

ON THE USE OF PARAMETER AND MODULI SPACES IN CURVE COUNTING

RAGNI PIENE

In order to solve problems in enumerative algebraic geometry, one works with various kinds of parameter or moduli spaces: Chow varieties, Hilbert schemes, Kontsevich spaces. In this note we give examples of such spaces. In particular we consider the case where the objects to be parametrized are algebraic curves lying on a given variety. The classical problem of enumerating curves of a given type and satisfying certain given conditions has recently received new attention in connection with string theories in theoretical physics. This interest has led to much new work—on the one hand, within the framework of more traditional algebraic geometry, on the other hand, with rather surprising results, using new methods and ideas, such as the theory of quantum cohomology and generating functions.

1. Introduction

Enumerative geometry has a long history. Apollonius of Perga (262–200 B.C.) considered and solved problems like the following: construct all circles tangent to three given circles in the plane. The enumerative part of this problem is to determine the *number* of solutions: there is one such circle containing (or circumscribing) the three circles, three containing precisely two, three containing only one, and one containing none, hence the answer is eight.

Similar questions can be asked for arbitrary conics (curves of degree 2) in the complex projective plane $\mathbb{P}^2 := \mathbb{P}_{\mathbb{C}}^2$ —e.g., how many conics are tangent to five given conics. A conic in $\mathbb{P}^2$ is given by the six coefficients of its defining equation (up to multiplication by a nonzero scalar), hence the parameter space of conics can be identified with $\mathbb{P}^5$. The points corresponding to *degenerate* conics (pairs of lines) form a hypersurface in $\mathbb{P}^5$, and the points corresponding to “double” lines form a 2-dimensional subvariety V in this hypersurface. For a given conic, the points corresponding to conics tangent to that conic, form a hypersurface of degree 6. Hence one might be led to think (as Jakob Steiner did in 1848), that there are $6^5 = 7776$ conics tangent to five given conics—in other words, that

the points corresponding to these conics are the points of intersection of five hypersurfaces, each of degree 6. This argument is wrong, however, because the parameter space is not “complete” with respect to the given problem in the following sense: any such “tangency condition” hypersurface contains the subvariety V of double lines, hence their intersection is never finite. The correct solution to the problem is 3264, as was found by de Jonquières (1859) and Chasles (1864). Essentially, what Chasles did, was to replace the parameter space $\mathbb{P}^5$ with a space B , whose points correspond to pairs consisting of a conic and its dual conic, and all limits of such pairs. The space B is the blow-up of $\mathbb{P}^5$ along V , and on B the intersection of the “tangency conditions” is finite. (See [11] for the history and the details.) More generally, one can ask to determine the *characteristic numbers* $N_{a,b}$ of a given family of plane curves; here $N_{a,b}$ is, by definition, the number of curves passing through a given points and tangent to b given lines, where $a + b$ is equal to the dimension of the family. Classically, this problem was solved—by Schubert and Zeuthen—for curves of degree at most 4, and these numbers have been verified by modern, rigorous methods.

To solve problems like the ones above, for example, to count curves lying on a given variety and satisfying certain conditions, a natural procedure is to represent the curves as points in some space, and then to represent the conditions as cycles on this parameter space. If the intersection theory of the parameter space is known, then the solution to a given enumerative problem can be obtained as the intersection number of the cycles corresponding to the given conditions (at least up to multiplicities of the solutions)—provided the cycles intersect properly.

A particular problem of this kind, which goes back to Severi and Zariski, is the following: given an r -dimensional family of curves on a surface, determine the number of curves in that family having r nodes (a *node* is an ordinary double point, that is, a singular point formed by two branches of the curve meeting transversally). The family can, for example, be a subsystem of a complete linear system, given by imposing the curves to pass through a certain number of points on the surface. There has been a lot of work on this problem in the last few years—here are just a very few sample references: [22], [3], [8], [2], [23], [6], [12], [21], [1].

The more nodes (or other singularities) a curve has, the smaller geometric genus it has. Therefore, one can also consider enumerative problems where instead of fixing the number (and type) of singularities, one fixes the geometric genus of the curves. For example, if one considers *rational* curves (i.e., curves of geometric genus zero) in the projective plane, then such an irreducible nodal curve of degree d must have $(d-1)(d-2)/2$ nodes. The question of enumerating rational curves is the one that first came up in the context of string theory in theoretical physics, and it is also one that has been central to many problems in algebraic and symplectic geometry.

In Section 2, we give a very brief presentation of three parameter spaces: the *Chow variety*, the *Hilbert scheme*, and the *Kontsevich moduli space of stable maps*. We elaborate a little on the last, which is the newest of the three, and we also define Gromov-Witten invariants in certain cases. Section 3 gives a simple example of a situation where the three above spaces lead to different compactifications of the same space, namely that of twisted cubic curves. In Section 4, we give a short introduction to quantum cohomology and show how the associativity of the quantum product can be used to deduce a

recursive formula for the number of rational plane curves of given degree d , passing through $3d - 1$ general points.

This note is only meant as a tiny introduction to what has recently become a very lively area of research. No proofs are given, and not all statements are completely true the way they are written. I refer to the papers in the bibliography for precise statements and proofs, more material, and further references. In particular, parts of Sections 2 and 4 draw heavily on [5] and [10].

Notation. I use standard notation from algebraic geometry (as in [7]). Note that a complex, projective, nonsingular *variety* can be considered as a complex analytic manifold, and more generally, a *projective scheme* can be considered as a complex analytic space (see [7, p. 438]). A *curve* (resp. a *surface*) is a variety of (complex) dimension 1 (resp. 2).

2. Parameter and moduli spaces

Let $X \subset \mathbb{P}^n$ be a complex, projective, nonsingular variety. There are at least three approaches to representing the set of curves $C \subset X$:

- (1) Chow variety: its points correspond to 1-dimensional cycles on X .
- (2) Hilbert scheme: its points correspond to 1-dimensional subschemes of X .
- (3) Kontsevich space: its points correspond to stable maps from a curve to X .

The first approach is the oldest; it goes back to Cayley, but was developed by Chow (see [14, p. 40]). The idea is to parametrize effective 1-dimensional cycles $C = \sum n_i C_i$ (the C_i are reduced and irreducible curves and $n_i \geq 0$) on X by associating to each such C a hypersurface $\Phi(C)$ in $\mathbb{P}^{n*} \times \mathbb{P}^{n*}$ (where $\mathbb{P}^{n*}$ is the *dual* projective space whose points are hyperplanes in $\mathbb{P}^n$): intuitively, $\Phi(C)$ is the set of pairs of hyperplanes (H, H') such that $C \cap H \cap H' \neq \emptyset$. For each C , the coefficients of $\Phi(C)$ determine a point in an appropriate projective space, and the union of these points, as C varies, is $\text{Chow}_1(X)$. In order to get something of reasonable size, we restrict the set of curves we consider by fixing the degree, say d , of the cycle. The corresponding parameter space is denoted $\text{Chow}_{1,d}(X)$.

The second approach is due to Grothendieck. It gives a projective scheme, $\text{Hilb}(X)$, which parametrizes all closed subschemes of a given projective variety X . The advantage with this approach is that there exists a universal flat family of subschemes having the Hilbert scheme as a base. In fact, to give a morphism from a scheme T to the Hilbert scheme, is equivalent to giving a flat family of schemes over T , where each fiber is a subscheme of X . Since the Hilbert polynomial is constant in a flat family, the Hilbert scheme splits into (not necessarily irreducible) components $\text{Hilb}^{P(t)}(X)$ according to the Hilbert polynomial $P(t)$. For a projective curve, the Hilbert polynomial is of the form $P(t) = dt + 1 - g_a$, where d is the degree and g_a the arithmetic genus of the curve.

The last approach is relatively new and is part of the “revolution” in enumerative geometry due to the appearance of the physicists on the scene. The Kontsevich moduli space of pointed morphisms can be defined as follows (see [15], [5]). Fix an element $\beta \in A_1 X := H_2(X; \mathbb{Z})$ and consider the set of isomorphism classes of pointed morphisms:

$$M_{g,n}(X, \beta) = \{(\mu : C \longrightarrow X; p_1, \dots, p_n) \mid \mu_*([C]) = \beta\} / \sim,$$

where C is an irreducible smooth curve of genus g , the p_i 's are distinct points on C , and

$$(\mu : C \longrightarrow X; p_1, \dots, p_n) \sim (\mu' : C' \longrightarrow X; p'_1, \dots, p'_n)$$

if there exists an isomorphism $\nu : C \rightarrow C'$ with $\mu' \circ \nu = \mu$ and $\nu(p_i) = p'_i$, for $i = 1, \dots, n$. By adding the so-called *stable* pointed morphisms from not necessarily irreducible curves, one obtains a compactification $\overline{M}_{g,n}(X, \beta)$ of this space, which is a *coarse moduli space* (see [9] and [18]). In particular, if X is a point (so that $\beta = 0$), then $\overline{M}_{g,n}(\{\text{point}\}, 0) = \overline{M}_{g,n}$ is the usual Deligne-Mumford moduli space of stable, n -pointed curves of genus g .

In what follows, we shall only consider the case where $g = 0$ (so that $C = \mathbb{P}^1$) and X is convex (e.g., X is a projective space, a Grassmannian, a flag variety, ...). Then one can show that $M_{0,n}(X, \beta)$ is a normal projective variety of dimension

$$\dim X + \int_{\beta} c_1(T_X) + n - 3,$$

where $c_1(T_X)$ denotes the first Chern class of the tangent bundle of X , and $\int_{\beta} \alpha$ is the degree of the zero cycle $\alpha \cap \beta$. Set $\overline{M} = \overline{M}_{0,n}(X, \beta)$, and let

$$\rho_i : \overline{M} \longrightarrow X$$

denote the i th evaluation map, given by

$$\rho_i(\mu : C \longrightarrow X; p_1, \dots, p_n) = \mu(p_i).$$

If $\gamma_1, \dots, \gamma_n \in A^*X := H^*(X; \mathbb{Z})$ are cohomology classes, we define *Gromov–Witten invariants* as follows:

$$I_{\beta}(\gamma_1, \dots, \gamma_n) = \int_{\overline{M}} \rho_1^* \gamma_1 \cup \dots \cup \rho_n^* \gamma_n.$$

Assume each γ_i is effective, that is, γ_i is equal to the class $[\Gamma_i]$ of some subvariety Γ_i of X . Assume moreover that $\sum_i \text{codim } \Gamma_i = \dim \overline{M}$. If the Γ_i are in “general position”, the Gromov-Witten invariants have enumerative significance:

$$I_{\beta}(\gamma_1, \dots, \gamma_n) = \deg \rho_1^{-1}(\Gamma_1) \cap \dots \cap \rho_n^{-1}(\Gamma_n)$$

is the number of pointed maps $(\mu : C \rightarrow X; p_1, \dots, p_n)$ such that $\mu_*([C]) = \beta$ and $\mu(p_i) \in \Gamma_i$. This is the same as the number of rational curves in X of class β and meeting all the subvarieties Γ_i .

Example. Let $X = \mathbb{P}^2$ and $\beta = d$ [line]. We shall write $M_{0,n}(\mathbb{P}^2, d)$ instead of $M_{0,n}(\mathbb{P}^2, d$ [line]). This space has dimension $2 + 3d + n - 2 = 3d - 1 + n$, since $\int_{\beta} c_1(T_X) = 3d$. Consider the case $n = 3d - 1$. Take points $x_1, \dots, x_{3d-1} \in X$ in general position, and set $\Gamma_i = \{x_i\}$. Then

$$\sum_i \text{codim } \Gamma_i = 2(3d - 1) = \dim M_{0,n}(\mathbb{P}^2, d)$$

holds, and

$$I_\beta([\Gamma_1], \dots, [\Gamma_{3d-1}]) = N_d$$

is the number of rational plane curves of degree d passing through the $3d - 1$ points $x_1, \dots, x_{3d-1}$.

There exist linear relations between the boundary components of the compactification $\overline{M}_{0,n}(\mathbb{P}^2, d)$ of $M_{0,n}(\mathbb{P}^2, d)$, and these can be used to find a recursive formula for the numbers N_d in terms of N_{d_i} with $d_i < d$ (see [5, 0.6]). In Section 4, we shall indicate how this formula also can be derived from quantum cohomology.

Consider the case of plane conics, that is, take $d = 2$. In this case, both $\text{Chow}_{1,2}(\mathbb{P}^2)$ and $\text{Hilb}^{2r+1}(\mathbb{P}^2)$ are equal to the space $\mathbb{P}^5$ of plane conics. Hence, as we have seen, neither is good for enumerative problems involving tangency conditions. Classically, one considered the variety B of *complete* conics: $B \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ is the set of pairs of a conic and its dual conic (the conic in the dual projective plane whose points are the tangent lines of the original conic) and limits of such pairs; one shows that B is equal to the blow-up of $\mathbb{P}^5$ in the locus V corresponding to double lines. The limits of a pair consisting of a conic and its dual conic can be identified with the following three types of configurations: a pair of lines (the limit of the dual conic in this case is the “double line” consisting of all lines through the point of intersection of the line pair), a line with two marked points (the limit of the dual is the union of the sets of lines through each of these points), and a line with one marked point (the dual is the set of lines through the point, considered as a “double line” in the dual plane).

The space $M_{0,0}(\mathbb{P}^2, 2)$ is the set of isomorphism classes of maps $\mu : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ such that the image cycle $\mu_*(\mathbb{P}^1)$ has degree 2. The class of a map μ which is one-to-one is determined by its image, $\mu(\mathbb{P}^1)$, which is a nonsingular conic. A map which is two-to-one is a degree 2 map from $\mathbb{P}^1$ to some line in $\mathbb{P}^2$. Its isomorphism class is determined by that line together with two distinct points on it (the two branch points of the map). The maps corresponding to points on the boundary of the compactification $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ are maps from the union of two $\mathbb{P}^1$'s intersecting in a point; if the map is an immersion, its isomorphism class is determined by its image, the union of two lines—otherwise, it maps the two $\mathbb{P}^1$'s onto the same line, and its class is determined by that line together with the point which is the image of the intersection point of the two $\mathbb{P}^1$'s. Hence we can indeed identify $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ with the space B of complete conics (see [5, 0.4]).

The example above is typical for hypersurfaces, in the sense that the (relevant components of the) Chow variety and Hilbert scheme are equal for hypersurfaces, e.g., for curves on surfaces. We shall see in the next section that this does not hold when we consider curves on higher dimensional varieties.

3. Twisted cubic curves

A *twisted cubic* is a nonsingular, rational curve of degree 3 in $\mathbb{P}^3$. The set $\mathcal{T}$ of twisted cubics has a natural structure as a homogeneous space of dimension 12: since any twisted

cubic is projectively equivalent to the image of the Veronese embedding $\mathbb{P}^1 \rightarrow \mathbb{P}^3$, given by sending a point with homogeneous coordinates (s, t) to the point (s^3, s^2t, st^2, t^3) , we get $\mathcal{T} = \mathrm{SL}(4; \mathbb{C})/\mathrm{SL}(2; \mathbb{C})$. Consider the following three compactifications of the variety $\mathcal{T}$.

(1) A twisted cubic is a 1-cycle of degree 3 in $\mathbb{P}^3$, so $\mathcal{T} \subset \mathrm{Chow}_{1,3}(\mathbb{P}^3)$. Let $\mathcal{C}$ denote the irreducible component containing $\mathcal{T}$.

(2) A twisted cubic is a curve of degree 3 and arithmetic genus zero in $\mathbb{P}^3$, hence $\mathcal{T} \subset \mathrm{Hilb}^{3r+1}(\mathbb{P}^3)$. Let $\mathcal{H}$ denote the irreducible component containing $\mathcal{T}$.

(3) A twisted cubic is the image of a map $\mathbb{P}^1 \rightarrow \mathbb{P}^3$, hence $\mathcal{T} \subset \overline{M}_{0,0}(\mathbb{P}^3, 3)$. Let $\mathcal{M}$ denote the irreducible component of the Kontsevich space containing $\mathcal{T}$.

These three spaces, $\mathcal{C}$, $\mathcal{H}$, and $\mathcal{M}$, are birationally equivalent, but they are not equal. We have a map $\phi : \mathcal{H} \rightarrow \mathcal{C}$, which “forgets” the scheme structure except for the multiplicities of the components—e.g., any scheme structure of multiplicity 3 on a line L in $\mathbb{P}^3$ maps to the same point $3L \in \mathcal{C}$. Similarly, there is a map $\psi : \mathcal{M} \rightarrow \mathcal{C}$, but no obvious maps between $\mathcal{H}$ and $\mathcal{M}$.

As an example, consider the point $2L + L' \in \mathcal{C}$, where L, L' are lines in $\mathbb{P}^3$ intersecting in a point. Points in $\phi^{-1}(2L + L')$ correspond to double structures of genus -1 on the line L , and one can show that there is a 2-dimensional family of such structures. Points in $\psi^{-1}(2L + L')$ correspond to stable maps from a union of $\mathbb{P}^1$'s onto $L \cup L'$, of degree 2 on L and 1 on L' , and where some $\mathbb{P}^1$'s may map to points. The set of isomorphism classes of such maps also contains a 2-dimensional set, but there seems to be no natural relation between the fibres $\phi^{-1}(2L + L')$ and $\psi^{-1}(2L + L')$ (cf. [17]).

There are also other natural compactifications of $\mathcal{T}$. The ideal of a twisted cubic in the homogeneous coordinate ring is generated by three quadrics, and one can show that $\mathcal{T}$ has a “minimal” compactification $\overline{\mathcal{T}}$ equal to the moduli space of nets (i.e., 2-dimensional linear systems) of quadrics. In fact, one can show that $\mathcal{H}$ is the blow up of $\overline{\mathcal{T}}$ along the boundary $\overline{\mathcal{T}} - \mathcal{T}$; points in the boundary correspond to degenerate nets, i.e., nets with a plane as fixed component (see [4]).

For enumerative problems, one is led to consider a space of “complete” twisted cubics, similarly to the case of plane curves, by taking triples consisting of the curve, its tangent developable surface, and its strict dual curve, and limits of such triples. Depending on whether one takes these limits in the Hilbert schemes or in the Chow varieties, one gets different spaces, and they also differ from the ones considered above (see [19] and [20]).

4. Quantum cohomology and rational curves

The quantum cohomology ring of a projective variety can be thought of as a deformation of the ordinary cohomology ring, where the deformation parameters—or “quantum variables”—are “dual” to a basis of the cohomology groups (viewed as a complex vector space.) To get a ring structure, one deforms the ordinary cup product to get a “quantum product.” The structure of this ring has quite surprising implications in enumerative geometry. In particular we shall see how the recursive formula for the number N_d can be deduced from the associativity of the quantum product.

Let $T_0, \dots, T_m$ be a basis for A^*X such that $T_0 = 1$, $T_1, \dots, T_p$ is a basis for A^1X , and $T_{p+1}, \dots, T_m$ is a basis for the sum of the other cohomology groups. Consider the

“universal element”

$$\gamma = \sum_{i=0}^m y_i T_i,$$

where the coefficients y_i are the “quantum variables”. The main idea from physics is to form a generating function (or “potential” or “free energy” function) for the Gromov-Witten invariants as follows:

$$\Phi(y_0, \dots, y_m) = \sum_{n_0 + \dots + n_m \geq 3} \sum_{\beta \in A_1 X} I_\beta(T_0^{n_0}, \dots, T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \dots \frac{y_m^{n_m}}{n_m!}.$$

One can show that for each β there are only finitely many nonzero Gromov-Witten invariants, hence $\Phi(y_0, \dots, y_m) \in \mathbb{Q}[[y_0, \dots, y_m]]$ is a power series ring.

The part of Φ corresponding to $\beta = 0$ (corresponding to maps $\mathbb{P}^1 \rightarrow X$ with image a point) is the “classical” part—the rest is the “quantum” part:

$$\Phi = \Phi_{\text{cl}} + \Gamma.$$

Define numbers g_{ij} by

$$g_{ij} = \int_X T_i \cup T_j$$

and let (g^{ij}) denote the inverse matrix $(g_{ij})^{-1}$. Then we define the quantum product:

$$T_i * T_j = \sum_{k,l} \Phi_{ijk} g^{kl} T_l,$$

where $\Phi_{ijk} = \delta^3 \Phi / \delta y_i \delta y_j \delta y_k$. By extending this product $\mathbb{Q}[[y_0, \dots, y_m]]$ -linearly to the $\mathbb{Q}[[y_0, \dots, y_m]]$ -module $A^* X \otimes \mathbb{Q}[[y_0, \dots, y_m]]$ we obtain a $\mathbb{Q}[[y_0, \dots, y_m]]$ -algebra that we denote by $QA^* X$ —this is our “quantum cohomology” ring. Obviously, the above product is commutative, and one sees easily that T_0 is a unit. On the contrary, it takes a lot more effort to prove that the product is associative! In view of the consequences of associativity, this is not so surprising (see [5], [10]).

The case $X = \mathbb{P}^2$. In this case, $p = 1$ and $m = 2$: $T_0 = [X]$, $T_1 = [\text{line}]$, $T_2 = [\text{point}]$, and $\beta = dT_1$, for $d \in \mathbb{Z}$. We have $(g_{ij}) = 1$ if $i + j = 2$ and $g_{ij} = 0$ otherwise, so that

$$g_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (g^{ij}).$$

Hence we get

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0.$$

The classical part of Φ becomes

$$\Phi_{\text{cl}} = \sum_{n_0 + n_1 + n_2 = 3} \left(\int_{\mathbb{P}^2} T_0^{n_0} \cup T_1^{n_1} \cup T_2^{n_2} \right) \frac{y_0^{n_0}}{n_0!} \cdot \frac{y_1^{n_1}}{n_1!} \cdot \frac{y_2^{n_2}}{n_2!} = \frac{1}{2} y_0 y_1^2 + \frac{1}{2} y_0^2 y_2.$$

Hence

$$(\Phi_{\text{cl}})_{ijk} = 1$$

if (i, j, k) is a permutation of $(0, 1, 1)$ or $(0, 0, 2)$, and zero otherwise.

From the definition of the Gromov-Witten invariants it follows that the quantum part Γ does not contain the variable y_0 . In fact, we get

$$\begin{aligned} \Gamma(y_0, y_1, y_2) &= \sum_{n_1+n_2 \geq 3} \sum_{d>0} I_{dT_1}(T_1^{n_1}, T_2^{n_2}) \frac{y_1^{n_1}}{n_1!} \cdot \frac{y_2^{n_2}}{n_2!} \\ &= \sum_{n_1} \sum_{d>0} \left(\int_{dT_1} T_1 \right)^{n_1} I_{dT_1}(T_2^{3d-1}) \frac{y_1^{n_1}}{n_1!} \cdot \frac{y_2^{3d-1}}{(3d-1)!} \\ &= \sum_{d>0} N_d e^{dy_1} \cdot \frac{y_2^{3d-1}}{(3d-1)!}, \end{aligned}$$

where $(\int_{dT_1} T_1)^{n_1} = d^{n_1}$, $\sum_{n_1} d^{n_1} \frac{y_1^{n_1}}{n_1!} = e^{dy_1}$, and $N_d := I_{dT_1}(T_2^{3d-1})$ is, as observed earlier, the number of rational plane curves of degree d passing through $3d-1$ general points.

We now deduce a recursive relation for the numbers N_d . Since the quantum part Γ of Φ does not contain the variable y_0 , we have $\Phi_{ij0} = (\Phi_{\text{cl}})_{ij0}$. We can therefore compute

$$\begin{aligned} T_1 * T_1 &= T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0, \\ T_1 * T_2 &= \Gamma_{121} T_1 + \Gamma_{122} T_0, \\ T_2 * T_2 &= \Gamma_{221} T_1 + \Gamma_{222} T_0. \end{aligned}$$

Hence we get

$$\begin{aligned} (T_1 * T_1) * T_2 &= (\Gamma_{221} T_1 + \Gamma_{222} T_0) + \Gamma_{111} (\Gamma_{121} T_1 + \Gamma_{122} T_0) + \Gamma_{112} T_2, \\ T_1 * (T_1 * T_2) &= \Gamma_{121} (T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0) + \Gamma_{122} T_1. \end{aligned}$$

The associativity of the product now gives the following differential equation for the function Γ :

$$\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111} \Gamma_{122}.$$

We now plug in the power series expression for the function Γ in this differential equation and solve for N_d :

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right].$$

The initial condition is $N_1 = 1$ —there is exactly one line through two given points in the plane—and so one can compute all N_i recursively—here are the first 8:

$$\begin{aligned} N_1 &= 1, \\ N_2 &= 1, \\ N_3 &= 12, \\ N_4 &= 620, \\ N_5 &= 87304, \\ N_6 &= 26312976, \\ N_7 &= 14616808192, \\ N_8 &= 13525751027392. \end{aligned}$$

Acknowledgement

Finally, I would like to thank the organizers of the EWM workshop on “Moduli spaces in mathematics and physics” for a very interesting meeting.

References

- [1] J. Bryan and N. C. Leung, *Counting curves on irrational surfaces*, Surveys in Differential Geometry: Differential Geometry Inspired by String Theory, Int. Press, Boston, MA, 1999, pp. 313–339. [CMP 1 772 273](#).
- [2] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392. [MR 99i:14064](#). [Zbl 934.14040](#).
- [3] P. Di Francesco and C. Itzykson, *Quantum intersection rings*, The Moduli Space of Curves (Texel Island, 1994) (Boston, MA), Birkhäuser Boston, 1995, pp. 81–148. [MR 96k:14041a](#). [Zbl 868.14029](#).
- [4] G. Ellingsrud, R. Piene, and S. A. Strømme, *On the variety of nets of quadrics defining twisted cubics*, Space curves (Rocca di Papa, 1985) (Berlin), vol. 1266, Springer, 1987, pp. 84–96. [MR 88h:14034](#). [Zbl 659.14027](#).
- [5] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995 (Providence, RI), vol. 62, Amer. Math. Soc., 1997, pp. 45–96. [MR 98m:14025](#). [Zbl 898.14018](#).
- [6] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196** (1998), no. 3, 523–533. [MR 2000f:14085](#).
- [7] R. Hartshorne, *Algebraic Geometry*, vol. 52, Springer-Verlag, New York, 1977. [MR 57#3116](#). [Zbl 367.14001](#).
- [8] C. Itzykson, *Counting rational curves on rational surfaces*, Internat. J. Modern Phys. B **8** (1994), no. 25–26, 3703–3724. [MR 96e:14061](#).
- [9] F. Kirwan (ed.), *Moduli spaces in algebraic geometry*, these proceedings.
- [10] S. L. Kleiman, *Applications of QH^* to enumerative geometry*, Quantum Cohomology, Mittag-Laffler Institute 1996–1997 (P. Aluffi, ed.), www.math.fsu.edu/~aluffi/eprint.archive.html, 79–01.
- [11] ———, *Chasles’s enumerative theory of conics: a historical introduction*, Studies in Algebraic Geometry (Washington, D.C.), vol. 20, Math. Assoc. America, 1980, pp. 117–138. [MR 82i:14033](#). [Zbl 444.14001](#).
- [12] S. L. Kleiman and R. Piene, *Node polynomials for curves on surfaces*, to appear.

- [13] ———, *Enumerating singular curves on surfaces*, Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998) (Providence, RI), Amer. Math. Soc., 1999, pp. 209–238. [CMP 1 718 146](#). [Zbl 991.19224](#).
- [14] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. [A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1996. [MR 98c:14001](#). [Zbl 877.14012](#).
- [15] M. Kontsevich, *Enumeration of rational curves via torus actions*, Progr. Math., vol. 129, pp. 335–368, Birkhäuser Boston, Boston, MA, 1995. [MR 97d:14077](#).
- [16] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562. [MR 95i:14049](#). [Zbl 853.14020](#).
- [17] M. Martin-Deschamps and R. Peine, work in progress.
- [18] R. M. Miró-Roig (ed.), *Moduli spaces of vector bundles on algebraic varieties*, these proceedings.
- [19] R. Piene, *Degenerations of complete twisted cubics*, Enumerative Geometry and Classical Algebraic Geometry (Nice, 1981) (Mass.) (P. Le Barz and Y. Hervier, eds.), Birkhäuser Boston, 1982, pp. 37–50. [MR 84h:14010](#). [Zbl 498.14029](#).
- [20] ———, *On the problem of enumerating twisted cubics*, Algebraic Geometry, Sitges (Barcelona), 1983 (Berlin) (E. Casas-Alvero, G. Welters, and S. Xambò-Descamps, eds.), vol. 1124, Springer, 1985, pp. 329–337. [MR 86m:14039](#). [Zbl 578.14045](#).
- [21] ———, *On the enumeration of algebraic curves—from circles to instantons*, First European Congress of Mathematics, Vol. II (Paris, 1992) (Basel), Progr. Math., vol. 24, Birkhäuser, 1994, pp. 327–353. [MR 96g:14046](#). [Zbl 835.14021](#).
- [22] I. Vainsencher, *Enumeration of n -fold tangent hyperplanes to a surface*, J. Algebraic Geom. **4** (1995), no. 3, 503–526. [MR 96e:14063](#). [Zbl 928.14035](#).
- [23] R. Vakil, *Counting curves on rational surfaces*, Preprint, 1999.

RAGNI PIENE: MATEMATISK INSTITUTT, UNIVERSITETET I OSLO, P.B. 1053 BLINDERN, NO–0316 OSLO, NORWAY