi(2) = 3$$

$$3 = 3,\ \begin{array}{c|c} 2 & 1 \end{array},\ \begin{array}{c} 2 \\ \hline 1 \end{array},\ \begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array},\ \begin{array}{c|c|c} 1 & 1 & 1 \end{array},\ \begin{array}{c|c} 1 & 1 \\ \hline 1 & \end{array} \Rightarrow \pi(3) = 6$$

The generating function for $\pi(n)$ is the MacMahon function:

$$M(q) = \prod_{m \geq 1} (1 - q^m)^{-m}$$

$$M(q) = (1+q+q^2+q^3+\cdots)(1+q^2+\cdots)^2(1+q^3+\cdots)^3\cdots$$
$$= 1+q+q^2+2q^2+q^3+q\cdot 2q^2+3q^3+\cdots$$

$$q \leftrightarrow 1 \leftrightarrow \{1\}$$

$$q^2 \leftrightarrow 2 \leftrightarrow \{1, z\}$$

$$2q^2 \leftrightarrow \begin{array}{c|c} 1 & 1 \end{array} \text{ and } \begin{array}{c} 1 \\ \hline 1 \end{array} \leftrightarrow \{1, y\} \text{ and } \{1, x\}$$

$$q^3 \leftrightarrow 3 \leftrightarrow \{1, z, z^2\}$$

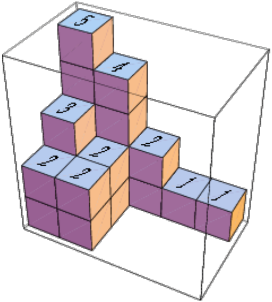
$$q \cdot 2q^2 \leftrightarrow \begin{array}{c|c} 2 & 1 \end{array} \text{ and } \begin{array}{c} 2 \\ \hline 1 \end{array} \leftrightarrow \{1, y, z\} \text{ and } \{1, x, z\}$$

$$3q^3 \leftrightarrow \begin{array}{c|c|c} 1 & 1 & 1 \end{array}, \begin{array}{c|c} 1 & 1 \\ \hline 1 & \end{array}, \begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array} \leftrightarrow \{1, y, y^2\}, \{1, x, y\}, \{1, x, x^2\}$$

A plane partition of 22

5	4	2	1	1
3	2			
2	2			

$$\longleftrightarrow \quad \{1, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2y, x^2yz, y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, z, z^2, z^3, z^4\}$$

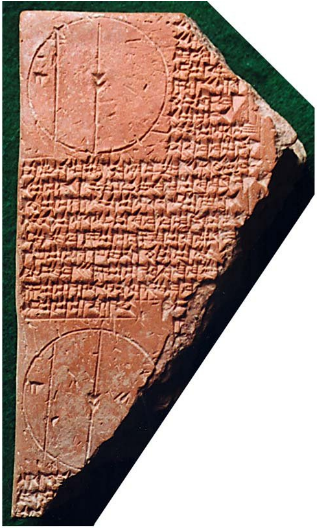


We want to count:

- The number of ways to insert n pairs of parentheses in a word of $n+1$ letters.

  For $n = 3$ there are 5 ways:
  ((ab)(cd)), (((ab)c)d), ((a(bc))d), (a((bc)d)), (a(b(cd))).

- The number of ways to join $2n$ points on a circle to form n nonintersecting chords.

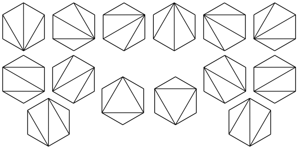


MS 3049
Properties of chords of circles, in the Sumerian sexagesimal system.
Babylonia, ca. 17th c. BC

We want to count:

- The number of ways to insert n pairs of parentheses in a word of $n+1$ letters.

  For $n = 3$ there are 5 ways:
  ((ab)(cd)), (((ab)c)d), ((a(bc))d), (a((bc)d)), (a(b(cd))).
- The number of ways to join $2n$ points on a circle to form n nonintersecting chords.
- The number of ways to triangulate a regular $(n+2)$-gon.
  For $n = 4$ there are 14 ways:



The answer: the Catalan numbers

The nth Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

They satisfy the convolution formula

$$C_{n+1} = \sum_{i=0}^{n} C_i \, C_{n-i}$$

From this one deduces the equation $q\,C(q)^2 - C(q) + 1 = 0$ for the generating function:

$$C(q) := \sum_{n=0}^{\infty} C_n \, q^n = \frac{1 - \sqrt{1-4q}}{2q}$$

Schubert calculus

- How many lines intersect four given lines in $\mathbb{P}^3$?

  Degenerate the four lines to two pairs of intersecting lines. Then the line through the two points of intersection, and the line which is the intersection of the two planes are the lines meeting all four lines – the answer is 2.
- How many lines meet six given planes in $\mathbb{P}^4$?

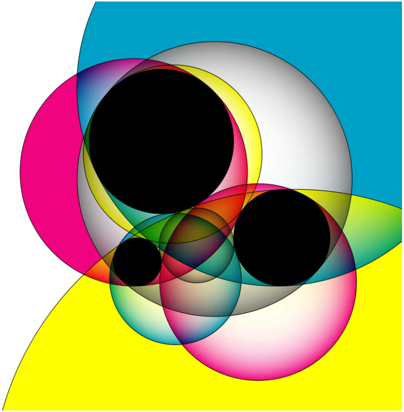
  The answer is 5.
- How many lines meet $2n$ given $(n-1)$-planes in $\mathbb{P}^{n+1}$?

  The answer is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The function $C(q) = \frac{1-\sqrt{1-4q}}{2q}$ displays the sequence

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

Apollonius's 8 circles

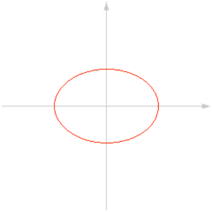


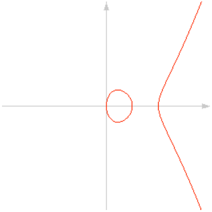
Melchoir,

Plane curves

A plane curve of degree d is given as the set of zeros of a (homogeneous) polynomial $F(x_0, x_1, x_2)$ in three variables of degree d.

The curve on the left is a conic ($d = 2$). It is a *rational* curve (it has genus 0). The one to the right is a cubic ($d = 3$). It is *elliptic* (it has genus 1).

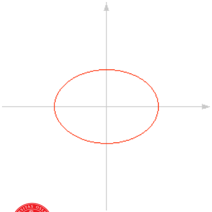


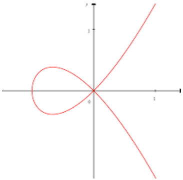


Plane curves

A plane curve of degree d is given as the set of zeros of a (homogeneous) polynomial $F(x_0, x_1, x_2)$ in three variables of degree d.

The curve on the left is a conic ($d = 2$). It is a *rational* curve (it has genus 0). The one to the right is a cubic ($d = 3$) with a node. It is *rational*.





Counting rational plane curves ("Gromov–Witten")

The space of all curves of degree d in $\mathbb{P}^2$ is a projective space of dimension $\binom{d+2}{2} - 1$.

The *rational* curves are those which have $\binom{d-1}{2}$ singular points. They form a sub-"space" of dimension $3d - 1$.

Problem: Find the number N_d of plane rational curves of degree d passing through $3d - 1$ points.

Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \big(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \big)$$

What is the generating function $\sum N_d \, q^d = ??$

A better generating function?

Set $n_d := \frac{N_d}{(3d-1)!}$. Then

$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} \frac{d_1 d_2((3d_1-2)(3d_2-2)(d+2)+8(d-1))}{6(3d-1)(3d-2)(3d-3)}$$

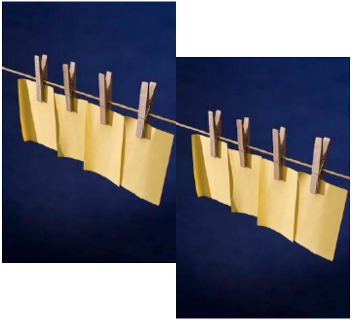
gives a generating function (in two variables)

$$\Gamma(y_1, y_2) := \sum_d n_d e^{y_1 d} y_2^{3d-1},$$

which satisfies the differential equation

$$\partial^3\Gamma/\partial y_2^3 = (\partial^3\Gamma/\partial y_1^2 \partial y_2)^2 - (\partial^3\Gamma/\partial y_1^3)(\partial^3\Gamma/\partial y_1 \partial y_2^2).$$

To display on the clothesline:



…this sequence of numbers:

$$
\begin{aligned}
N_1 &= 1 \\
N_2 &= 1 \\
N_3 &= 12 \\
N_4 &= 620 \\
N_5 &= 87304 \\
N_6 &= 26312976 \\
N_7 &= 14616808192 \\
N_8 &= 13525751027392 \\
N_9 &= 19385778269260800 \\
N_{10} &= 40739017561997799680
\end{aligned}
$$

Rational curves on a quartic surface

Let $S \subset \mathbb{P}^3$ be a quartic surface. If $H \subset \mathbb{P}^3$ is a plane, then $C := H \cap S \subset H \cong \mathbb{P}^2$ is a plane quartic curve.

Problem: How many H are such that C has three nodes? Or, how many H are triply tangent to S?

Answer 1: Set $d = 4$ in the formula

$$t(d) = \frac{1}{6}d(d-2)(d^7 - 4d^6 + 7d^5 - 45d^4 + 111d^2 + 548d - 960)$$

to get $t(4) = 3200$.

Answer 2: $P(q)^{24} = 1 + 24q + 324q^2 + 3200q^3 + \cdots$

K3 and the partition function

Let S be a K3 surface (e.g., $S \subset \mathbb{P}^3$ is a quartic surface) and $\mathcal{L}$ a line bundle. The *rational* (genus 0) curves in the linear system $|\mathcal{L}|$ are the curves with $r := (\mathcal{L}^2 + 2)/2$ singularities. Their number, N_r, is independent of S and $\mathcal{L}$.

Yau–Zaslow Conjecture: The generating function is

$$\sum_{r=0}^{\infty} N_r \, q^r = \prod_{m \geq 1} (1 - q^m)^{-24} = \left(\sum p(n) q^n\right)^{24} = P(n)^{24}.$$

Geometric reason (Bryan–Leung): The Euler number $c_2(S)$ is 24, and S can be degenerated (in symplectic geometry) to an elliptic fibration over $\mathbb{P}^1$ with 24 singular (hence rational) fibers. There are $p(j_i)$ degree j_i maps from a stable rational curve (a bunch of $\mathbb{P}^1$'s) to a singular fiber, hence $N_r = \sum_{j_1 + \cdots + j_{24} = r} p(j_1) \cdots p(j_{24})$.

Bell numbers

The *Stirling* numbers $S_{n,k}$ count the number of partitions of a *set* with n elements into k blocks. The *Bell* numbers count *all* partitions: $B_n = \sum_{k=1}^{n} S_{n,k}$. They can be defined recursively:

$$B_0 = 1 \text{ and } B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$$

Their *exponential* generating function

$$B(q) := \sum_{n=0}^{\infty} \frac{1}{n!} B_n \, q^n$$

satisfies the differential equation

$$\frac{B'(q)}{B(q)} = e^q,$$

and hence

$$B(q) = e^{e^q - 1}.$$

Polydiagonals

Let X be a space, and consider

$$X^n = X \times \cdots \times X = \{(x_1, \ldots, x_n) \mid x_i \in X\}.$$

For each $k \leq n$ and each partition $\mathbf{i}_k$ of $\{1, 2, \ldots, n\}$ into k disjoint subsets $\{i_1^j, i_2^j, \ldots, i_{l(j)}^j\}$, $j = 1, \ldots, k$, $\sum l(j) = n$, the corresponding *polydiagonal* is defined as

$$\Delta(\mathbf{i}_k) = \{(x_1, \ldots, x_n) \mid x_{i_r^j} = x_{i_s^j} \text{ for all } r, s, j\}.$$

Then $\#\{\Delta(\mathbf{i}_k)\} = S_{n,k}$, and $B_n = \# \cup_{k=1}^n \{\Delta(\mathbf{i}_k)\}$ is the number of all polydiagonals in X^n, and there are

$$\frac{n!}{j_1! \cdots j_n!} \left(\frac{1}{1!}\right)^{j_1} \cdots \left(\frac{1}{n!}\right)^{j_n}$$

polydiagonals of each type ($j_i = \#$blocks of i elements).

Example

$n = 4$: $X^4 = X \times X \times X \times X$

$k = 2$

$\mathbf{i}_2$:

$\{1\}, \{2,3,4\}; \{2\}, \{1,3,4\}; \{3\}, \{1,2,4\}; \{4\}, \{1,2,3\};$

$\{1,2\}, \{3,4\}; \{1,3\}, \{2,4\}; \{1,4\}, \{2,3\}.$

There are $\frac{4!}{1!1!}(\frac{1}{1!})^1(\frac{1}{3!})^1 = 4$ of the first type,

and $\frac{4!}{2!}(\frac{1}{2!})^2 = 3$ of the second type.

$\#\{\Delta(\mathbf{i}_2)\} = S_{4,2} = 4 + 3 = 7$

$\sum_k \#\{\Delta(\mathbf{i}_k)\} = \sum_k S_{4,k} = 1 + 7 + 6 + 1 = 15 = B_4$

Bell polynomials

The *partial Bell polynomials* are

$$B_{n,k}(a_1,\ldots,a_{n-k+1}) = \sum \binom{n}{j_1\cdots j_{n-k+1}} \left(\tfrac{a_1}{1!}\right)^{j_1} \cdots \left(\tfrac{a_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

summing over $j \vdash n, \sum j_i = k$.

The *complete (exponential) Bell polynomials* are

$$B_n(a_1,\ldots,a_n) = \sum_{k=1}^n B_{n,k}(a_1,\ldots,a_{n-k+1}).$$

$$\begin{aligned}
B_1(a_1) &= a_1,\\
B_2(a_1,a_2) &= a_1^2 + a_2,\\
B_3(a_1,a_2,a_3) &= a_1^3 + 3a_1a_2 + a_3\\
B_4(a_1,a_2,a_3,a_4) &= a_1^4 + 6a_1^2a_2 + 4a_1a_3 + 3a_2^2 + a_4
\end{aligned}$$

Bell polynomials – other definitions

Recursively defined by $B_0 = 1$ and

$$B_{n+1}(a_1, \ldots, a_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(a_1, \ldots, a_{n-i}) a_{i+1},$$

or by the formal identity

$$\sum_{i \geq 0} \frac{1}{i!} B_i(a_1, \ldots, a_i) q^i = \exp\left(\sum_{j \geq 1} \frac{1}{j!} a_j q^j\right).$$

Note that $S_{n,k} = B_{n,k}(1, \ldots, 1)$ and $B_n = B_n(1, \ldots, 1)$.

Faà di Bruno's formula

Let $h(t) = f(g(t))$ be a composed function.

Differentiate once: $h'(t) = f'(g(t))g'(t)$

and twice: $h''(t) = f''(g(t))g'(t)^2 + f'(t)g''(t)$.

Set $h_i := h^{(i)}(t)$, $f_i := f^{(i)}(g(t))$, $g_i := g^{(i)}(t)$. Then

$$\begin{aligned} h_1 &= f_1\, g_1 \\ h_2 &= f_1\, g_2 + f_2\, g_1^2 \\ h_3 &= f_1\, g_3 + 3 f_2\, g_1\, g_2 + f_3\, g_1^3 \end{aligned}$$

and indeed

$$h_n = \textstyle\sum_{k=1}^{n} f_k\, B_{n,k}(g_1, \ldots, g_{n-k+1})$$

Nodal curves on families of surfaces

Given a family of curves on a family of surfaces, find an expression N_r for the class of curves that have r nodes.

Conjecture (Kleiman–P.): There exist universal linear polynomials a_1, a_2, … in four variables such that

$$\sum N_r\, q^r = \sum \frac{1}{r!} B_r(a_1(m,k,s,x), \ldots, a_r(m,k,s,x))\, q^r$$

where B_r is the rth Bell polynomial and m, k, s, x are the Landweber–Novikov Chern classes of the family.

Proved it for $r \leq 8$.

Proved by T. Laarakker (2018) for all r.

The recursion

Express the *class* N_r of r-nodal curves in terms of $(r-1)$-nodal and lower.

Blow up the family of surfaces to get rid of one node in each curve, then use the formula for $(r-1)$-nodal curves on the blown up family and push it down.This creates a "derivation formula" à la Faà di Bruno:

$$r!N_r = (r-1)!N_{r-1}N_1 + \partial((r-1)!N_{r-1})$$

Set $a_1 = N_1$ and $a_2 = \partial(a_1)$. Then

$$2!N_2 = a_1^2 + a_2 = B_2(a_1, a_2)$$

and, pretendig ∂ is a derivation and setting $a_3 = \partial(a_2)$:

$$3!N_3 = (a_1^2 + a_2)a_1 + \partial(a_1^2 + a_2) = a_1^3 + 3a_1a_2 + a_3 = B_3(a_1, a_2, a_3).$$

Why Bell polynomials? Another reason!

Let $D \subset F \stackrel{\pi}{\to} Y$ be a family of curves on surfaces, and let $X \subset D$ denote the critical locus: the set of points that are singular in their fibre.

The class a_i represents an intersection class supported on the small diagonal of $X^i = X \times_Y \cdots \times_Y X$ and each product of a_i's to a class on a corresponding polydiagonal (Qviller).

For example, $a_1 = \pi_*[X] = N_1$ and $-a_2$ is the sum of the (excess) contribution of the diagonal in $X \times_Y X$ and of the locus of cuspidal curves. Therefore

$$N_2 = \frac{1}{2}(a_1^2 + a_2) = \frac{1}{2}B_2(a_1, a_2).$$

Other ways of counting curves

- Count tropical curves (Mikhalkin, Gathman–Markwig, Brugallé, Block–Göttsche, . . .)
- Count floor diagrams (Block, Göttsche, Fomin, . . .)
- Count *real* curves (Welschinger, . . .)
- Count curves on (some) *singular* toric surfaces (Liu–Osserman)
- Count curves (or sheaves, or cycles, . . .) on higher dimensional varieties, e.g. threefolds (Donaldson–Thomas)

Back to MacMahon

A *plane partition* of n can be viewed as a set of n monomials in three variables x, y, z such that all the other monomials generate an ideal in $\mathbb{C}[x, y, z]$, of colength n.

Example

The plane partition of 4:

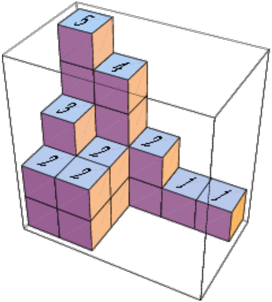
$$\begin{array}{c|c} 2 & 1 \\ \hline 1 & \end{array}$$

corresponds to the set of monomials $\{1, x, y, z\}$, which gives the ideal $\langle x^2, xy, xz, y^2, yz, z^2, \dots \rangle$ of colength 4.

Recall the plane partition of 22:

5	4	2	1	1
3	2			
2	2			

$$\longleftrightarrow \quad \{1, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2y, x^2yz, y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, z, z^2, z^3, z^4\}$$



Donaldson–Thomas and MacMahon

We have seen that the integer $\pi(n)$ counts the number of monomial ideals in $\mathbb{C}[x, y, z]$ of colength n.

These correspond to the colength n ideals that are invariant under the action of the torus $(\mathbb{C}^*)^3$,

or to the length n fixed points of $\mathbb{C}^3$ under the torus action.

This is the *virtual count* of the degree zero Donaldson–Thomas invariants:

$$DT_0(\mathbb{C}^3)(q) = M(-q) = \prod_{m \geq 1} (1 - (-q)^m)^{-m}$$

THANK YOU FOR YOUR ATTENTION!