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NUMERICAL CHARACTERS OF A CURVE IN PROJECTIVE n -SPACE

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§1. Introduction

The enumerative properties of a curve embedded in projective n -space can be studied by considering curves in lower-dimensional projective spaces, derived from the given curve in a natural way. Such was the point of view of Veronese ([14]), who used the 'principle of projections and sections' to establish relations among the numerical characters of a curve in n -space by applying Plücker's formulas to $n-1$ plane curves derived from the given one. This method was first used by Cayley ([3]) in the case of a curve in 3-space.

The numerical characters we shall consider are the *ranks* – the number of osculating spaces to the curve which satisfy a given Schubert condition – and the *stationary indices* – the number of *hyperosculating* spaces (i.e. osculating spaces which have 'too high' order of contact with the curve). The points of hyperosculation were called singular points of higher order by Pohl ([13]); in fact, he represented the osculating spaces by vector bundles (dual to the jet bundles) and got the points of hyperosculation as the singularities of maps of bundles. In this paper we show how (the dual of) his representation can be used to prove the geometric statements about projections and sections made by Veronese. Another application of Pohl's viewpoint was given by Kato

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([9]), who described the higher order singularities of an elliptic curve of degree $n+1$ in $\mathbb{P}^n$ as the points of order n of the curve considered as abelian variety.

We shall assume, for simplicity, that the base field k is algebraically closed, rather than just infinite. By a *curve* we mean a purely 1-dimensional, reduced, proper, algebraic scheme. Any embedding $j: X_0 \hookrightarrow P$ of a curve X_0 in projective n -space is assumed to be such that X_0 spans P . Given an embedding j , we let $f: X \rightarrow P$ denote the finite map obtained by composing j with the normalization $X \rightarrow X_0$. In §2 we define, for each integer m , $0 \leq m \leq n$, the osculating m -space $S(x, m)$ to the branch of X_0 at $f(x)$ corresponding to a point $x \in X$. We show that these spaces, for *ordinary* (non-hyperosculating) points x , are the projective fibers of the bundle $\mathcal{P}_X^m(\mathcal{L})$ of principal parts (or jets) of order m of the canonical line bundle $\mathcal{L} = f^*\mathcal{O}_P(1)$. A modification $\mathcal{P}^m$ of this bundle, called the osculating bundle of order m , gives (if $\text{char } k = 0$ or $\text{char } k$ is sufficiently large) also the hyperosculating m -spaces.

The numerical characters of the curve – the ranks and the stationary indices – are defined and interpreted in §3, and the formulas relating them (and the genus of the curve) are deduced (3.2). We then proceed to treat the principle of projections and sections, in §4. A *q-projection* of f is defined as a generic linear projection of f to $\mathbb{P}^q$, while an *m-section* $X_m \rightarrow \mathbb{P}^{n-m}$ of f is defined to be the intersection of a (general) subspace $\mathbb{P}^{n-m}$ of P with the *osculating developable* of X_0 , i.e. with the ruled $(m+1)$ -dimensional variety generated over X_0 by the m -spaces $S(x, m)$. We prove associativity (4.2, 4.4) and commutativity (4.6) of these two operations and relate the numerical characters of the derived curves to those of X_0 (4.7). Finally, in §5, Weyl's proof of duality for analytic curves is adapted. Our key result is an isomorphism of exact sequences (5.2) which shows that all the natural duality statements for a curve and its derived curves hold. As an application we deduce the Cayley-Plücker equations for a curve in 3-space.

Three examples are carried through; the first is the rational n -ic R_n in n -space. This curve is particularly simple: it has no points of hyperosculating and it is self-dual. The second example is a linearly normal, elliptic curve of degree $d \geq 3$. The third curve, Γ_d , was discussed by Dye ([4], [5]). It is the complete intersection of $n-1$ hypersurfaces of degree d which are in special position.

In the appendix (§6) we recall the definition of the functor of principal parts. We show the existence, as well as some general properties, of a natural transformation of functors which will be used to define the osculating bundles.

As far as the characteristic of k is concerned, let us remark that in small,

positive characteristic the osculating bundles $\mathcal{P}^m$ (though well defined) may not represent the *hyperosculating* spaces. For $r_m = \deg \mathcal{P}^m$ still to hold the osculating spaces could be *defined* as the fibers of $\mathcal{P}^m$. This definition also makes sense for any finite map (not necessarily birational onto its image) $f: X \rightarrow P$ such that $a^n: V_x \rightarrow \mathcal{P}_X^n(\mathcal{L})$ (where $V = H^0(P, \mathcal{O}_P(1))$), see §2) is generically surjective. If $f': X' \rightarrow f(X)$ denotes the normalization and $g: X \rightarrow X'$ the induced map one can show that in this case one has $r_m(f) = (\deg g)r_m(f')$; the restriction to the case of an embedded curve ($\deg g = 1$) is made in order to simplify notations.

§2. The osculating bundles

Let $X_0 \hookrightarrow P = \mathbb{P}(V)$ be a curve embedded in, and spanning, projective n -space. Let $X \rightarrow X_0$ denote the normalization and $f: X \rightarrow P$ the induced map. For each point $x \in X$ we let $B(x)$ denote the corresponding branch at $f(x)$ of X_0 . For each integer m , $0 \leq m \leq n$, we define the *osculating m -space* $S(x, m)$ to $B(x)$ to be the linear m -dimensional subspace of P which has the highest order of contact with $B(x)$ at $f(x)$. To be more precise, put $A = \mathcal{O}_{X,x}$ and let $t \in A$ denote a uniformizing parameter. There is a parametrization

$$x_i = a_i t^{i+1} + (\text{higher order terms}), \quad (*)$$

$i = 0, 1, \dots, n$, with $0 = l_0 \leq l_1 \leq \dots \leq l_n$ (where the x_i 's are considered as elements of the completion $\hat{A}$ of A), and $a_i \in k^*$. With this choice of coordinates in P , $S(x, m)$ is the space defined by $X_{m+1} = X_{m+2} = \dots = X_n = 0$ and has $(l_{m+1} + m + 1)$ -order contact with $B(x)$ at $f(x)$.

If $l_n = 0$ (and hence $l_i = 0$, $0 \leq i \leq n$) holds, we say that x is an *ordinary point*. The *points of hyperosculating* are points with $l_{m+1} > 0$ for some m . Pohl called such a point singular of order m ([13]). Classically, a hyperosculating space was also called *stationary*, e.g. a cusp is a stationary point, at a flex there is a stationary tangent, a stall is a stationary osculating hyperplane.

The osculating spaces $S(x, m)$ are the (projective) fibers of an $(m+1)$ -bundle on X , which we will call the osculating bundle of order m ; the formulation we use is dual to Pohl's ([13]). Let $\mathcal{P}_X^m(-)$ denote the functor of principal parts (or jets) of order m (§6). Put $\mathcal{L} = f^*\mathcal{O}_P(1)$. Since X is a smooth curve, $\mathcal{P}_X^m(\mathcal{L})$ is an $(m+1)$ -bundle – in fact, there are exact sequences, for $m \geq 1$,

$$0 \longrightarrow \Omega_X^m \otimes \mathcal{L} \xrightarrow{\quad} \mathcal{P}_X^m(\mathcal{L}) \xrightarrow{\quad} \mathcal{P}_X^{m-1}(\mathcal{L}) \longrightarrow 0$$

([7], 16.10.1, 16.7.3). Let $g: X \rightarrow \text{spec}(k)$ denote the structure map. There are functorial maps (§6)

$$\alpha^m(\mathcal{L}): g^*g_*\mathcal{L} \rightarrow \mathcal{P}_X^m(\mathcal{L})$$

compatible with the maps b^m . Let $a^m: V_X \rightarrow \mathcal{P}_X^m(\mathcal{L})$ denote the composition of $\alpha^m(\mathcal{L})$ with the map $V_X = g^*H^0(P, \mathcal{O}_P(1)) \rightarrow g^*H^0(X, \mathcal{L}) = g^*\mathcal{L}$.

PROPOSITION (2.1). Let $\mathcal{P}^m \subseteq \mathcal{P}_X^m(\mathcal{L})$ denote the image of the homomorphism a^m . For each ordinary point $x \in X$, the embedded linear space $\mathbb{P}(\mathcal{P}^m(x)) \xrightarrow{\pi(a^m(x))} \mathbb{P}(V_X(x)) = P$ is equal to the osculating m -space $S(x, m)$; if $\text{char } k$ is 0 or sufficiently big, then this holds for all points of X .

PROOF. Let $x \in X$ and choose a parametrization $(*)$ of $B(x)$. Locally, the map a^m consists of taking the m -jets (or the Taylor series development up to order m) of the coordinate functions (§6): Consider the basis $\{1, dt, \dots, (dt)^m\}$ for the free A -module $\mathcal{P}_X^m(\mathcal{L}) \otimes A \cong P_A^m$; let $\{d^0, d^1, \dots, d^m\}$ denote the dual basis for $(P_A^m)^\vee$. Then

$$a^m \otimes A: V_X \otimes A \cong A^{n+1} \rightarrow P_A^m \cong A^{m+1}$$

is given by the matrix

$$w_x^m = \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ d^1 x_0 & d^1 x_1 & \cdots & d^1 x_n \\ \vdots & \vdots & \ddots & \vdots \\ d^m x_0 & d^m x_1 & \cdots & d^m x_n \end{pmatrix}$$

Since d^i is a truncated Taylor series, we have $d^i(t^j) = \binom{j}{i} t^{j-i}$, for all i, j . Hence we get (considering the entries as elements of $\hat{A}$),

$$w_x^m = \begin{pmatrix} 1 + \cdots & a_1 t^{1+1} + \cdots & \cdots & a_n t^{n+1} + \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & (l_1 + 1)a_1 t^{l_1+1} + \cdots & \cdots & (l_n + n)a_n t^{l_n+1} + \cdots \\ \vdots & \vdots & \vdots & \vdots \\ * & \binom{l_m + m}{m} a_m t^{l_m+1} + \cdots & \cdots & \binom{l_n + n}{m} a_n t^{l_n+1} + \cdots \end{pmatrix}$$

(where $+\dots$ indicates terms of higher powers of t , and $*$ indicates elements in the maximal ideal of $\hat{A}$, see also ([9], p. 101)).

CASE 1. x is an ordinary point, i.e. $l_0 = l_1 = \cdots = l_n = 0$. Then a^m is surjective at x , so that $\mathcal{P}^m(x) = \mathcal{P}_X^m(\mathcal{L})(x)$, and $a^m(x)$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \end{pmatrix},$$

hence it defines ([6], §4) the subspace of P given by $X_{m+1} = X_{m+2} = \cdots = X_n = 0$.

CASE 2. $l_m > 0$ and $\binom{l_i+i}{i} \neq 0$ ($\text{char } k$) for $i = 1, \dots, n$. The inclusion $\mathcal{P}^m \hookrightarrow \mathcal{P}_X^m(\mathcal{L})$ is given, locally at x , by the matrix

$$\begin{pmatrix} 1 & t^{l_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t^{l_n} \end{pmatrix},$$

and $a^m(x)$ is given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \binom{l_1+1}{1} & \cdots & \binom{l_m+m}{m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Hence it defines the same linear m -space as in case 1.

The bundle $\mathcal{P}^m = \text{Im}(a^m)$ will be referred to as the *osculating bundle of order m* of the curve $f: X \rightarrow P$. From now on we assume $\text{char } k$ to be such that $\mathcal{P}^m$ represents the osculating m -spaces at all points of X .

EXAMPLE 1. The rational n -ic in n -space.

The curve $R_n \subset P$ of degree n and genus 0 has only ordinary points. Every point $x \in R_n$ has a parametrization $(1, t, t^2, \dots, t^n)$, hence $a^m: V_X \rightarrow \mathcal{P}_X^m(\mathcal{L})$ is everywhere surjective and the osculating bundle of order m is equal to $\mathcal{P}_{R_n}^m(\mathcal{L})$ for $m = 0, 1, \dots, n$.

EXAMPLE 2. A linearly normal, elliptic curve.

Let $E_d \subset P$ be a curve of genus 1 and degree $d \geq 3$, which is linearly normal. From Riemann-Roch it follows that at any point x , the parametrization (*) satisfies

$$l_i = 0, \quad i = 0, \dots, n-1, \\ l_n \leq 1.$$

Hence the curve is non-singular up to order $n-1$.

EXAMPLE 3. Dye's special curve ([4], [5]).

Suppose $\Gamma_d \subseteq P$ is the complete intersection of $n-1$ smooth surfaces S_i of degree d with a common self-polar simplex S , i.e., assume S_i has equation $\sum_{j=0}^n a_j^{(i)} X_j^2 = 0$, where all $(n-1)$ -minors of the matrix $(a_j^{(i)})$ are non-zero. Note that Γ_d has degree d^{n-1} and its genus g is given by

$$2g - 2 = d^{n-1}((d-1)(n-1) - 2).$$

The points of hyperosculation on Γ_d are its $(n+1)d^{n-1}$ intersections with the faces of the simplex S ; the osculating m -spaces at these points have order of contact $l_{m+1} + m + 1 = md$ and so are hyperosculating (except when $m=1$ and $d=2$).

§3. Numerical characters

Given a curve in 3-space, we define its *rank* to be the number of tangents that meet a given line. The number of osculating planes that pass through a given point is called the *class* of the curve. In general, for a curve $X_0 \hookrightarrow P = \mathbb{P}(V)$ in n -space, we define its m -rank to be the number r_m of its osculating m -spaces that meet a given (general) sub- $(n-m-1)$ -space of P . As in §2, let $X \rightarrow X_0$ denote the normalization and $f: X \rightarrow P$ the induced map. We defined (§2) the osculating bundle of order m of f as an $(m+1)$ -bundle $\mathcal{O}^m$ on X together with a surjection $a^m: V_X \rightarrow \mathcal{O}^m$, which represents the osculating m -spaces of X_0 . The $(m+1)$ -quotient $a^m: V_X \rightarrow \mathcal{O}^m$ also defines the m th osculating developable of X_0 as the image of the map $f_{(m)}: \mathbb{P}(\mathcal{O}^m) \rightarrow P$ obtained by composing $\mathbb{P}(a^m): \mathbb{P}(\mathcal{O}^m) \hookrightarrow \mathbb{P}(V_X) = X \times P$ with the projection to P . Moreover, the 1-quotient $\Lambda^{m+1} V_X \rightarrow \Lambda^{m+1} \mathcal{O}^m$ defines the m th associated curve $f^{(m)}: X \rightarrow \mathbb{P}(\Lambda^{m+1} V)$. The corresponding interpretations of the ranks are given in the next proposition.

PROPOSITION (3.1). The following numbers are equal (for $0 \leq m \leq n-1$):

- (i) the m -rank r_m ,
- (ii) the degree $\deg(\mathcal{O}^m)$ of the osculating bundle of order m ,
- (iii) the degree of the m th osculating developable $Y_m = f_{(m)}(\mathbb{P}(\mathcal{O}^m)) \subset P$,
- (iv) the degree of the m th associated curve $f^{(m)}: X \rightarrow \mathbb{P}(\Lambda^{m+1} V)$.

PROOF. Since the quotient $a^m: V_X \rightarrow \mathcal{O}^m$ represents the osculating m -spaces of the curve, we see that the m -rank r_m is equal to the degree of the pullback to X of the Schubert cycle $\sigma \in \text{Grass}_{m+1}(V)$ of m -spaces of $P = \mathbb{P}(V)$ which meet a given $(n-m-1)$ -space, via the map $G(a^m): X \rightarrow \text{Grass}_{m+1}(V)$ defined by a^m . Consider the Plücker embedding, $\text{Grass}_{m+1}(V) \hookrightarrow \mathbb{P}(\Lambda^{m+1} V)$; then σ is a hyperplane section, and the m th associated map $f^{(m)}$ is the composition of $G(a^m)$ with j . Hence we get $r_m = \deg(f^{(m)*} \mathcal{O}_{\mathbb{P}(\Lambda^{m+1} V)}(1)) = \deg(\Lambda^{m+1} \mathcal{O}^m) = \deg(\mathcal{O}^m)$. This proves (i) = (ii) = (iv).

We have $\deg(Y_m) = (c_1(\mathcal{O}_P(1)))^{m+1} \cdot Y_m / P$, and since $X \rightarrow X_0$ is birational, so is $\mathbb{P}(\mathcal{O}^m) \rightarrow Y$, therefore we have $\deg(Y_m) = (c_1(f_{(m)}^* \mathcal{O}_P(1)))^{m+1} / \mathbb{P}(\mathcal{O}^m) = (c_1(\mathcal{O}_{\mathbb{P}(\mathcal{O}^m)}(1)))^{m+1} / \mathbb{P}(\mathcal{O}^m)$. By Chern class theory this is equal to $(c_1(\mathcal{O}^m) \cdot c_1(\mathcal{O}_{\mathbb{P}(\mathcal{O}^m)}(1)))^m / \mathbb{P}(\mathcal{O}^m)$, which, by the projection formula, equals $(c_1(\mathcal{O}^m))_X = \deg(\mathcal{O}^m)$.

EXAMPLE 1 ($n=3$). The twisted cubic $R_3 \subseteq \mathbb{P}^3$. We can write R_3 as the intersection of three quadrics Q_1, Q_2, Q_3 , with its tangent developable defined by $Q_2^2 - 4Q_1Q_3 = 0$. Hence R_3 has 1-rank $r_1 = 4$.

Let $x \in X$ and consider, as in §2, a parametrization (*) $x_i = a_i t^{l_i+1} + \dots$, $a_i \neq 0(i)$, $i = 0, \dots, n$, $0 = l_0 \leq l_1 \leq \dots \leq l_n$. The integer $l_{m+1} - l_m$ is called the m th stationary index of the curve $f: X \rightarrow P$ at x , and we define the m th stationary index of f to be $k_m = \sum_{x \in X} (l_{m+1} - l_m)$. Hence, in the case that f is non-singular of order $m-1$, k_m is the number (counted properly) of hyperosculating (or stationary) m -spaces.

THEOREM (3.2). There are formulas

$$r_m = (m+1)(r_0 + m(g-1)) - \sum_{i=0}^{m-1} (m-i)k_i,$$

for $m = 0, 1, \dots, n-1$, and

$$\sum_{i=0}^{n-1} (n-i)k_i = (n+1)(r_0 + n(g-1)),$$

where g denotes the genus of X (see [2], p. 200).

PROOF. Put $\mathcal{Q}_m = \text{Coker}(a^m)$. The exact sequences $0 \rightarrow \mathcal{O}^m(\mathcal{Q}) \rightarrow \mathcal{Q}_m \rightarrow 0$ gives $r_m = \deg(\mathcal{O}^m) = \deg(\mathcal{Q}_m(\mathcal{Q})) - \deg(\mathcal{Q}_m^1)$, where $\mathcal{Q}_m^1$ is the 0th Fitting ideal $F^0(\mathcal{Q}_m)$ of $\mathcal{Q}_m$, i.e. $\mathcal{Q}_m^1$ is the image of the homomorphism $(\Lambda^{m+1} \mathcal{O}^m) \otimes (\Lambda^{m+1} \mathcal{O}_X^m(\mathcal{Q}))^{-1} \rightarrow \mathcal{O}_X$. Locally at a point $x \in X$, in terms of a parametrization (*), $\mathcal{Q}_m^1$ is generated by an element of the form $\prod_{i=0}^m (a_i t^{l_i+1})^{k_i} + \dots$. Hence we get $\deg(\mathcal{Q}_m^1) = \sum_{x \in X} l_i = \sum_{i=0}^{m-1} (m-i)k_i$. The

exact sequences (§2)

$$0 \rightarrow S^m \Omega_X \otimes \mathcal{L} \rightarrow \mathcal{P}_X^m(\mathcal{L}) \rightarrow \mathcal{P}_X^{m-1}(\mathcal{L}) \rightarrow 0$$

give

$$\deg(\mathcal{P}_X^m(\mathcal{L})) = \sum_{i=1}^m \deg(S^i \Omega_X \otimes \mathcal{L}) + \deg \mathcal{L}.$$

Hence we get

$$\begin{aligned} r_m &= \frac{m(m+1)}{2} \deg \Omega_X + (m+1) \deg \mathcal{L} - \sum_{i=0}^{m-1} (m-i) k_i \\ &= (m+1)(r_0 + m(g-1)) - \sum_{i=0}^{m-1} (m-i) k_i. \end{aligned}$$

(Since $r_0 = \deg \mathcal{L}$ and $\deg \Omega_X = 2(g-1)$.) The last equation follows from the fact that $r_n = 0$ holds: the surjection $a^n: V_X \rightarrow \mathcal{P}^n$ is necessarily an isomorphism, since the bundles have the same rank.

We note that the map $a^1: V_X \rightarrow \mathcal{P}^1(\mathcal{L})$ is canonically isomorphic to the map $f^* \mathcal{P}_P^1(1) \rightarrow \mathcal{P}_X^1(\mathcal{L})$ (6.4). Hence we get Coker $(a^1) = \Omega_{X/P}(\mathcal{L})$ ([7], 16.4.18), and k_0 is the degree of the ramification divisor; we have $k_0 =$ the number of cusps of $f(X)$, if $\text{char} \neq 2$ (if $\text{char} = 2$, then $k_0 = 2\#\text{cusps}$).

COROLLARY (3.3). There are equalities

- (i) $(r_{m+1} - r_m) - (r_m - r_{m-1}) = 2(g-1) - k_m$
- (ii) $(r_{m+1} + k_m) - (r_{m-2} + k_{m-1}) = 3(r_m - r_{m-1})$
- (iii) $(r_{n-2} + k_{n-1}) - (r_1 + k_0) = 2(r_{n-1} - r_0).$

The formulas (i) are usually referred to as the generalized Plücker formulas for curves (see [13], p. 207; [8], 4.26; [15], p. 43). In the case that the curve is plane ($n=2$), it is the classical formula

$$r_1 = r_0 + 2g - 2 - k_0 \quad (\text{I})$$

giving the class in terms of the degree, genus, and number of cusps. The other Plücker formula for the class of a plane curve can be written

$$r_1 = r_0(r_0 - 1) - e \quad (\text{II})$$

where $e = \deg(\mathcal{J}^{-1})$ and $\mathcal{J}$ denotes the pullback to X of the Jacobian ideal of $f(X)$. (This formula follows from the fact that, for a plane curve, $\text{Ker}(a^1)$ is equal to $f^* \mathcal{N} \otimes \mathcal{J}^{-1}$, where $\mathcal{N}$ is the conormal bundle of $f(X)$ in P ([12]).) One shows $e = 2\delta + k_0$, where 2δ denotes the degree of the conductor of X in $f(X)$. In particular, if $f(X)$ has no other singularities than D (ordinary double points and K (simple) cusps, we get $e = 2D + 3K$, if $\text{char} \neq 2, 3$).

EXAMPLE 1. R_n has no hyperosculating points. We get

$$r_m = (m+1)(n-m) = r_{n-m-1},$$

hence the degree of the m th associated curve $f^{(m)}: X \rightarrow \text{Grass}_{m+1}(V) \hookrightarrow \mathbb{P}(\Lambda^{m+1} V)$ is equal to the dimension of $\text{Grass}_{m+1}(V)$.

EXAMPLE 2. We deduce, from the Riemann-Roch theorem, $k_m = 0$, $m = 0, \dots, n-2$. Hence we get $r_m = (m+1)d$, $m = 0, \dots, n-1$ and $k_{n-1} = (n+1)d$, so that the curve E_d has exactly $(n+1)d$ hyperosculating hyperplanes.

EXAMPLE 3. For the curve Γ_d we have $k_0 = 0$, $k_i = (n+1)d^{n-1}(id - i - 1)$ for $i \geq 1$, from which we can compute the formulas for r_m .

Let us consider the case $d=2$, $n=4$. Then $r_0=8$, $g=5$, and $k_0=0$, $k_1=0$, $k_2=40$, $k_3=40$. Hence we get $r_1=24$, $r_2=48$, and $r_3=40$. The number of hyperosculating points on $\Gamma=\Gamma_2$ is 40, but their 'weighted number' $\sum_{i=0}^3 (4-i)k_i$ is 120. In fact the curve $\Gamma \hookrightarrow \mathbb{P}^4$ is canonical, i.e. the embedding is given by the canonical bundle Ω_Γ . So the 40 hyperosculating points are the Weierstrass points of Γ ; the above remark about their number shows that these points are not ordinary Weierstrass points.

§4. Projections and sections

If we project a curve in 3-space from a (general) point, we obtain a plane curve, of the same degree, whose class is equal to the rank of the space curve. Another way of obtaining a plane curve from the space curve is to take the intersection of its tangent developable with a (general) plane. We expect this plane curve to have the same class as the space curve, while its degree is that of the tangent developable, hence equal to the rank of the space curve. Veronese ([14]) used the principle of projections and sections to obtain $n-1$ plane curves from a given curve in n -space; he then applied the formulas (I), (II) (and their duals) of §3 to these curves, deducing relations among the numerical characters of the given curve (interpreting these in terms of the characters of the plane curves).

Assume from now on that X is a smooth curve, given with a finite map $f: X \rightarrow P = \mathbb{P}(V)$ such that $a^n: V_X \rightarrow \mathcal{P}_X^n(\mathcal{L})$ is generically surjective (§1, §2). A $(q+1)$ -dimensional subspace $V' \subseteq V$ such that $f(X) \cap (V/V') = \emptyset$ (i.e., such that the induced map $V'_X \rightarrow \mathcal{L} = f^* \mathcal{O}_P(1)$ is surjective) defines a projection $X \rightarrow P' = \mathbb{P}(V')$. If, in addition, the induced map $V'_X \rightarrow \mathcal{P}^{q-1}$ is surjective (here $\mathcal{P}^{q-1}$ denotes the osculating bundle of order $q-1$ of f , as in §2), we say that $p_q(f) = p_q: X \rightarrow \mathbb{P}(V')$ is a q -projection of f . In fact, a general subspace $V' \subseteq V$ has this property, as the following lemma shows.

LEMMA (4.1). If Y is a d -dimensional variety and $V_Y \rightarrow \mathcal{E}$ is an r -quotient on Y , then a general $(r+d)$ -dimensional subspace V' of V has the property that the induced map $V'_Y \rightarrow \mathcal{E}$ is surjective.

PROOF. Let $\phi: Y \rightarrow \text{Grass}(V)$ be the morphism defined by the quotient. We want to show ([10], 2.6) that for a general $V' \subseteq V$, we have $\phi^{-1}(\sigma_{d+1}(V')) = \emptyset$, where $\sigma_{d+1}(V') \subseteq \text{Grass}(V)$ denotes the $(d+1)$ th special Schubert cycle defined by V' . Note that $\sigma_{d+1}(V')$ has codimension $d+1$.

Let G denote the general linear group acting on V . Given $V' \subseteq V$ there is an open dense subset U of G such that $\phi^{-1}(\sigma_{d+1}(gV')) = \emptyset$ for all $g \in U$ ([11], Cor. 4). There is a surjective map $\psi: G \rightarrow G' = \text{Grass}_{n+1-r-d}(V)$ and (by the theorem of generic flatness) there is an open dense subset U' of G' such that $\psi^{-1}(U') \subseteq U$. Hence gV' satisfies the condition of the lemma, for all $g \in U'$.

PROPOSITION (4.2). *The osculating bundles of $p_q: X \rightarrow P'$ are equal to the osculating bundles (of order $\leq q-1$) of $f: X \rightarrow P$, and a q' -projection of p_q is also a q' -projection of f .*

PROOF. From the general properties of the maps a^m (6.1) it follows that $a^m(p_q): V'_X \rightarrow \mathcal{P}_X^m(\mathcal{L})$ is equal to the composition of $a^m(f): V_X \rightarrow \mathcal{P}_X^m(\mathcal{L})$ with the inclusion $V'_X \hookrightarrow V_X$. Therefore the image of $a^m(p_q)$ is equal to $\mathcal{P}^m$, since this is true for $m = q-1$ and since the canonical surjections $b^m: \mathcal{P}_X^m(\mathcal{L}) \rightarrow \mathcal{P}_X^{m-1}(\mathcal{L})$ induce surjections $\mathcal{P}^m \rightarrow \mathcal{P}^{m-1}$. To prove the last assertion, we observe that if $W' \hookrightarrow V'$ defines a q' -projection of p_q , then $W' \hookrightarrow V$ defines a q' -projection of f .

Assume now that $V' \subseteq V$ is an m -dimensional subspace. Applying lemma (4.1) to the $(n-m)$ -quotient $V'_X \rightarrow (\text{Ker}(a^m))^V$ we see that for a general such subspace V' , the induced map $V'_X \rightarrow \mathcal{P}^m$ is locally split. As in §2, let $f_{(m)}: \mathbb{P}(\mathcal{P}^m) \rightarrow P$ denote the m th osculating developable of f . Suppose $V' \subseteq V$ is general and put $W = V/V'$. We set $X_m = f_{(m)}^{-1}(\mathbb{P}(W))$ and call $f_m = f_{(m)}|_{X_m}: X_m \rightarrow \mathbb{P}(W)$ an m -section of f . We let $\mathcal{L}_m$ denote the line bundle $\text{Coker}(V'_X \rightarrow \mathcal{P}^m)$.

PROPOSITION (4.3). *With the above notations, the map $\gamma_m: X_m \rightarrow X$ induced by $\mathbb{P}(\mathcal{P}^m) \rightarrow X$ is equal to the canonical isomorphism $\mathbb{P}(\mathcal{L}_m) \rightarrow X$.*

PROOF. Put $\mathcal{H}_m = \text{Ker}(a^m)$. From the commutative diagram of bundles on X

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}_m & = & \mathcal{H}_m & & \mathcal{H}_m & & \mathcal{H}_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow V'_X \rightarrow V_X \rightarrow W_X \rightarrow 0 & & & & & & \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow V'_X \rightarrow \mathcal{P}^m \rightarrow \mathcal{L}_m \rightarrow 0 & & & & & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 & & 0 \end{array}$$

it follows that the curves $\mathbb{P}(\mathcal{L}_m) \subseteq \mathbb{P}(\mathcal{P}^m)$ and $X_m = f_{(m)}^{-1}(\mathbb{P}(W)) \subseteq \mathbb{P}(\mathcal{P}^m)$ are equal.

The next result we will prove is that an m' -section of an m -section of f is an $(m+m')$ -section of f .

THEOREM (4.4). *Suppose $f_m: X_m \rightarrow \mathbb{P}(W)$ is an m -section of f such that $a^{m'}(f_m)$ is generically surjective, let $\mathcal{P}_m^{m'}$ denote the osculating bundle of order m' of f_m , and let $f_{m,m'}: \mathbb{P}(\mathcal{P}_m^{m'}) \rightarrow \mathbb{P}(W)$ denote the m' th osculating developable of f_m . Suppose*

$$f_{m,m'}: X_{m,m'} = (f_{m,(m')})^{-1}(\mathbb{P}(W')) \rightarrow \mathbb{P}(W)$$

is an m' -section of f_m . Then

$$f_{m+m'}: X_{m+m'} = (f_{m,(m+m')})^{-1}(\mathbb{P}(W')) \rightarrow P$$

is an $(m+m')$ -section of f , and there is a cartesian diagram

$$\begin{array}{ccc} X_{m,m'} & \xrightarrow{u} & X_{m+m'} \times_X X_m \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{P}_m^{m'}) & \xrightarrow{v} & \mathbb{P}(\mathcal{P}^{m+m'}) \times_X X_m \end{array}$$

where u is an isomorphism and v is an embedding.

PROOF. We will show that there is a canonical exact sequence

$$0 \rightarrow V'_X \rightarrow \gamma^* \mathcal{P}^{m+m'} \rightarrow \mathcal{P}_m^{m'} \rightarrow 0, \quad (4.5)$$

where $V' = \text{Ker}(V \rightarrow W)$, and $\gamma = \gamma_m: X_m \rightarrow X$ is the isomorphism of (4.3). Once (4.5) is established, the theorem follows: Put $\mathcal{H}_{m+m'} = \text{Ker}(a^{m+m})$. We have $\text{Ker}(W_{X_m} \rightarrow \mathcal{P}_m^{m'}) = \text{Ker}(V'_X \rightarrow \gamma^* \mathcal{P}^{m+m'}) = \gamma^* \mathcal{H}_{m+m'}$. Put $W'' = \text{Ker}(W \rightarrow W')$ and $\mathcal{L}' = \text{Coker}(W'_X \rightarrow \mathcal{P}_m^{m'})$. Then $\mathcal{L}'$ is invertible (since V defines an m' -section of f_m) and there is an induced surjection $W_{X_m} \rightarrow \mathcal{L}'$ with kernel $\gamma^* \mathcal{H}_{m+m'}$. Hence we get a commutative diagram of bundles on X , where all arrows are surjective and the three vertical arrows all have kernel equal to $\gamma^* \mathcal{H}_{m+m'}$,

$$\begin{array}{ccccc} V_{X_m} & \rightarrow & W_{X_m} & \rightarrow & W'_X \\ \downarrow & & \downarrow & & \downarrow \\ \gamma^* \mathcal{P}^{m+m'} & \rightarrow & \mathcal{P}_m^{m'} & \rightarrow & \mathcal{L}' \end{array}$$

Put $V'' = \text{Ker}(V \rightarrow W')$. Then V'_X is equal to $\text{Ker}(\mathcal{P}^{m+m'} \rightarrow \gamma^* \mathcal{L}')$, hence a sub- $(m+m')$ -bundle of $\mathcal{P}^{m+m'}$. This means that V'' defines an $(m+m')$ -section $f_{m+m'}: X_{m+m'} \rightarrow \mathbb{P}(W')$ of f . The existence of the cartesian diagram follows from the existence of the above commutative diagram of bundles. It remains to establish (4.5).

STEP 1. Put $\mathcal{E} = \text{Coker}(\mathcal{H}_{m+m'} \xrightarrow{c} W_X)$. Then $\mathcal{E}$ is an $(m+1)$ -bundle.

This follows since c is equal to the composition of the locally split map $\mathcal{H}_{m+m'} \hookrightarrow \mathcal{H}_m$ (induced by the surjection $\mathcal{P}^{m+m'} \rightarrow \mathcal{P}^m$) with the map $\mathcal{H}_m \rightarrow W_X$ (which is locally split because W defines an m -section of f).

STEP 2. The map $a: W_X \rightarrow \mathcal{P}_X^m(\mathcal{L}_m)$, obtained by composing $\beta^{m'}: W_X \rightarrow \mathcal{P}_X^{m+m'}(W_X)$ with $\mathcal{P}_X^{m+m'}(a^0): \mathcal{P}_X^{m+m'}(W_X) \rightarrow \mathcal{P}_X^m(\mathcal{L}_m)$ (§6) factors through $b: W_X \rightarrow \mathcal{E}$.

Consider the following commutative diagram of exact rows (since X is smooth, $\mathcal{P}_X^m(-)$ is an exact functor ([7], 16.7.3)),

$$\begin{array}{ccccccc} 0 & \rightarrow & V_X' & \rightarrow & V_X & \rightarrow & W_X \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{P}_X^{m+m'}(V_X) & \rightarrow & \mathcal{P}_X^{m+m'}(V_X) & \rightarrow & \mathcal{P}_X^{m+m'}(W_X) \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{P}_X^{m+m'}(V_X) & \rightarrow & \mathcal{P}_X^{m+m'}(\mathcal{P}^m) & \rightarrow & \mathcal{P}_X^{m+m'}(\mathcal{L}_m) \rightarrow 0. \end{array}$$

The injection $\mathcal{P}^m \hookrightarrow \mathcal{P}_X^m(\mathcal{L})$ gives an injection $j: \mathcal{P}_X^m(\mathcal{P}^m) \hookrightarrow \mathcal{P}_X^{m+m'}(\mathcal{P}_X^m(\mathcal{L}))$. The map $e': V_X \rightarrow \mathcal{P}_X^{m+m'}(\mathcal{L})$, obtained by composing e with j , factors through $a^{m+m'}: V_X \rightarrow \mathcal{P}_X^{m+m'}(\mathcal{L})$ via $\delta^{m,m'}: \mathcal{P}_X^{m+m'}(\mathcal{L}) \rightarrow \mathcal{P}_X^m(\mathcal{P}_X^m(\mathcal{L}))$ (6.2(iv)). Thus we get an equality $e'(V_X) = e(V_X)$ in $\mathcal{P}_X^m(\mathcal{P}^m)$ (since j is injective), also we see that $a^{m+m'}(V_X) = \mathcal{P}_X^{m+m'}(e)$. Therefore the map $\delta^{m,m'}$ induces a map $\delta: \mathcal{P}_X^{m+m'} \rightarrow e(V_X)$ of $(m+m'+1)$ -quotients of V_X . It follows that there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & V_X' & \rightarrow & V_X & \rightarrow & W_X \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & V_X' & \rightarrow & \mathcal{P}_X^{m+m'} & \rightarrow & \mathcal{E} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{P}_X^{m+m'}(V_X) & \rightarrow & \mathcal{P}_X^{m+m'}(\mathcal{P}^m) & \rightarrow & \mathcal{P}_X^{m+m'}(\mathcal{L}_m) \rightarrow 0. \end{array}$$

STEP 3. The map $\gamma^*a: W_X \rightarrow \gamma^*\mathcal{P}_X^{m+m'}(\mathcal{L}_m)$ is equal to the map $a^{m'}(f_m)$.

Observe first that proposition (4.3) is a special case ($m'=0$) of the lemma. Therefore we know that the canonical 1-quotient $a^0(f_m)$ on X_m is equal to $\gamma^*a^0: W_{X_m} \rightarrow \gamma^*\mathcal{L}_m$. Applying (6.2(iii)) to $\gamma: X_m \rightarrow X$ then gives a commutative diagram

$$\begin{array}{ccc} W_{X_m} & \xrightarrow{a^{m'}(f_m)} & \mathcal{P}_{X_m}^{m'}(\gamma^*\mathcal{L}_m) \\ \parallel & & \uparrow \\ W_{X_m} & \xrightarrow{\gamma^*a} & \gamma^*\mathcal{P}_X^{m+m'}(\mathcal{L}_m). \end{array}$$

This shows that (4.5) exists: Since $a^{m'}(f_m)$ is generically surjective, the quotients $W_X \rightarrow \gamma^*\mathcal{E}$ and $W_X \rightarrow \mathcal{P}_X^{m+m'}$ are equal. This completes the proof of theorem (4.4).

THEOREM (4.6). A q -projection of an m -section of $f: X \rightarrow P$ is the same as an m -section of a $(q+m)$ -projection of f .

PROOF. Let $f_m: X_m \rightarrow P(V^m)$ be an m -section of f , defined by $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$. Let $p_q^m: X_m \rightarrow P(W')$ be a q -projection of f_m , defined by $0 \rightarrow W' \rightarrow V'' \rightarrow W'' \rightarrow 0$. Put $W = \text{Ker}(V \rightarrow W'')$ (note that W has dimension $m+q+1$). We want to show that $0 \rightarrow W \rightarrow V \rightarrow W'' \rightarrow 0$ defines a $(q+m)$ -projection of f , i.e., that the induced map $c: W_X \rightarrow \mathcal{P}^{q+m-1}$ is surjective.

Since p_q^m is a q -projection of f_m , we have a commutative diagram

$$\begin{array}{ccc} W_X' & \xrightarrow{\quad} & V_X'' \\ & \searrow & \downarrow \\ & & \gamma_*\mathcal{P}_m^{q-1} \end{array}$$

which fits into the following commutative diagram (with exact rows)

$$\begin{array}{ccccccc} 0 & \rightarrow & V_X' & \rightarrow & V_X & \rightarrow & V_X'' \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & V_X' & \rightarrow & W_X & \rightarrow & W_X' \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & V_X' & \rightarrow & \mathcal{P}^{q+m-1} & \rightarrow & \gamma_*\mathcal{P}_m^{q-1} \rightarrow 0. \end{array}$$

An easy diagram chase shows that c is surjective.

The exact sequence $0 \rightarrow V' \rightarrow V \rightarrow W'' \rightarrow 0$ defines an m -section of the $(q+m)$ -projection given above; in fact, we get a commutative diagram

$$\begin{array}{ccc} 0 & \rightarrow & V_X' \rightarrow W_X \rightarrow W_X' \rightarrow 0 \\ & & \parallel & & \downarrow \\ 0 & \rightarrow & V_X' \rightarrow \mathcal{P}^m \rightarrow \gamma_*\mathcal{P}_m^0 \rightarrow 0 \end{array}$$

which also shows that this m -section is equal to p_q^m .

If we suppose, on the other hand, that we are given an m -section of a $(q+m)$ -projection of f , a similar argument shows that this curve is also a q -projection of an m -section of f .

If we put this last result together with (4.2) and (4.4), we see that any curve $f': X' \rightarrow P'$ which is obtained from f by projections and sections can be viewed as a q -projection of an m -section of f , p_q^m , where $q = \dim P'$ and m is uniquely determined. In particular, from the given curve $f: X \rightarrow P$ in n -space, we obtain by these methods $n-1$ (and not more) plane curves p_2^m , $m = 0, 1, \dots, n-2$.

PROPOSITION (4.7). Let $(r_0, r_1, \dots, r_{n-1})$ and $(k_0, \dots, k_{n-1})$ denote the ranks and stationary indices of the curve $f: X \rightarrow P$. The ranks and stationary

indices of a q -projection of an m -section $p_q^m: X_m \rightarrow P'$ of f are given by:

$$\begin{aligned} r_i(p_q^m) &= r_{m+i}, & i &= 0, \dots, q-1, \\ k_0(p_q^m) &= r_{m-1} + k_m & (\text{if } q > 1) \\ k_i(p_q^m) &= k_{m+i}, & 1 \leq i < q-1, \\ k_{q-1}(p_q^m) &= r_{m+q} + k_{m+q-1}. \end{aligned}$$

PROOF. The osculating bundle of order i , $\mathcal{O}_i^m$, of an m -section f_m fits into an exact sequence (see (4.5))

$$0 \rightarrow V_{X_m} \rightarrow \gamma^* \mathcal{O}^{m+i} \rightarrow \mathcal{O}_i^m \rightarrow 0.$$

Hence we get $r_i(f_m) = \deg(\mathcal{O}_i^m) = \deg(\mathcal{O}^{m+i}) = r_{m+i}$ (3.1). From (4.2) it follows that the osculating bundle of p_q^m is equal to that of f_m , hence we get also $r_i(p_q^m) = r_{m+i}$.

The formulas for $k_i(p_q^m)$ are straightforward applications of the formula (3.3) (ii) to the curves p_q^m and f .

EXAMPLE 1. The i th rank of an m -section of a q -projection of the rational n -ic R_n is $(m+i+1)(n-m-i)$; it has $m(n-m+1)$ cusps (if $q > 1$), $(m+q+1)(n-m-q)$ stationary hyperplanes, and no other points of hyperosculation.

EXAMPLE 2. For the curve $p_q^m(E_d)$ we get $r_i = (m+i+1)d$, $k_0 = md$ (if $q > 1$), $k_i = 0$ for $0 < i < q-1$ (if $m+i \leq n-2$), and $k_{q-1} = (m+q+1)d$. In particular, if we take $q=2$, $m=n-2$, we get a plane curve of degree $(n-1)d$, class nd , with $(n-2)d$ cusps and $2(n+1)d$ flexes.

EXAMPLE 3. Starting with the curve $\Gamma_2 \subset \mathbb{P}^4$, we obtain 3 plane curves p_2^i , $i=0, 1, 2$, with the following numerical characters:

	$r_0(\text{degree})$	$r_1(\text{class})$	$k_0(\text{cusps})$	$k_1(\text{flexes})$
p_2^0	8	24	0	48
p_2^1	24	48	8	80
p_2^2	48	40	64	40.

§5. Duality

Suppose $f: X \rightarrow P$ is finite and $a^n: V_X \rightarrow \mathcal{O}_X^n(\mathcal{L})$ is generically surjective. Assume also that we have either $\text{char } k=0$ or $\text{char } k > n$. The dual curve

$\check{f}: X \rightarrow P^\vee = \mathbb{P}(V^\vee)$ of $f: X \rightarrow P$ is defined by the 1-quotient $V_X^\vee \rightarrow \mathcal{H}_{n-1}^\vee$, where we have put $\mathcal{H}_{n-1} = \text{Ker}(a^{n-1}: V_X \rightarrow \mathcal{O}^{n-1})$. Hence a point $\check{f}(x)$ of the dual curve is equal to the osculating hyperplane to f at x , considered as a point in the dual projective space.

THEOREM (5.1). (i) The m -rank of $\check{f}$ is equal to the $(n-1-m)$ -rank of f , i.e., $r_m(\check{f}) = r_{n-1-m}(f)$, for $m=0, 1, \dots, n-1$, and similarly for the stationary indices: $k_m(\check{f}) = k_{n-1-m}(f)$ for $m=0, 1, \dots, n-1$.

(ii) The dual curve of $\check{f}$ is equal to f (we write $(\check{f})^\vee = f$).

(iii) The dual of a q -projection of an m -section of f is equal to a q -projection of an $(n-m-q)$ -section of $\check{f}$ (we write $p_q^m(f)^\vee = p_q^{n-m-q}(\check{f})$).

PROOF. Let $\mathcal{O}_m^m$ denote the m th osculating bundle of $\check{f}$ and $a_*^m: V_X^\vee \rightarrow \mathcal{O}_m^m$ the canonical quotient. Put $\mathcal{H}_*^m = \text{Ker}(a_*^m)$ and $\mathcal{H}_{n-1-m} = \text{Ker}(a^{n-1-m})$. The key result is the following:

LEMMA (5.2). There are canonical isomorphisms of exact sequences (for $m=0, \dots, n-1$)

$$\begin{aligned} 0 \rightarrow \mathcal{H}_*^m \rightarrow V_X^\vee \rightarrow \mathcal{O}_m^m \rightarrow 0 \\ \uparrow \quad \parallel \quad \uparrow \\ 0 \rightarrow (\mathcal{O}^{n-1-m})^\vee \rightarrow V_X^\vee \rightarrow (\mathcal{H}_{n-1-m})^\vee \rightarrow 0. \end{aligned}$$

Assuming for a moment that this holds, let us see how it implies the theorem. First of all, (i) follows:

$$\begin{aligned} r_m(\check{f}) &= \deg \mathcal{O}_m^m \text{ (by (3.1))} = \deg(\mathcal{H}_{n-1-m}^\vee) \\ &= -\deg \mathcal{H}_{n-1-m} = \deg \mathcal{O}^{n-1-m} = r_{n-1-m}(f). \end{aligned}$$

The curve $(\check{f})^\vee$ is defined by the 1-quotient $V_X^\vee \rightarrow \mathcal{H}_*^{n-1-\vee}$, i.e. by $V_X \rightarrow \mathcal{O}^0 = \mathcal{L}$, therefore (ii) holds.

Suppose $p_q^m(f)$ is defined by an $(n+1-m)$ -quotient $V \rightarrow V''$ and a $(q+1)$ -subspace $W' \subset V''$. We obtain (as in the proof of (4.6)) a commutative diagram

$$\begin{array}{ccccccc} 0 & \downarrow & 0 & \downarrow & 0 & \downarrow & 0 \\ 0 & \rightarrow & \mathcal{H}_{m+q-1} & \rightarrow & W_X'' & \rightarrow & 0 \\ & & \downarrow & & \parallel & & \\ 0 & \rightarrow & W_X' & \rightarrow & V_X'' & \rightarrow & W_X'' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_m^{q-1} = \mathcal{O}_m^{q-1} & \rightarrow & 0 & & 0. \end{array}$$

The dual curve $p_q^m(f)^\vee$ is defined by the 1-quotient $(W_X^\vee)^\vee \rightarrow \mathcal{M}^\vee$. Put $W = \text{Ker}(V \rightarrow W^\vee)$. We have an induced surjection $W \rightarrow W'$, and we note $\dim W = m+1+q$. The quotient $V^\vee \rightarrow W^\vee$ defines an $(n-m-q)$ -section of f ; we get an exact sequence $0 \rightarrow \mathcal{H}_*^{n-m-q} \rightarrow W_X^\vee \rightarrow \mathcal{M}^\vee \rightarrow 0$ defining this section. The isomorphism (5.2) then shows that the 1-quotient $W_X^\vee \rightarrow \mathcal{M}^\vee$ defining the projection $p_q^{n-m-q}(\tilde{f})$ corresponding to the subspace $W^\vee \subset W'$ is canonically isomorphic to the 1-quotient $W_X^\vee \rightarrow \mathcal{M}^\vee$. This proves (iii).

PROOF OF (5.2). (The proof is essentially the one given by Weyl ([15], p. 47) for analytic curves.) We will first show that the map between the sequences exists, by showing that the composition $a_*^\vee \circ (a^{n-1-m})^\vee$ is 0. Then we show that $\mathcal{H}_*^m$ has rank $m+1$, for $m=0, 1, \dots, n$ (so that the dual curve spans $\tilde{P}$). From these two facts it then follows that the generically surjective map $a^{n-1-m}(\tilde{f})$ factors through the quotient $V_X^\vee \rightarrow \mathcal{H}_{n-1-m}^\vee$, hence its image $\mathcal{H}_*^m$ is isomorphic to this quotient.

It suffices to show that $a_*^\vee \circ (a^{n-1-m})^\vee = 0$ holds locally on X . Let $x \in X$ and put $A = \mathcal{O}_{x,x}$. With the notations of the proof of (2.1), the map $(a^{n-1-m})^\vee$ is given by $(w_x^{n-1-m})^t: A^{m+1} \rightarrow A^{n+1}$. Let $(y): A \rightarrow A^{n+1}$ be the $(n+1, n)$ -matrix corresponding to the (locally split) map $\mathcal{H}_{n-1} \rightarrow V_X$ at x , so that the 1-quotient $V_X^\vee \rightarrow \mathcal{H}_*^m = (\mathcal{H}_{n-1})^\vee$ defining $\tilde{f}$ is given, locally at x , by $(y)^t$. It follows (2.1) that the map $a_*^\vee$ is given by the matrix w_y^m . The entries of the matrix $w_y^m \cdot (w_x^{n-1-m})^t$ are $\sum_{v=0}^m d^i y_v d^j x_v$. Hence we want to show:

$$D^{i,j} = \sum_{v=0}^m d^i y_v d^j x_v = 0 \quad \text{holds for} \quad 0 \leq i \leq m, 0 \leq j \leq n-1-m, \quad (*)$$

$$0 \leq m \leq n-1.$$

By definition of (y) , $D^{0,j} = 0$ holds for $j=0, 1, \dots, n-1$. Assume $D^{i,j} = 0$ holds for some i , and for $j=0, \dots, n-1-i$. Then we can apply the differential operator d^i :

$$0 = d^i(D^{i,j}) = \sum_{v=0}^m [(i+1)d^{i+1} y_v d^j x_v + (j+1)d^i y_v d^{j+1} x_v] \\ = (i+1)D^{i+1,j} + (j+1)D^{i,j+1}$$

Hence we get $D^{i+1,j} = 0$ for $j=0, \dots, n-1-(i+1)$, and we can prove (*) by induction.

It remains to prove that $a^m(\tilde{f})$ is generically surjective, for $0 \leq m \leq n$; since we have surjections $\mathcal{H}_*^m \rightarrow \mathcal{H}_*^{m-1}$, it is enough to show that $a^n(\tilde{f})$ is generically surjective. Hence we may assume that $x \in X$ is an ordinary point (see §2), so that $w(x) = \det(w_x^2)$ is a unit in $A = \mathcal{O}_{x,x}$. Suppose we can prove the equality

$$w(x)w(y) = (-1)^{\frac{1}{2}n(n+1)} w(x)^{n+1} \prod_{i=0}^n \binom{n}{i}, \quad (**)$$

where we have put $w(y) = \det(w_y^2)$. It would follow that $w(y)$ is a unit in A , hence $a^n(\tilde{f})$ is surjective at $x \in X$, as desired.

Consider the matrix $w_y^n \cdot (w_x^2)^t$. Since $D^{i,j} = 0$ holds for $0 \leq i+j \leq n-1$, this matrix is lower right triangular; its anti-diagonal entries are $D^{i,n-i}$. In order to prove (**), it thus suffices to show

$$D^{i,n-i} = (-1)^i \binom{n}{i} w(x), \quad (***)$$

Using the definition of (y) (and linear algebra) we see that $(***)_0$ holds. Assume $(***)_i$ holds and apply d^1 to $D^{i,n-1-i}$. From this we obtain $D^{i+1,n-(i+1)} = 0$ and the proof can be completed by induction.

Together with the results of §4, theorem (5.1) shows that from a curve $f: X \rightarrow P$ in n -space we obtain, by the method of projections, sections and dual curves, $n-1$ (and not more) plane curves $p_2^m(f)$ and their duals. Applying formulas (I) and (II) of §3 to these $2(n-1)$ plane curves we obtain the formulas of Veronese ([14]), if we (similarly to what he does) interpret the character e of (II) for each curve $p_2^m(f)$, $p_2^m(f)^\vee$ in terms of (actual and apparent) double osculating spaces of the curve f . We will not discuss this interpretation here; let us only, as an application of (5.1), give the Cayley-Plücker formulas for a curve in 3-space ([2], p. 191).

COROLLARY (5.3). Suppose $f: X \rightarrow P = \mathbb{P}^3$ is a curve such that $a^3: V_X \rightarrow \mathcal{P}_X^3(\mathcal{L})$ is generically surjective. Suppose $\text{char } k \neq 2, 3$. Let (r_0, r_1, r_2) and (k_0, k_1, k_2) denote the ranks and stationary indices of f , and let e_i (resp. $\tilde{e}_i$) denote the degree of the jacobian ideal of the plane curve $p_2^i(f)$ (resp. $p_2^i(f)^\vee$), for $i=0, 1$. There are formulas

- (1) $k_1 = 2r_1 - (r_0 + r_2) + 2g - 2$
- (2) $k_0 = 2r_0 - r_1 + 2g - 2$
- (3) $k_2 = 2r_2 - r_1 + 2g - 2$
- (4) $e_1 - 3(r_0 + k_1) = (r_1 - 1)(r_1 - 6) + 2r_2 - 6g$
- (5) $\tilde{e}_0 - 3(r_2 + k_1) = (r_1 - 1)(r_1 - 6) + 2r_0 - 6g$
- (6) $e_0 - 3k_0 = (r_0 - 1)(r_0 - 6) + 2r_1 - 6g$
- (7) $\tilde{e}_1 - 3k_2 = (r_2 - 1)(r_2 - 6) + 2r_1 - 6g$.

EXAMPLE 1. The rational n -ic is selfdual: its dual curve is also a rational n -ic.

EXAMPLE 2. $E_d \subset \mathbb{P}^3$. We have:

$$e_0 = d(d-3), \quad \tilde{e}_0 = d(4d-3) \\ e_1 = d(4d-5), \quad \tilde{e}_1 = d(9d-5).$$

EXAMPLE 3. $\Gamma_2 \subseteq \mathbb{P}^3$ is an elliptic curve of degree 4, with no cusps and no flexes. It has rank 8 and class 12. The number of apparent double points is $\frac{1}{2}e_0 = 2$, and the degree of the double curve of its tangent developable is $\frac{1}{2}(e_1 - 3(r_0 + k_1)) = 16$. The corresponding characters for the dual curve are $\frac{1}{2}(\tilde{e}_1 - 3k_2) = 38$ and $\frac{1}{2}(\tilde{e}_0 - 3(r_2 + k_1)) = 8$.

§6. Appendix. The functor of principal parts

Let $f: X \rightarrow S$ be a morphism of schemes. For each integer $m \geq 0$, consider the functor of relative principal parts (or jets) of order m

$$\mathcal{P}_{X/S}^m = p_{1*} p_2^{(m)*} : (\mathcal{O}_X - \text{mod}) \rightarrow (\mathcal{O}_X - \text{mod}).$$

Here $p_i^{(m)}$ denotes the composition of the i th projection $p_i: X \times_S X \rightarrow X$ with the m th infinitesimal neighborhood $h_m: X^{(m)} \rightarrow X \times_S X$ of the diagonal ([7], 16.7.1).

Let $\rho(f): 1 \rightarrow f_* f^* \rightarrow 1$ denote the canonical natural transformations. There is a natural transformation

$$p_{1*} \rho(h_m) p_2^*: p_{1*} p_2^* \rightarrow p_{1*} h_m^* h_m^* p_2^* \cong \mathcal{P}_{X/S}^m.$$

Define $\alpha^m = \alpha^m(f): f^* f^* \rightarrow \mathcal{P}_{X/S}^m$ to be the natural transformation obtained by composing the one above with the base change transformation

$$(f_* \rho(p_2))^\# : f^* f^* \rightarrow p_{1*} p_2^*$$

(where $\#$ denotes the isomorphism $\text{Hom}_S(-, f_* -) \cong \text{Hom}_X(f^* -, -)$). Define also a natural transformation of functors from $(\mathcal{O}_S - \text{mod})$ to $(\mathcal{O}_X - \text{mod})$

$$\beta^m = \beta^m(f): f^* \rightarrow \mathcal{P}_{X/S}^m \circ f^*$$

by composing $\rho(p_1^{(m)}) f^*: f^* \rightarrow p_{1*} p_1^{(m)*} f^*$ with the isomorphism $p_{1*} p_1^{(m)*} f^* \cong p_{1*} p_2^{(m)*} f^*$ (which exists since we have $f \circ p_1^{(m)} = f \circ p_2^{(m)}$).

Note that

$$\beta^m(\mathcal{O}_S): \mathcal{O}_X \rightarrow \mathcal{P}_{X/S}^m(\mathcal{O}_X),$$

is the homomorphism which defines the (left) $\mathcal{O}_X$ -algebra structure on $\mathcal{P}_{X/S}^m(\mathcal{O}_X)$.

PROPOSITION (6.1). There is a factorization $\alpha^m = (\mathcal{P}_{X/S}^m \sigma(f)) \circ (\beta^m f_*)$.

PROOF. By the adjoint property ([6] 0, 3.5.3.4) we know that α^m factors through $\beta^m f_*$,

$$\alpha^m = (p_{1*}^{(m)} \alpha^{\#}) \circ (\beta^m f_*).$$

(here $\#$ denotes the isomorphism $\text{Hom}_X(-, p_{1*}^{(m)} -) \cong \text{Hom}_{X^{(m)}}(p^{(m)*} -, -)$). Thus it suffices to show that we have $\mathcal{P}_{X/S}^m \sigma(f) = p_{1*}^{(m)} \alpha^{\#}$, or, equivalently, that $\alpha^{\#} = p_2^{(m)*} \sigma(f)$. The proof of this equality is straightforward.

The following proposition lists various properties of α^m . The proofs a straightforward (using the naturality of the transformations involved) and will therefore be omitted.

PROPOSITION (6.2).

- (i) We have $\alpha^0 = \sigma: f^* f^* \rightarrow 1$.
- (ii) If $m' \leq m$, the canonical diagram

$$\begin{array}{ccc} & \alpha^m & \\ f^* f^* & \searrow & \mathcal{P}_{X/S}^m \\ & \alpha^{m'} & \\ & \searrow & \mathcal{P}_{X/S}^{m'} \end{array}$$

commutes.

- (iii) If $g: Y \rightarrow X$ is an S -morphism, there is a canonical commutative diagram

$$\begin{array}{ccc} g^* f^* f^* & \xrightarrow{g^* \alpha^m} & g^* \mathcal{P}_{X/S}^m \\ \downarrow g^* f^* f_*(g) & & \downarrow u_g \\ g^* f^* f^* g^* g^* & \xrightarrow{\alpha^{m'} g^*} & \mathcal{P}_{Y/S}^{m'} g^* g^*. \end{array}$$

- (iv) Let $\delta^{m, m'}: \mathcal{P}_{X/S}^{m+m'} \rightarrow \mathcal{P}_{X/S}^m \mathcal{P}_{X/S}^{m'}$ denote the natural transformation defined in ([7] 16.8.2.2). The following diagram commutes

$$\begin{array}{ccccc} f^* f^* & \xrightarrow{\beta^{m'}} & \mathcal{P}_{X/S}^{m'} f^* f^* & \xrightarrow{\mathcal{P}_{X/S}^{m'} \beta^m} & \mathcal{P}_{X/S}^{m+m'} f^* f^* \\ \downarrow \beta^{m+m'} f_* & & \downarrow \delta^{m, m'} f_* & & \downarrow \delta^{m, m'} f_* \\ \mathcal{P}_{X/S}^{m+m'} f^* f^* & \xrightarrow{\beta^{m+m'}} & \mathcal{P}_{X/S}^{m+m'} f^* f^* & \xrightarrow{\mathcal{P}_{X/S}^{m+m'} \beta^0} & \mathcal{P}_{X/S}^{m+m'} f^* f^* \\ \downarrow \mathcal{P}_{X/S}^{m+m'} \alpha^0 & & \downarrow \mathcal{P}_{X/S}^{m+m'} \alpha^0 & & \downarrow \mathcal{P}_{X/S}^{m+m'} \alpha^0 \\ \mathcal{P}_{X/S}^{m+m'} & \xrightarrow{\delta^{m, m'}} & \mathcal{P}_{X/S}^m \mathcal{P}_{X/S}^{m'} & \xrightarrow{\delta^{m, m'}} & \mathcal{P}_{X/S}^{m+m'} \end{array}$$

PROPOSITION (6.3). Let $\mathcal{E}$ be a bundle on S . Put $P = \mathbb{P}(\mathcal{E})$ and $\mathcal{L} = \mathcal{O}_P(1)$. The map $\alpha^1(\mathcal{L}): \mathcal{E}_P \rightarrow \mathcal{P}_P^1(\mathcal{L})$ is an isomorphism.

PROOF. It suffices to show that $\alpha^1(\mathcal{L})$ is surjective (since it is a morphism between bundles of the same rank). The question is local on S , so we may assume $S = \text{Spec}(A)$ and $E = \bigoplus_{i=0}^n A e_i$. Write $\text{Sym}(E) = A[T_0, \dots, T_n]$ and let $U \subseteq P$ denote the open affine scheme defined by $T_i \neq 0$. Then $B = A[T_0/T_i, \dots, T_n/T_i]$ is the ring of U . The map $\alpha^0(\mathcal{L}): \mathcal{E}_B \rightarrow \mathcal{P}_B^0(\mathcal{L})$ is the map which sends e_i to $(T_i/T_i) e_i$. Therefore it follows from the factorization $\alpha^1(\mathcal{L}) = \mathcal{P}_P^1(\alpha^0(\mathcal{L})) \circ \beta^1(f_* \mathcal{L})$ (6.1) that $\alpha^1(\mathcal{L}): \mathcal{E}_B \rightarrow \mathcal{P}_B^1(\mathcal{L})$ is the map which sends e_i to $1 \otimes (T_i/T_i) e_i$ (here we have $(B \otimes B)/I^2 \cong B$).

put $I = \{b \otimes 1 - 1 \otimes b\} B \otimes B$. Since these elements generate $(B \otimes B)/I^2 \otimes BT_i$, we have shown that $\alpha^1(\mathcal{L})$ is surjective.

REMARK (6.4). Put $\mathcal{K} = \text{Ker}(\alpha^1(\mathcal{L}))$. The lemma shows that there is a canonical isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_P & \longrightarrow & \mathcal{L} \longrightarrow 0 \\ & & \alpha^1 \downarrow & & \alpha^1 \downarrow & & \parallel \\ 0 & \longrightarrow & \Omega_P(\mathcal{L}) & \longrightarrow & \mathcal{P}_P^1(\mathcal{L}) & \xrightarrow{v} & \mathcal{L} \longrightarrow 0. \end{array}$$

The isomorphism α is, of course, the canonical isomorphism called the second fundamental form in ([1], I, 3.1). (The isomorphism α can be described as follows: We see that $\mathcal{K} \mid U$ is the free B -module generated by $\{e_i^1\}_{i \neq h}$, where $e_i^1 = (T_i/T_i)e_i - e_i$. Hence

$$\begin{aligned} \alpha(e_i^1) &= \alpha^1(\mathcal{L})((T_i/T_i)e_i - e_i) \\ &= (T_i/T_i \otimes 1)(1 \otimes T_i) - (1 \otimes (T_i/T_i)T_i) \\ &= (T_i/T_i \otimes 1 - 1 \otimes T_i/T_i)(1 \otimes T_i). \end{aligned} \quad ([1], \text{I}, 3.1).$$

Note that the map $v \mid U: (B \otimes B)/I^2 \otimes BT_i \rightarrow BT_i$ is given by

$$v((1 \otimes (T_i/T_i))(1 \otimes T_i)) = v(1 \otimes (T_i/T_i)T_i) = (T_i/T_i)T_i.$$

We shall use $\alpha^1(\mathcal{L})$ to identify $\mathcal{P}_P^1(\mathcal{L})$ (as a left $\mathcal{O}_P$ -module) with $\mathcal{E}_P$, and we will identify $\Omega_P(\mathcal{L})$ with $\mathcal{K}$ via α .

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