Planar polypols, their adjoints and canonical forms

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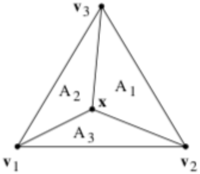
Algebraic geometry and its interactions

- Classical enumerative geometry: "so old-fashioned" until the arrival of string theory in theoretical physics (Gromov–Witten, Donaldson–Thomas, etc.).
- The physics of scattering amplitudes (probability that certain particles are produced in collision with other particles): positive geometries and canonical forms. (Calculating scattering amplitudes reduces to determining the canonical form.)
- Algebraic statistics: statistical models and likelihood equations.
- Combinatorics: polytopes and toric varieties.
- Less unexpected: algebraic geometry in geometric modeling.

Barycentric coordinates



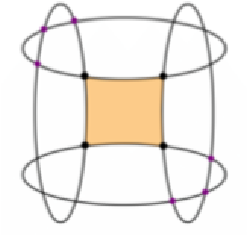
$$x = \phi_1(x)v_1 + \phi_2(x)v_2 + \phi_3(x)v_3$$
$$\phi_i(x) \geq 0$$
$$\phi_1(x) + \phi_2(x) + \phi_3(x) = 1$$

Barycentric coordinates were introduced by August Ferdinand Möbius in 1827.

Generalized barycentric coordinates

Generalized barycentric coordinates for *polygons* were introduced by Eugene Wachspress (1975), with a view towards applications for solving PDE's by the finite element method (further work by Warren, Floater, ...).

Wachspress defined *polycons* as polygons with sides being segments of conics.



Polypols

A (planar) *polypol* P is given by irreducible curves $C_1, \ldots, C_k \subset \mathbb{P}^2_{\mathbb{C}}$ and points $v_i \in C_i \pitchfork C_{i+1}$.

Set $d_i := \deg C_i$, $C := \cup C_i$, and $d := \sum d_i = \deg C$.

The *residual scheme* $R(P)$ of P is $\operatorname{Sing} C \setminus \{v_1, \ldots, v_k\} \subset C$ fattened by the conductor ideal of C at each point.

Recall that if $\mathcal{O}$ is a local ring of dimension 1 and $\overline{\mathcal{O}}$ its normalization, then the conductor ideal is

$$\mathcal{C} := \{f \in \mathcal{O} | f\overline{\mathcal{O}} \subseteq \mathcal{O}\}.$$

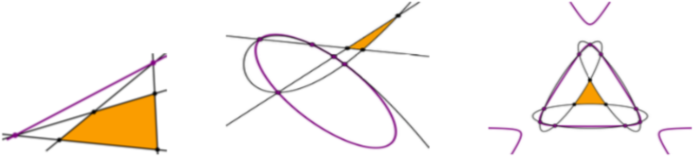
The conductor ideal of a node is the maximal ideal which is also the Jacobian ideal – that of a cusp is also the maximal ideal, which in this case is different from the Jacobian ideal.

Adjoints

If $D \subset \mathbb{P}^2_{\mathbb{C}}$ is a curve of degree d, then, classically, an *adjoint curve* is a curve of degree $d-3$ that cuts out the complete canonical series on D (or its normalization $\overline{D}$ if D is singular).

Definition

An adjoint curve A_P of a polypol P is a curve of degree $d-3$ that contains $R(P)$.



Regular rational polypols

A polypol is *rational* if all curves C_i are rational. We show: *a rational polypol has precisely one adjoint curve.*

Wachspress used the adjoint curve A_P to propose generalized barycentric coordinates on a polycon (or even rational polypol) P. These coordinates should be rational functions on P that are positive on the interior of P and have poles on A_P.

A polypol is *real* if the C_i and v_i are real, with a choice of real segments (sides) from v_i to v_{i+1} and a closed semialgebraic set $P_{\geq 0} \subset \mathbb{P}^2_{\mathbb{R}}$ with interior $P_{>0}$ a union of simply connected sets with boundary the union of the sides.

A real polypol is *regular* if all points on the sides of P except the vertices v_i are nonsingular on C and $C \cap P_{>0} = \emptyset$.

Wachspress coordinates for a polygon

Let P be a polygon, let $C_i \colon f_i = 0$, and $A_P \colon \alpha_P = 0$. Set $p_i := \frac{f_1 \cdots f_k}{f_i f_{i+1}}$. The Wachspress coordinates are the rational functions on P given by

$$\phi_i(v) := \frac{\alpha_P(v_i)}{p_i(v_i)} \cdot \frac{p_i(v)}{\alpha_P(v)}.$$

They satisfy

- $\phi_i(v_j) = \delta_{i,j}$
- $v = \sum_{i=1}^{d} \phi_i(v) v_i$
- $\phi_i(v) \geq 0$
- $\sum_{i=1}^{d} \phi_i(v) = 1$

Wachspress's conjecture

Let P be a regular rational polypol.

Wachspress (and we) showed that $A_P \cap \partial P = \emptyset$.

Wachspress conjectured: $A_P \cap P_{>0} = \emptyset$.

He claimed the conjecture holds (1) for convex polygons, and (2) for polypols with $d \leq 5$.

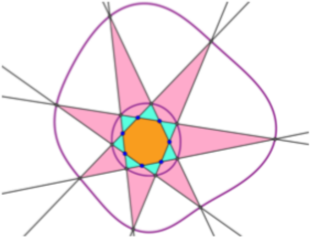
We proved (1) – and more.

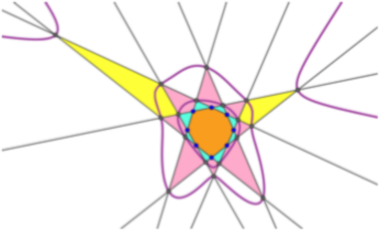
The first open case of (2) is a P formed by three ellipses. We show there are 44 configurations and prove the conjecture for 33 of these.

Convex polygons

Theorem

Let P be a convex real polygon with k sides. Then A_P is hyperbolic with respect to any $v \in P_{\geq 0}$. More precisely, $A_P(\mathbb{R})$ consists of $\lfloor \frac{k-3}{2} \rfloor$ nested ovals plus, if k is even, a pseudoline.





Positive geometries and their canonical forms

Let X be a projective complex variety of dimension n, $X_{\geq 0} \subset X(\mathbb{R})$ a closed semialgebraic subset such that its Euclidean interior $X_{>0}$ is an open oriented manifold with closure $X_{\geq 0}$. Let $C_1, \ldots, C_k$ be the irreducible components of ∂X. Let $C_{i,\geq 0}$ be the closure of the interior of $C_i \cap X_{\geq 0}$ in $C_i(\mathbb{R})$.

We say that $(X, X_{\geq 0})$ is a *positive geometry* if there exists a unique meromorphic n-form $\Omega(X, X_{\geq 0})$, called its canonical form, such that

(a) if $n = 0$, $X_{\geq 0}$ is a point and $\Omega(X, X_{\geq 0}) = \pm 1$,

(b) if $n > 0$, $(C_i, C_{i,\geq 0})$ is a positive geometry with canonical form $\Omega(C_i, C_{i,\geq 0}) = \mathrm{Res}_{C_i}(\Omega(X, X_{\geq 0}))$,

(c) $\Omega(X, X_{\geq 0})$ is holomorphic on $X \setminus \cup C_i$.

Positive geometries in dimension 1

Observe that it follows from the definition that if $(X, X_{\geq 0})$ is a positive geometry, then X can have no non-zero holomorphic n-forms.

Example ($n = 1$)

The curve X must be rational, hence $X = \mathbb{P}^1_{\mathbb{C}}$. Then $X_{\geq 0}$ is a finite union of closed segments in $\mathbb{P}^1_{\mathbb{R}}$.

Assume $X_{\geq 0} = [a, b] \subset \mathbb{R}$. Consider $\Omega := (\frac{1}{t-a} - \frac{1}{t-b})dt$. Then $\operatorname{Res}_a \Omega = 1$ and $\operatorname{Res}_b \Omega = -1$ and Ω is holomorphic on $\langle a, b \rangle$. Hence

$$\Omega(\mathbb{P}^1_{\mathbb{C}}, [a, b]) = \frac{t - b - t + a}{(t-a)(t-b)} dt = \frac{b - a}{(t-a)(b-t)} dt.$$

Regular rational polypols give positive geometries

Theorem

A regular rational polypol P gives a positive geometry with canonical form

$$\Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \frac{\alpha_P}{f_1 \cdots f_k} dx \wedge dy.$$

Proof. We have

$$\frac{\alpha_P}{f_1 \cdots f_k} dx \wedge dy = \frac{\alpha_P}{f_1 \cdots \hat{f}_i \cdots f_k} dx \wedge \frac{dy}{f_i}$$

and $df_i = (f_i)_x dx + (f_i)_y dy$ and $dx \wedge df_i = (f_i)_y dx \wedge dy$, hence

$$\operatorname{Res}_{C_i} \Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \frac{\alpha_P}{f_1 \cdots \hat{f}_i \cdots f_k \cdot (f_i)_y} dx.$$

Take a rational parameterization $t \mapsto (x(t), y(t))$ of C_i with $v_i = (x(a_i), y(a_i))$ and $v_{i+1} = (x(b_i), y(b_i))$.
We get

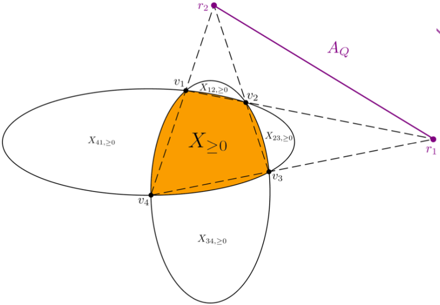
$$\operatorname{Res}_{C_i} \Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \frac{F(t)}{G(t)(t-a_i)(b_i-t)} dt$$

with $\deg F = \deg G = d_i(d-1) - 2$. Check that F and G have the same roots with the same multiplicities (Jacobian ideal = conductor ideal $\cdot$ ramification ideal). Thus $\gamma_i := F(t)/G(t)$ is a constant, and we get

$$\operatorname{Res}_{C_i} \Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \gamma_i \Omega(\mathbb{P}^1_{\mathbb{C}}, [a_i, b_i]) = \gamma_i \frac{b_i - a_i}{(t-a_i)(b_i-t)}.$$

With the correct scaling of α_P, we show $\gamma_i = 1$ for all i.

Other positive geometries



Open questions

- Wachspress's conjecture for regular rational polypols is wide open.
- The adjoint curve of a convex polygon is hyperbolic. For convex polypols (or for polytopes in higher dimension) the adjoint can be hyperbolic or not. Find a condition for hyperbolicity.
- When is the adjoint curve of a polypol singular?
- The *adjoint map* that sends a polypol to its adjoint curve is a finite map in some cases (we found all). What is the degree of this map? For convex heptagons, we conjecture the degree is 864.
- Prove the pushforward conjecture: for nice maps between positive geometries, the pushforward of the canonical form is the canonical form. (We proved this in dimension 1.)

Thank you for your attention!