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§1. INTRODUCTION

Fix an algebraically closed field k of characteristic 0, and let V be a vector space over k of dimension 4. Set $\mathbb{P}^3 = \mathbb{P}(V)$, so that $V = \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and $S_2(V) = \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

The main objective of this paper is the study of the variety $X \subset \text{Grass}_3(S_2(V))$ (the Grassmann variety parametrizing subspaces of dimension 3) consisting of the nets of quadrics generated by the 2-minors of (3×2) -matrices with linear forms as entries. The interest in X stems from the fact that the space of twisted cubic curves may be considered as an open subset of X ; in fact, any twisted cubic curve is defined by the vanishing of the 2-minors of a matrix as above.

We shall prove that X is smooth and compact. Hence it gives a natural compactification of the space of twisted cubics. Moreover, it will follow from the construction that X is a minimal compactification in the sense that the complement in X of the space of twisted cubics is an irreducible divisor. Furthermore, we compute - at least in principle - the Chow ring of X by giving algebra generators and relations.

Another compactification of the space of twisted cubics in the Hilbert scheme, or more precisely, the component H of $\text{Hilb}^{3m+1}(\mathbb{P}^3)$ containing the points corresponding to twisted cubics. In [P-S] it is shown that H is smooth. Furthermore, the complement in H of the space of twisted cubics is the union of H_c and H_e , where H_c consists of points corresponding to degenerate twisted cubics

that are arithmetically Cohen-Macaulay, and H_e to those that are not, i.e., that consist of a plane, singular cubic curve with an embedded point at a singular point. It is easy to see that all these degenerate curves are contained in exactly three linearly independent quadrics (for the Cohen-Macaulay curve this follows e.g. from [E], for curves with an embedded point this is shown in [P-S]). Hence there is a map $f: H + X \subset \text{Grass}_3(S_2(V))$ which sends a curve to the net of quadrics containing it. Outside H_e , f is an isomorphism, because a curve $C \in H - H_e$ is the intersection of the quadrics in $f(C)$.

If $C \in H_e$, then C is a plane cubic with an embedded point, and $f(C)$ is the net generated by $L_0^2, L_0 L_1, L_0 L_2$, where $L_0 = 0$ is an equation of the plane, and $L_0 = L_1 = L_2 = 0$ are equations of the point. Hence $f(H_e)$ is isomorphic to the point-plane incidence correspondence I (which is embedded in $\text{Grass}_3(S_2(V))$ as indicated above).

At the Rocca di Papa conference we asserted that $f: H + X$ is the blow-up of X along I . However, when writing up this paper, we discovered a gap in that proof. Only in the last minute we succeeded filling the gap, so that the proof of this fact, together with its "corollary" - the computation of the Chow ring of H - will be the subject of a forthcoming paper.

The restriction of f to $f^{-1}(I)_{\text{red}} = H_e + I$ is isomorphic to a map $P(N^V) + I$, for some rank 7 bundle N on I . Let $g: \tilde{\Pi} + I$ denote the universal plane. Then N is the subbundle of $g_* \mathcal{O}_{\tilde{\Pi}}(3)$ with fiber at (P, Π) consisting of the cubics in Π that are singular at P .

Knowing the Betti numbers of X this suffices to compute the Betti numbers of H (see also [Sch]). We remark that because the natural action of a maximal torus in $\text{PGL}(V)$ on X and on H has isolated fixed points, the Chow groups are equal to the homology groups, and they are all free abelian [B1, B2].

The talks by the two first authors at Rocca di Papa concerned respectively three points in the plane and twisted cubics, i.e., determinantal nets of quadrics in $\mathbb{P}^2$ and in $\mathbb{P}^3$. In the present paper we have chosen to treat the case of

twisted cubics; though our methods work in all dimensions, we hope to make the exposition clearer by treating only one case.

Finally, we would like to thank all the people involved in the organization of this interesting and inspiring conference.

§2. THE CONSTRUCTION OF X

The points of the compactification X of the space of twisted cubics are nets of quadrics that can be generated by the 2 -minors of a (3×2) -matrix with linear forms as entries. We shall now make this connection explicit by exhibiting X as a quotient space.

Let E and F be vector spaces of dimensions 3 and 2 respectively. Set $W = \text{Hom}_k(F, E \otimes V)$. After a choice of bases for E and F we may consider an element $A \in W$ as a matrix (a_{ij}) , with $1 \leq i \leq 3$, $1 \leq j \leq 2$, with entries linear forms, i.e., $a_{ij} \in V$.

For any matrix representation (a_{ij}) of $A \in W$ the maximal minors generate the same subspace E_A of $S_2(V)$. An intrinsic way of constructing E_A is as follows: The map A induces a map $E^V + F^V \otimes V$, hence a map $\Lambda^2 E^V + \Lambda^2(F^V \otimes V)$. Now there is a canonical map $\Lambda^2(F^V \otimes V) \rightarrow \Lambda^2 F^V \otimes S_2(V)$ and a canonical isomorphism $\Lambda^2 E^V \cong E \otimes \Lambda^3 E^V$, hence - after identifying the two 1 -dimensional vector spaces $\Lambda^3 E^V$ and $\Lambda^2 F^V$ - we obtain a map $\lambda_A: E \rightarrow S_2(V)$, whose image is E_A . Note that λ_A is uniquely defined up to a scalar, due to the choice of isomorphism $\Lambda^3 E^V \cong \Lambda^2 F^V$.

The group $G_1 = \text{GL}(E) \times \text{GL}(F)$ acts on W by $(g, h)A = g \otimes \text{id}_V \cdot A \cdot h^{-1}$. Clearly the subgroup $\Gamma = \{(\alpha \cdot \text{id}_E, \alpha \cdot \text{id}_F) : \alpha \in k^*\}$ acts trivially on W , hence the group $G = G_1/\Gamma$ acts on W .

For technical reasons we shall consider $P = P(W^V)$ and the action of $S = \text{SL}(E) \times \text{SL}(F)$ on P induced by the action of G on W . If $A \in W$, let

$\bar{A} \in P$ denote the corresponding element. Denote by $U \subset W$ the set of maps A such that $\dim E_A = 3$, and denote by $\bar{U} \subset P$ the image of U . There is a map $\psi: U \rightarrow \text{Grass}_3(S_2(V))$ which sends A to the net E_A . Clearly ψ factors through X and is G -invariant.

Proposition 1: *There exists a projective, smooth geometric quotient $\bar{U}/S$ of $\bar{U}$ by S . The map $\bar{U}/S \rightarrow X$ induced by ψ is an isomorphism and $U \rightarrow X$ is a principal homogeneous bundle under G .*

The rest of this section is devoted to the proof of the above proposition.

Lemma 1: *The following statements are equivalent.*

- (i) $\bar{A}$ is a semistable point under the action of S .
- (ii) $\bar{A}$ is a stable point under the action of S .
- (iii) $\dim E_A = 3$.

Proof: Assume $\bar{A} \in P$ is not stable. Then there exists an element $(g, h) \in S$ and a 1 -parameter subgroup λ of S , on standard form, such that $\mu_\lambda(g \otimes \text{id}_V \cdot A \cdot h^{-1}) \leq 0$ (see [N], Prop. 4.11). That λ is on standard form means that

$$\lambda(t) = \left(\begin{pmatrix} t^{\alpha_1} & 0 & 0 \\ 0 & t^{\alpha_2} & 0 \\ 0 & 0 & t^{\alpha_3} \end{pmatrix}, \begin{pmatrix} t^{-\beta_1} & 0 \\ 0 & t^{-\beta_2} \end{pmatrix} \right),$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3$, $\beta_1 \geq \beta_2$, and $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 = 0$. Let $A' = (a'_{ij}) = g \otimes \text{id}_V \cdot A \cdot h^{-1}$. Clearly $\lambda(t) \cdot A = (t^{\alpha_1 + \beta_j} a'_{ij})$. If $\alpha_1 + \beta_j < 0$, we have $a'_{ij} = 0$, because $\bar{A}$ is not stable. Hence $a'_{32} = 0$ because $\alpha_3 + \beta_2 < 0$. Suppose $\alpha_2 + \beta_2 \geq 0$ and $\alpha_3 + \beta_1 \geq 0$. Then, adding the two inequalities, we get $-\alpha_1 = \alpha_2 + \alpha_3 + \beta_1 + \beta_2 \geq 0$, which contradicts the fact that α_1 is the largest of three nonzero numbers whose sum is zero. Hence either $a'_{22} = 0$ or $a'_{31} = 0$, so A is equivalent to a matrix of one of the following types:

$$\begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ * & 0 \\ * & 0 \end{pmatrix}.$$

On the other hand, it is easy to see that matrices of the above types give points of P that are not semistable; in fact, we can use a 1-parameter subgroup with $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = -5$, $\beta_1 = 1$, $\beta_2 = -1$, in the first case, and one with $\alpha_1 = 5$, $\alpha_2 = -2$, $\alpha_3 = -3$, $\beta_1 = 4$, $\beta_2 = -4$, in the second case. The following lemma then finishes the proof of Lemma 1.

Lemma 2: Let $A \in W$. Then $\dim E_A \leq 2$ if and only if A is equivalent under G to a matrix of one of the above types, i.e., with $a_{32} = a_{31} = 0$ or with $a_{32} = a_{22} = 0$.

Proof: The maximal minors of such matrices are clearly not independent. Hence we may assume that $\dim E_A \leq 2$. By performing row operations on A , we may assume that $A = (a_{ij})$, with $a_{21}a_{32} = a_{31}a_{22}$. If $a_{32} = 0$, then either $a_{31} = 0$ or $a_{22} = 0$, and we are done. If $a_{21} = 0$, then either $a_{31} = 0$ or $a_{22} = 0$, and we are done by interchanging the two columns or the last two rows. If all four a_{ij} 's are nonzero, we can write $a_{21} = \gamma a_{31}$ and $a_{32} = \gamma^{-1} a_{22}$, or $a_{21} = \gamma a_{22}$ and $a_{32} = \gamma^{-1} a_{31}$, with $\gamma \in k^*$; in both cases an obvious row operation puts A on the desired form. $\square$

We conclude from Lemma 1 and the theory of $[M]$ (see $[N]$, Thm.3.14) that there exists a projective geometric quotient $\bar{U}/S$.

Lemma 3: Let $A \in W$ and let $\mathcal{I}_A \subset \mathcal{O}_{\mathbb{P}^3}$ denote the sheaf of ideals generated by the quadrics of E_A . Assume $\dim E_A = 3$ and that $V(\mathcal{I}_A)$ is not a curve. Then A can be represented by a matrix on the form

$$\begin{pmatrix} 0 & -L_0 \\ L_0 & 0 \\ -L_1 & L_2 \end{pmatrix},$$

where $L_0, L_1, L_2 \in V$ are linearly independent.

Proof: If $V(\mathcal{I}_A)$ is not a curve, then the 2-minors of A have a common factor which is not a quadric, since otherwise $\dim E_A \leq 1$. Hence they must have a common linear factor. Because $V(\mathcal{I}_A)$ has no non-embedded component of codimension greater than 2, the minors are $L_0^2 L_2$, $L_0 L_1^2$, L_0^2 , for some linearly independent forms L_0, L_1, L_2 . Since the relations between these minors obviously are the same as the relations between L_2, L_1, L_0 , the columns of A are linear combinations of the columns of the Koszul matrix

$$K = \begin{pmatrix} 0 & -L_0 & L_1 \\ L_0 & 0 & -L_2 \\ -L_1 & L_2 & 0 \end{pmatrix}$$

Hence $A = K \cdot (\alpha_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 2}$, where $\alpha_{ij} \in k$. Working modulo L_1 and L_2 we see that $\det(\alpha_{ij})_{1 \leq i \leq 2, 1 \leq j \leq 2} \neq 0$, and modulo L_0 we obtain $\alpha_{31} = \alpha_{32} = 0$. $\square$

The map $\bar{U}/S \rightarrow X$, induced by ψ , is bijective on closed points. In fact, on points corresponding to nets defining curves, this is clear because of the following: If a net E_A defines a curve, this curve is (a possible degeneration of) a twisted cubic, which in turn determines the matrix A up to the action by G . If E_A does not define a curve, it defines by Lemma 3 a point-plane, and all matrices A defining this point-plane are equivalent under G , again by Lemma 3.

Lemma 4: For any point $A \in U$, the derivative $d\psi_A$ of ψ at A has rank 12.

Before proving this lemma we observe that this finishes the proof of Proposition 1. In fact, since the map $\bar{U}/S \rightarrow X$ is bijective, the map $U \rightarrow X$ has connected fibers. Hence, by Lemma 4, X is smooth, and thus $\bar{U}/S \rightarrow X$ is an isomorphism because of Zariski's Main Theorem. Now it is easy to see that $\bar{U}/S$ is a quotient of U by G ($U \rightarrow \bar{U}$ is a k^* -bundle, and G is an extension of k^* by S), so we may identify $U/G = \bar{U}/S = X$. To show that $U \rightarrow X$ is a principal homogeneous bundle under G , it therefore suffices ($[M]$, 0.9) to check that G acts freely on U . There are two cases to consider. Assume first that $A \in U$ is such that $V(\mathcal{I}_A)$ is a curve. Then there is a resolution of $\mathcal{I}_A$ on $\mathbb{P}^3$,

$$0 + 2\mathcal{O}(-3) \xrightarrow{A} 3\mathcal{O}(-2) + \mathcal{J}_A + 0.$$

Assume $(g, h) \in G_1$ stabilizes A . Then, since $\text{Hom}_{\mathbb{P}^3}(\mathcal{J}_A, \mathcal{J}_A) = k$, we have

$$\begin{aligned} 0 + 2\mathcal{O}(-3) &\xrightarrow{A} 3\mathcal{O}(-2) + \mathcal{J}_A + 0 \\ &\quad \uparrow h \quad \uparrow g \quad \uparrow a \\ 0 + 2\mathcal{O}(-3) &\xrightarrow{A} 3\mathcal{O}(-2) + \mathcal{J}_A + 0. \end{aligned}$$

Then $g - \alpha \cdot \text{id}$ induces a map $3\mathcal{O}(-2) + 2\mathcal{O}(-3)$, which must be zero - hence $g = \alpha \cdot \text{id}$. This implies $h = \alpha \cdot \text{id}$, hence $(g, h) \in \Gamma$, i.e., A has trivial stabilizer in G . In the case $V(\mathcal{J}_A)$ is not a curve, by Lemma 3, A can be represented by a matrix on the form

$$\begin{pmatrix} 0 & -L_0 \\ L_0 & 0 \\ -L_1 & L_2 \end{pmatrix},$$

and one verifies by direct computation that such an A has trivial stabilizer in G .

Proof of Lemma 4: Since $\dim X = 12$ and $\dim U = 24$, $\text{rank } d_A \Psi = 12$ for general points $A \in U$. There are obvious actions on U and on $\text{Grass}_3(S_2(V))$ by $\text{GL}(V)$, under which Ψ is equivariant. Hence the set of points $A \in U$ where $\text{rank } d_A \Psi < 12$ is invariant under the action of $\text{GL}(V)$, and - if nonempty - contains at least one closed orbit. The only closed orbits in U are the orbit consisting of matrices defining point-planes and that of matrices defining the full second order neighborhood of a line. Therefore we may assume that A is one of the following matrices

$$A_1 = \begin{pmatrix} 0 & -X_0 \\ X_0 & 0 \\ -X_1 & X_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} X_0 & 0 \\ X_1 & X_0 \\ 0 & X_1 \end{pmatrix}.$$

The tangent space to U at A is just W , and that of $\text{Grass}_3(S_2(V))$ at $\Psi(A)$

is $\text{Hom}_k(E_A, S_2(V)/E_A)$. A tangent vector τ to U at A is given as $\tau = A + \epsilon L$, where $L \in W$ and $\epsilon^2 = 0$. The map $d_A \Psi(\tau)$ sends a minor of A to the ϵ -part of the corresponding minor of τ . Hence $d_A \Psi(\tau) = 0$ is equivalent to the three relations

$$(*) \quad \begin{vmatrix} a_{i1} & l_{i2} \\ a_{j1} & l_{j2} \end{vmatrix} + \begin{vmatrix} l_{i1} & a_{i2} \\ l_{j1} & a_{j2} \end{vmatrix} \in E_A$$

for $1 \leq i < j \leq 3$.

Now $\text{Lie}(G_1) = \text{Lie}(\text{GL}(E)) \oplus \text{Lie}(\text{GL}(F))$ acts on $\text{Ker}(d_A \Psi)$ via

$$(\text{id}_E + \epsilon B)(A + \epsilon L)(\text{id}_F - \epsilon C) = A + \epsilon(L + BA - AC)$$

where $B \in \text{Lie}(\text{GL}(E))$ and $C \in \text{Lie}(\text{GL}(F))$. It will be enough to show that $\text{Ker}(d_A \Psi)$ is the orbit of $A + \epsilon \cdot 0$, because $\dim \text{Ker}(d_A \Psi) \geq 12$ and the orbit is of dimension ≤ 12 since $\text{Lie}(\Gamma) = \{(\gamma \cdot \text{id}_E, -\gamma \cdot \text{id}_F) : \gamma \in k\}$ acts trivially.

We may replace $\tau \in \text{Ker}(d_A \Psi)$ by any other element in the orbit of τ . Hence, if $A = A_1$, we may assume

$$\tau = \begin{pmatrix} 0 & -X_0 \\ X_0 & 0 \\ -X_1 & X_2 \end{pmatrix} + \epsilon \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \end{pmatrix}$$

with $l_{11}, l_{21}, l_{31} \in k[X_2, X_3]$. Using the relations $(*)$ with $i = 2, j = 3$ and $i = 1, j = 3$, we see that $l_{11} = l_{21} = 0$, and that all $l_{ij} \in k[X_0, X_1, X_2]$. It is now easy to produce an element $(B, C) \in \text{Lie}(G_1)$ such that $-(l_{ij}) = BA_1 - A_1 C$.

If $A = A_2$, then we may assume

$$\tau = \begin{pmatrix} X_0 & 0 \\ X_1 & X_0 \\ 0 & X_1 \end{pmatrix} + \epsilon \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \end{pmatrix}$$

with $\varepsilon_{12}, \varepsilon_{22}, \varepsilon_{32} \in k[X_2, X_3]$. Using the relations (*) with $i = 2, j = 2$ and $i = 1, j = 3$, we see that $\varepsilon_{12} = \varepsilon_{32} = 0$ and that all $\varepsilon_{ij} \in k[X_0, X_1]$, so that $\varepsilon_{22} = 0$. As above we can then write $-(\varepsilon_{ij}) = B A_2 - A_2 C$ for some $(B, C) \in \text{Lie}(G_1)$. $\square$

§3. THE CHOW RING OF X

We start by constructing two bundles on X , $\mathcal{E}$ and $\mathcal{F}$, of ranks 3 and 2 respectively, whose Chern classes are algebra generators for the Chow ring $A(X)$ of X . The idea is to try to descend the bundles E_U and F_U on U to X . Clearly G_1 acts on E_U and F_U as follows. If $(g, h) \in G_1$ and $(e, u) \in E_U = E \times U$, then $(g, h)(e, u) = (ge, guh^{-1})$, and similarly on F_U . Since $\Gamma \subset G_1$ does not act trivially on these bundles, we do not get an induced action by $G = G_1/\Gamma$. However, if $\lambda = \text{Hom}_k(\Lambda^3 E, \Lambda^2 F)$, i.e., λ corresponds to the character $(g, h) \mapsto \frac{\det h}{\det g}$, it is easily seen that Γ acts trivially on the G_1 -bundles $E'_U = E_U \otimes_k \lambda$ and $F'_U = F_U \otimes_k \lambda$. Hence E'_U and F'_U are G -bundles, and because G acts freely on U , these bundles descend to bundles $\mathcal{E}$ and $\mathcal{F}$ on X .

Since by definition $W = \text{Hom}_k(F, E \otimes V)$, there is a universal map $\tilde{A}: F_U + E_U \otimes V$ on U , and hence also a map $\tilde{A} \otimes \text{id}_\lambda: F'_U + E'_U \otimes V$. This map is G -equivariant and descends to a map $\alpha: \mathcal{F} + \mathcal{E} \otimes V$ on X .

Proposition 2: *The Chern classes of $\mathcal{E}$ and $\mathcal{F}$ generate the Chow ring $A(X)$ as a $\mathbb{Z}$ -algebra.*

Proof: The group G_1 is a structure group for the bundle $\mathcal{E} \otimes \mathcal{F}$, so we may construct the principal G_1 -homogeneous bundle $\phi: T + X$ associated with $\mathcal{E} \otimes \mathcal{F}$. Two things should be observed. Firstly, the Chow ring of T is, via ϕ^* , isomorphic to $A(X)$ modulo the ideal generated by the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ ([C], Remarques, p.4-35). Secondly, since $\phi^*\mathcal{E}$ and $\phi^*\mathcal{F}$ are trivial, ϕ factors through $\psi: U + X$, and it is easily seen that the induced map $T + U$ is a k^* -bundle. Therefore the pullback map $A(U) + A(T)$ is onto. Now U is an open subset of the affine space W , and so $A(U) = \mathbb{Z}$. This proves the proposition. $\square$

Remark that $c_1(\mathcal{E}) = c_1(\mathcal{F})$; in fact, as G -bundles, $\Lambda^3 E'_U \cong \Lambda^2 F'_U$, hence we have $\Lambda^3 \mathcal{E} \cong \Lambda^2 \mathcal{F}$. Furthermore, by the definition of ψ , the restriction to X of the universal subbundle $\mathcal{Q}$ of $S_2(V)$ on $\text{Grass}_3(S_2(V))$ is $\mathcal{E}$.

Let $\pi: Y = \text{Grass}_2(\mathcal{Q} \otimes V) + \text{Grass}_3(S_2(V))$ denote the Grassmann bundle of rank 2 subbundles of $\mathcal{Q} \otimes V$, and let $\mathcal{R}$ denote the universal subbundle of $\mathcal{Q} \otimes V$ on Y . The map $\alpha: \mathcal{F} + \mathcal{E} \otimes V$ induces an embedding $i: X + Y$. In fact, α gives $\mathcal{F}$ as a subbundle of $\mathcal{E} \otimes V$ because if α is not injective at a point represented by a (3×2) -matrix A , the two columns of A are linearly dependent, hence all the 2-minors vanish. This is impossible, so we get a map $i: X + Y$, which - being a section over X of the projection $Y + \text{Grass}_3(S_2(V))$ - is an embedding.

Proposition 3: *The class of X in $A(Y)$ is given by*

$$[X] = \frac{1}{m} [c(\pi^* \mathcal{Q}^V)^{10} (1 + c_1(\mathcal{R}) - c_1(\pi^* \mathcal{Q}))^{-1}]_{29},$$

where m is some positive integer.

Proof: On Y there are two inclusions, $\pi^* \mathcal{Q} + S_2(V)_Y$ and $\pi^*(\mathcal{Q} \otimes \Lambda^3 \mathcal{Q}^{-1}) \otimes \Lambda^2 \mathcal{R} + S_2(V)_Y$, the latter being constructed from $\mathcal{R} + \pi^* \mathcal{Q} \otimes V$ in the same way as we constructed λ_A in §2. The points of X correspond to nets of quadrics that are generated by the 2-minors of a (3×2) -matrix, so it is clear that X consists of the points of Y where the two above maps of bundles are proportional. Hence X is set-theoretically the scheme Z defined by the 2-minors of the map

$$\mathcal{O}_Y \otimes (\Lambda^2 \mathcal{R} \otimes \Lambda^3 \pi^* \mathcal{Q}^{-1}) + \text{Hom}_Y(\pi^* \mathcal{Q}, S_2(V)).$$

Since X has the "right" codimension 29, the class of Z is given by Porteous' formula as $[Z] = [c(\pi^* \mathcal{Q}^V)^{10} (1 + c_1(\mathcal{R}) - c_1(\pi^* \mathcal{Q}))^{-1}]_{29}$. Since X is irreducible, $[Z] = m[X]$ for some positive integer m . This finishes the proof. (Probably Z is reduced, so that $m = 1$ holds, but we don't need this.) $\square$

Theorem 1: *The Chow ring of X is given by $A(X) = A(Y)/a$, where*

* see correction

$$\alpha = \text{Ann}([c(\pi^* \mathcal{Q})^{10} (1+c_1(\mathcal{R})-c_1(\pi^* \mathcal{Q}))^{-1}]_{29}).$$

Proof: Recall that $\mathcal{E} = i^* \pi^* \mathcal{Q}$ and $\mathcal{F} = i^* \pi^* \mathcal{R}$, where $i: X \rightarrow Y$ is the embedding. Since $A(Y)$ is generated by the Chern classes of $\pi^* \mathcal{Q}$ and $\mathcal{R}$, and $A(X)$ by those of $\mathcal{E}$ and $\mathcal{F}$ (by Proposition 2), the map $i^*: A(Y) \rightarrow A(X)$ is surjective, hence ([S], proof of 5.2) $\text{Ker}(i^*) = \text{Ann}([X])$. The theorem then follows from Proposition 3 and the fact that $A(Y)$ is a free abelian group. $\square$

As a byproduct of the fact that the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ generate $A(X)$ (Proposition 2) and the knowledge of the topological Euler-Poincaré characteristic $e(X)$ of X , we obtain the Betti numbers of X :

i	0	1	2	3	4	5	6
$b_{2i} = b_{2(12-i)}$	1	1	3	4	7	8	10

To see this, we use that $e(X) = 58$ (this will be shown below). Let $R = \bigoplus_i R_i$ denote the free, graded $\mathbb{Z}$ -algebra with one generator in degree 1, two in degree 2, and one in degree 3. Then $A(X)$ is a quotient of R by a graded ideal $J = \bigoplus_i J_i$. The dimensions of the R_i are:

i	0	1	2	3	4	5	6
$\dim R_i$	1	1	3	4	7	9	14

Then $\sum_{i=0}^5 \dim R_i + \dim R_6 = 64$. Hence, if $x = \sum_{i=0}^5 \dim J_i$ and $y = \dim J_6$, we get $2x+y = 6$. Clearly $x = 0$, $y = 6$ is impossible, since $b_{10} \leq b_{12}$. Furthermore, $y = 0$ is impossible because if $J_i \neq 0$ for some $i < 6$, then $J_6 \neq 0$. Assume $x = y = 2$. If $J_4 \neq 0$, then $\dim J_6 \geq 3$, so $J_4 = 0$ and $\dim J_5 = 2$. It follows that $J_6 = tJ_5$, where t is the generator of degree 1. The locally split bundle map on X , $\mathcal{E}^V + \mathcal{F}^V \otimes V$, gives a relation of degree 6 between the Chern classes of $\mathcal{E}$ and $\mathcal{F}$, namely $[c(\mathcal{F}^V)^4 c(\mathcal{E}^V)^{-1}]_6 = 0$. This gives an element of J_6 which is not a multiple of t . So the only possibility left is $x = 1$, $y = 4$, and we are done.

The Euler-Poincaré characteristic of X is computed using the action of a maximal

torus of $\text{PGL}(V)$ on X . The fixed points are isolated and finite in number, and their number equals $e(X)$ [B1, B2]. If a fixed point of X corresponds to a curve, the support of this curve is contained in the tetrahedron of reference. Hence the curve is either three non coplanar edges (there are 16 such), one edge doubled in a plane (face) union a consecutive edge not contained in that plane (there are 24 such), or the full second order neighborhood of one edge (there are 6 of these). Since the fixed points that do not correspond to curves lie in I , there are $e(I) = 12$ of these. Adding up gives $e(X) = 58$.

Using the fact that the map $f: H \rightarrow X$ gives an isomorphism $H \xrightarrow{H_e} X-I$ and that the restriction of f to H_e is a bundle $\mathbb{P}(N^V) + I$, we obtain the Betti numbers of H .

i	0	1	2	3	4	5	6
$b_{2i} = b_{2(12-i)}$	1	2	6	10	16	19	22

CORRECTION: The formula of Proposition 3 is false. The correct value of $[X]$ is given as follows: Let $p: P := \mathbb{P}(Q_Y^V) \rightarrow Y$ with tautological exact sequence ($\text{rank } A=1$, $\text{rank } B=2$)

$$0 \rightarrow A \rightarrow Q_p \rightarrow B \rightarrow 0.$$

Also, put $L = (\Lambda^2 \mathcal{R})^{-1} \otimes (\Lambda^3 Q_Y)$. Then

$$[X] = [c(Q_Y^V \otimes L)^{10} c(L)^{-1}]_{29} \cdot p_* ([c(Q_p^V \otimes L_p)^{10} c(L_p)^{-1} c(\mathcal{R}_p^V \otimes B)^{-4} c(A^{-1} \otimes B)]_{15} c_4(\mathcal{R}_p^V \otimes B)^4).$$

The statement of Theorem 1 must be changed accordingly.

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GENRE D'UNE COURBE LISSE
TRACEE SUR UNE VARIETE REGLEE

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1 - FORMULE DE SEGRE.

Il est bien connu que si X est une courbe lisse et connexe qui admet une immersion non dégénérée dans $\mathbb{P}^r$ (i.e. l'espace linéaire $\langle X \rangle$ engendrée par les points de X coïncide avec $\mathbb{P}^r$), alors son genre est majoré par une fonction $w(r, d)$, calculée par G. Castelnuovo en 1889 [C], qui dépend du degré d de la courbe dans $\mathbb{P}^r$ et de la dimension de l'espace.

La démonstration de Castelnuovo se base essentiellement sur une propriété d'uniformité de la section hyperplane suivant laquelle r points quelconque d'une section hyperplane générale sont indépendents.

Deux ans avant, en 1887, C. Segre [S] calculait, sous hypothèses tout à fait naturelles, une borne semblable pour le genre d'une courbe contenue dans une famille ω^1 d'espaces projectifs. La démonstration de Segre repose sur une formule enumerative plutôt obscure que Schubert lui avait communiqué dans une lettre.

Le but de ce papier est de rappeler l'attention sur cette limitation de Segre-Schubert qui est peu connue dans la littérature contemporaine et, ensuite, considérer quelques applications à la théorie des courbes de $\mathbb{P}^r$.

Plus précisément, soit T une courbe lisse, irréductible de genre p . Sur T on fixe un fibré vectoriel F de rang $s \geq 2$ et soit $\mathbb{F} = \mathbb{P}(F)$ le fibré projectif associé. On a, alors, la projection

$$\pi : \mathbb{F} \rightarrow T$$

* Espace projectif sur un corps $k = \bar{k}$ de caractéristique zéro.