Envelopes and evolutes of curves and surfaces

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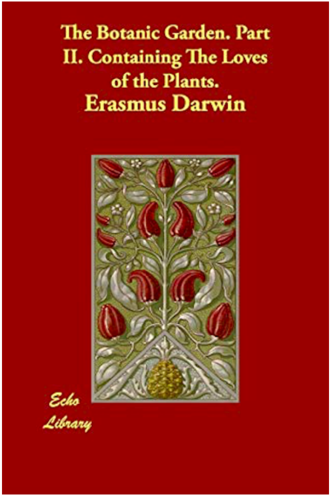
Erasmus Darwin 1791: The Loves of the Plants



George Canning, the Anti-Jacobin 1798

THE LOVES OF THE TRIANGLES.

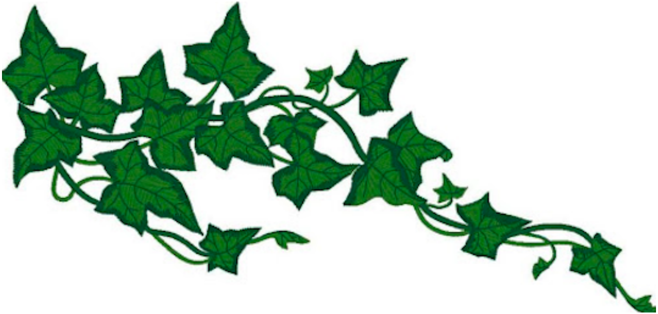
A MATHEMATICAL AND PHILOSOPHICAL POEM.

INSCRIBED TO DR. DARWIN.

Debased, corrupted, groveling, and confined,
No DEFINITIONS touch *your* senseless mind;
To *you* no POSTULATES prefer their claim,
No ardent AXIOMS *your* dull souls inflame;
For *you* no TANGENTS touch, no ANGLES meet,
No CIRCLES join in osculation sweet!

For *me*, ye CISSOIDS, round my temples bend
Your wandering Curves; ye CONCHOIDS extend;
Let playful PENDULES quick vibration feel,
While silent CYCLOIS rests upon her wheel;

Ivy ($\kappa\iota\sigma\sigma\acute{o}\varsigma$)



Ver. 11. *Cissois*--A Curve supposed to resemble the sprig of ivy, from which it has its name, and therefore peculiarly adapted to poetry.

Envelopes of families of linear spaces

Let $X \subset \mathbb{P}^n$ be a projective variety, $r = \dim X$, $\mathcal{F}$ a vector bundle of rank $n - r + 1$, and $\mathcal{O}_X^{n+1} \to \mathcal{F}$ a surjective map. Set $\pi : \mathbb{P}(\mathcal{F}) \to X$.

This gives a family of linear $(n - r)$-spaces

$$\psi : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}^n \to \mathbb{P}^n.$$

The *envelope* $E_{\mathcal{F}} \subset \mathbb{P}^n$ is the branch locus of ψ.

Example

The envelope of the family of tangents to a plane curve is the curve itself. The envelope of the family of "normal lines" is the *evolute* of the curve.

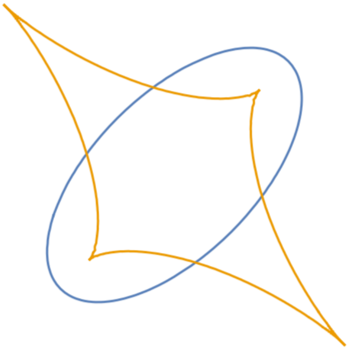
Apollonius of Perga, 262–200 BC

In his Treatise on Conic Sections, Book V, Apollonius constructs and studies *normals* to a conic – he finds these more interesting than the tangents:

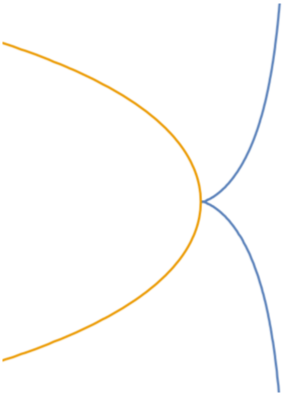
> *In the fifth book I have laid down propositions relating to maximum and minimum straight lines. You must know that our predecessors and contemporaries have only superficially touched upon this investigation of the shortest lines, and have only proved what straight lines touch the sections ...*

He spends 16 pages finding the number – 4 – of normals through a given point (the Euclidean distance degree!) and considers the points "where two normals fall together" – the evolute!

An ellipse and its evolute



The cissoid of Diocles $(x^2 + y^2)x - 4y^2 = 0$



Perpendicularity in projective space

Fix a hyperplane $H_\infty = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and a quadric $Q_\infty \subset H_\infty$. Let L be a linear space of dimension r and take $(L \cap H_\infty)^\perp$ with respect to Q_∞. For $p \in L$, the space *perpendicular* to L at p is

$$\langle p, (L \cap H_\infty)^\perp \rangle.$$

Example

$\mathbb{P}^2_{\mathbb{C}}$, $H_\infty = \{z = 0\}$, and

$$Q_\infty = \{x^2 + y^2 = 0\} \subset H_\infty,$$

gives the Euclidean geometry in $\mathbb{R}^2 \subset \mathbb{C}^2 = \mathbb{P}^2_{\mathbb{C}} \setminus H_\infty$.

The Euclidean normal bundle

Let T_p be the tangent space to $X \subset \mathbb{P}^n$ at $p \in X$. The *normal* space to X at p is the linear $(n-r)$-space

$$N_p := \langle p, (T_p \cap H_\infty)^\perp \rangle.$$

These normal spaces are the (projective) fibres of the *Euclidean normal bundle*

$$\mathcal{E} := \mathcal{K}^\vee \oplus \mathcal{O}_X(1),$$

where $\mathcal{K} := \mathrm{Ker}(\mathcal{O}_X^{n+1} \to \mathcal{P}_X^1(1))$.

The map $\mathcal{O}_X^{n+1} \to \mathcal{E} = \mathcal{K}^\vee \oplus \mathcal{O}_X(1)$ is the sum of the maps

$$\mathcal{O}_X^{n+1} \to \mathcal{O}_X^n \cong (\mathcal{O}_X^n)^\vee \to \mathcal{K}^\vee \text{ and } \mathcal{O}_X^{n+1} \to \mathcal{O}_X(1).$$

Evolutes

The *evolute* E_X of X is the envelope of its family of normal spaces

$$\psi : \mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n \to \mathbb{P}^n.$$

Example (Evolutes of plane curves)

If $X \subset \mathbb{P}^2$ is a plane curve of degree d, with δ nodes and k_0 cusps, we have:

$$\begin{aligned} \deg E_X &= 3d(d-1) - 6\delta - 8k_0 \\ \deg C_X &= 3\big(d(2d-3) - 4\delta - 5k_0\big) \\ \deg M_X &= d^2 - 4\delta - 5k_0 \end{aligned}$$

(where $M_X \subset (\mathbb{P}^2)^\vee$ is the curve of normals).

Real plane curves (with C. Riener and B. Shapiro)

We studied evolutes and curves of normals (in the dual projective plane) for *real* plane curves. Note that the evolute is the locus of the centers of curvature.

In particular, we were interested in finding Klein–Schuh type formulas for the singularities of these curves. For example, Klein showed (and Schuh generalized) that a plane curve of degree d can have at most $d(d-2)$ real inflection points, i.e., one third of the number of complex inflection points.

The $\mathbb{R}$-degree of a curve

The $\mathbb{R}$-degree of a curve $D \subset \mathbb{R}^2$ is

$$\mathbb{R}\deg(D) := \sup_L \#\,(D \cap L)$$

where $L \subset \mathbb{R}^2$ are all lines intersecting D transversally.

Problem 1: What are

$$e(d) := \max_D \mathbb{R}\deg(E_D) \text{ and } n(d) := \max_D \mathbb{R}\deg(M_D),$$

where the maximum is taken over all curves $D \subset \mathbb{R}^2$ of degree d?

Answer: $e(d) \geq d(d-2)$ and $n(d) = d^2$.

For complex curves: $\deg E_D = 3d(d-1)$ and $\deg M_D = d^2$.

Vertices

A *vertex* of a plane curve $D \subset \mathbb{R}^2$ is a critical point of the curvature function, i.e., a cusp of the evolute $E_D \cap \mathbb{R}^2$.

Problem 2: What is

$$v(d) := \max_D \kappa_{\mathbb{R}}(E_D),$$

with max taken over all curves $D \subset \mathbb{R}^2$ of degree d?

Answer: $v(d) \geq d(2d-3)$

For complex curves: $2d(3d-5)$.

Diameters

A *diameter* of a plane curve $D \subset \mathbb{R}^2$ is a line L which is the normal to D at two distinct points. The diameters are double points of the curve of normals M_D.

Problem 3: What is

$$\delta_M(d) := \max_D \delta^{\text{cru}}(M_D),$$

with max taken over all curves $D \subset \mathbb{R}^2$ of degree d?

Answer: $\delta_M(d) \geq \frac{1}{2}d^4 - d^3 + \frac{1}{2}d$

and conjecturally (for $d \geq 3$),

$$\delta_M(d) \leq \frac{1}{2}d^4 - 3d^2 + \frac{5}{2}d.$$

For complex nodes: $\delta_M(d) = \frac{1}{2}d^4 - \frac{5}{2}d^2 + 2d.$

Crunodes of the evolute

For a given curve $D \subset \mathbb{R}^2$, how many points in the plane are the centers of more than one circle of curvature, i.e., how many crunodes does the evolute have?

Problem 4: What is

$$c(d) := \max_D \delta^{\mathrm{cru}}(E_D),$$

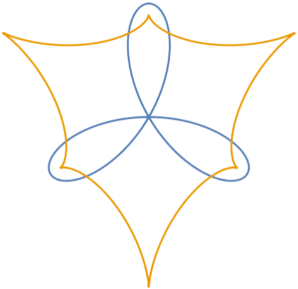
with max taken over all curves $D \subset \mathbb{R}^2$ of degree d?

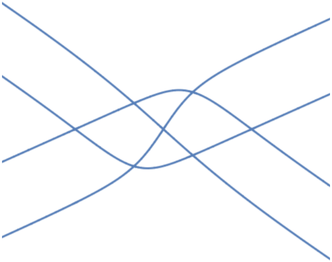
Answer: $c(d) \geq \binom{d(d-3)+1}{2} = \frac{1}{2}d^4 - 3d^3 + 5d^2 - \frac{3}{2}d$

For complex nodes: $c(d) = \frac{9}{2}d^4 - 9d^3 - \frac{13}{2}d^2 + 15d$

Trifolium pratense

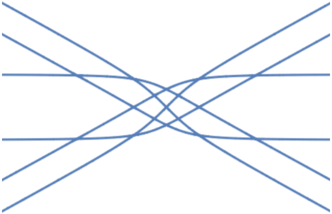
The trifolium $(x^2 + y^2)^2 - x^3 + 3xy^2 = 0$





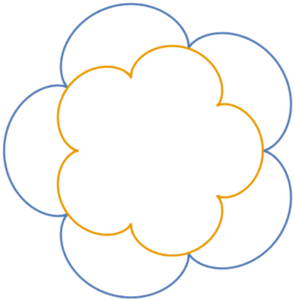
The fourleafed clover

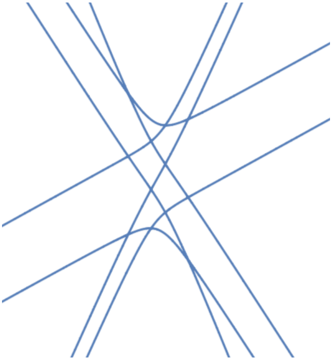
The quadrifolium $(x^2 + y^2)^3 - 4x^2y^2 = 0$



Ranunculus

The ranunculoid $x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + \cdots = 0$





Thom polynomials

Consider a family of linear $(n-r)$-spaces

$$\psi : \mathbb{P}(\mathcal{F}) \subset X \times \mathbb{P}^n \to \mathbb{P}^n.$$

Its envelope $E_{\mathcal{F}}$ is the image of the singularity locus Σ^1 of ψ. The cuspidal locus $C_{\mathcal{F}}$ of $E_{\mathcal{F}}$ is given by $\Sigma^{1,1}$, and its cuspidal locus $\kappa_{\mathcal{F}}$ by $\Sigma^{1,1,1}$. Set $\overline{c}_i := c_i(\psi^* T_{\mathbb{P}^n} - T_{\mathbb{P}(\mathcal{F})})$. Then

$$\begin{aligned} [E_{\mathcal{F}}] &= \psi_* \overline{c}_1 \cap [\mathbb{P}^n] \\ [C_{\mathcal{F}}] &= \psi_* (\overline{c}_1^2 + \overline{c}_2) \cap [\mathbb{P}^n] \\ [\kappa_{\mathcal{F}}] &= \psi_* (\overline{c}_1^3 + 3\overline{c}_1 \overline{c}_2 + 2\overline{c}_3) \cap [\mathbb{P}^n] \end{aligned}$$

The *degree* of these classes can be expressed in terms of the Chern classes of T_X, $\mathcal{O}_X(1)$, and $\mathcal{F}$.

Evolute of space curves

Let $X \subset \mathbb{P}^3$ be a curve. Its evolute E_X – the envelope of the family of its Euclidean normal planes – was denoted the *polar developable* by Monge (1871) and the *polar surface* by Darboux (1887).

The *space evolute* of X is the cuspidal edge $C_X \subset E_X$ of the evolute. It is the locus of the *centers of spherical curvature*. The osculating planes to the space evolute are the normal planes to X (Blaschke and Leichtweiss). The evolute E_X is the tangent developable of C_X.

Salmon (1862) considered projective plane and space curves and surfaces in $\mathbb{P}^3$ and found formulas for the degrees of their evolutes.

Numerical formulas for space curves

Let $X \subset \mathbb{P}^3$ be a curve of degree d, genus g, and with k_0 cusps. Evaluating the Thom polynomials with $\mathcal{F} = \mathcal{E}$ and taking degrees, gives

$$\begin{aligned} \deg E_X &= 6(d+g-1) - 2k_0 \\ \deg C_X &= 3(3d+4g-4-k_0) \\ \deg \kappa_X &= 4\big(3d+5g-5-k_0\big). \end{aligned}$$

Salmon found that the degree of the evolute E_X is equal to $3d + d^*$, where d^* is the *class* of X (the degree of its strict dual). He found the degree of the space evolute C_X to be $5d + k_2$, where k_2 is the number of hyperosculating planes to X. His formulas agree with the ones above.

Evolute of surfaces

Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. The Thom polynomials give

$$\begin{aligned}
[E_X] &= \psi_*\big(\pi^* c_1(\Omega^1_X) + \pi^* c_1(\mathcal{E}) + 2c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))\big) \cap [\mathbb{P}^3] \\
[C_X] &= \psi_*\big(2\pi^* c_1(\Omega^1_X)^2 + \dots\big) \cap [\mathbb{P}^3] \\
[\kappa_X] &= \psi_*\big(22\pi^* c_1(\Omega^1_X)^2 c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) + \dots\big) \cap [\mathbb{P}^3].
\end{aligned}$$

In terms of the degree d of X this gives

$$\begin{aligned}
\deg E_X &= 2d(d-1)(2d-1) \\
\deg C_X &= 2d(d-1)(11d-16) \\
\deg \kappa_X &= 4d(30d^2 - 97d + 78).
\end{aligned}$$

The first formula agrees with the one given by Salmon.

Umbilics

The local study of the geometry of the evolute a real surface is well known. Since the evolute is the locus of the focal points, i.e., the centers of spherical curvature, each normal to the surface will intersect the evolute in two points. When these two points come together, the curvature of the surface is the same in all directions. These points are the *umbilical points* of the surface.

According to Porteous, the umbilical points are the D_4 singularities of the real map $\psi_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}^3$.
The singularities come in different types.

Kazarian: $\#D_4^{++} - \#D_4^{+-} + \#D_4^{-+} - \#D_4^{--} = 4$,
Uribe-Vargas in $\mathbb{P}_{\mathbb{R}}$, ...

Lagrange maps

The map $\psi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^3$ is supposedly Lagrangian (only "locally"?).

According to Mikosz–Pragacz–Weber the Thom polynomial for D_4 for a Lagrangian map is

$$\tilde{Q}_{2,1} = a_1 a_2 - 2a_3,$$

for Chern classes a_i.

Question: Which bundles are the a_i the Chern classes of?

We know what to expect for $X \subset \mathbb{P}^3$ a surface of degree d, namely Salmon's 1882 formula:

$$\#D_4 = 2d(5d^2 - 14d + 11)$$

Other applications to enumerative geometry

- "Direct proof" of Noether's formula $\chi(X) = \frac{1}{12}(c_1^2 + c_2)$ for a surface X, by considering a generic map $X \to \mathbb{P}^3$ (P. 1979).
- "Direct proof" of the Hirzebruch Riemann–Roch formula $\chi(X) = \frac{1}{24}c_1 c_2$ for a threefold X, by considering a generic map $X \to \mathbb{P}^4$ (P.–Ronga 1981).
- Salmon's formula for the number of triple tangent planes of a smooth surface of degree d (P. 1978, Ohmoto 2024).
- Roberts's number of plane curves of degree d with three nodes passing through $\frac{1}{2}d(d+3) - 3$ points (Kleiman–P. 2004, Ohmoto 2024).
- Formulas for generic space curves (Nekarda–Ohmoto 2024) and surfaces in 3- and 4-space (Nekarda 2023).

Thank you for your attention!