



Counting curves on a surface

Ragni Piene

Centre of Mathematics for Applications
and Department of Mathematics,
University of Oslo

University of Pennsylvania,
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Enumerative geometry

Apollonius: How many circles in the plane are tangent to three given circles?

- ▶ The answer is 8, which can be seen combinatorially.

Three methods to solve a problem in enumerative geometry:

- ▶ Method 1: Specialize the objects and/or the conditions so that the problem has a combinatorial solution.
- ▶ Method 2: Represent the objects as points and the conditions as cycles, in some parameter space, and do intersection theory.
- ▶ Method 3: Physics!



Specialization

Question: How many lines in projective 3-space meet four given lines?

- ▶ Answer: Specialize the four lines into two line-pairs. Then the line of intersection of the two planes spanned by the line pairs meets all four lines, and the line joining the points of intersection of each line-pair meets all four lines. The answer is 2.





Intersection theory

Question: How many lines in projective 3-space meet four given lines?

- Answer: Represent the lines as points of the Grassmann variety of lines in $\mathbb{P}^3$. The condition “to intersect a line” is a Schubert cycle σ , and intersection theory on the Grassmannian gives $\sigma^4 = 2$. The answer is 2.





Plane rational curves

Let N_d denote the number of plane rational curves of degree d passing through $3d - 1$ given points.

- ▶ Kontsevich's recursion formula:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

- ▶ Kontsevich's original proof used a specialization argument, hence an example of Method 1. (For higher genus: Caporaso–Harris, Ran.)
- ▶ The formula can also be proved using quantum cohomology, as a consequence the associativity of the quantum product.





Generating functions

Consider an enumerative problem with answer N_α depending on an integer (or a tuple of integers) α .

- ▶ The generating function of the problem is

$$\sum_{\alpha} N_{\alpha} q^{\alpha}.$$

- ▶ For example, take $N_{d,g}$ to be the number of plane curves of degree d and genus g passing through $3d - 1 + g$ points:

$$f(q, t) := \sum_{d,g} N_{d,g} q^d t^g,$$

or, for fixed g ,

$$f_g(q) := \sum_d N_{d,g} q^d.$$





Curves in string theory

- ▶ Clemens' conjecture: There are only finitely many rational curves of degree d on a general quintic hypersurface in $\mathbb{P}^4$. (Proved for $d \leq 10$ — Clemens, Katz, Kleiman–Johnsen, Cotterill.)
- ▶ The physicists enter the scene:
Rational curves on Calabi–Yau threefolds are instantons.
There is something called mirror symmetry.
- ▶ By the principle of mirror symmetry, counting curves on a CY manifold can be done using integrals on the mirror manifold. This way Candelas, de la Ossa, Green, and Parkes predicted the generating function of the Clemens problem.





Partitions

Let p denote the partition function, i.e., for each positive integer n , $p(n)$ denotes the number of ways of writing $n = n_1 + \dots + n_k$, where $n_1 \geq \dots \geq n_k \geq 1$. (This is the same as the number of Young–Ferrers diagrams of size n .)

- The generating function for p is

$$\begin{aligned}\eta(q) &:= \sum_{n \geq 0} p(n) q^n \\ &= \prod_{m \geq 1} (1 - q^m)^{-1} \\ &= 1 + q + 2q^2 + 3q^3 + 5q^4 + \dots\end{aligned}$$





Curves on K3 surfaces

Assume S is a K3 surface and $C \subset S$ is a curve. For n, g such that $C^2/2 = g + n - 1$, let $N_{g,n}$ denote the number of curves in $|C|$ of geometric genus g , with n nodes, and passing through g points of S .

- Bryan and Leung proved: if S is general and $[C]$ is primitive,

$$f_g(q) := \sum N_{g,n} q^n = \frac{(DG_2)^g}{\Delta} q^{1-g},$$

where $\Delta = q \prod_{m \geq 1} (1 - q^m)^{24}$ is the discriminant (a modular form of weight 12) and DG_2 is the derivative of the Eisenstein series $G_2 = -\frac{1}{24} + \sum_{k \geq 1} \sigma(k) q^k$ (a quasimodular form), where $\sigma(k) = \sum_{d|k} d$.

- For $g = 0$, $f_0(q) = \Pi(1 - q^m)^{-24} = \eta(q)^{24}$, where η is the generating function of the partition function.



Proof by Method 1: Specialize (in symplectic geometry) S to an elliptic fibration $S' \rightarrow \mathbb{P}^1$ with 24 nodal fibers, and C to $C' = s(\mathbb{P}^1) + (n + g)\phi$, where s is a section and ϕ is a fiber. Fix g points on distinct, smooth fibers and work with the moduli space of maps of stable genus g curves with g marked points.

The case $g = 0$: The number of stable genus 0 maps for a given n can be identified with the product $\prod M_{a_i}$, where M_{a_i} is the number of such curves mapping $a_i: 1$ to the i th nodal fibre, and the product is taken over 24-tuples $(a_1, \dots, a_{24})$ with $\sum a_i = n$. One can see that M_{a_i} is equal to $p(a_i)$, the generating function is the 24th power of the generating function $\eta(q)$.

Gromov–Witten invariants

Let S be a smooth, projective surface. The Gromov–Witten invariants are the numbers of irreducible curves in a complete linear system $|C|$ of given genus g , or the numbers N_r of curves having r nodes, and passing through an appropriate number of points on S .

- ▶ Vainsencher gave explicit formulas for N_r as polynomials in the Chern numbers of C and S , for $r \leq 6$. His results and methods inspired Di Francesco–Itzykson, Göttsche, Kleiman–P., Ai-ko Liu, among others.



Göttsche conjectured that if $|C|$ is ample enough, then the generating function $\sum N_r q^r$ can be expressed in terms of two universal (but unknown and un-understood) formal power series and three quasimodular forms. If the canonical bundle of S is trivial, then only quasimodular forms appear.

- Proved by Bryan and Leung if S is a general K3 or Abelian surface and the class of C is primitive.



Bell polynomials

Consider the formal identity

$$\begin{aligned}\sum_{n=0}^{\infty} P_n q^n / n! &= \exp\left(\sum_{j=1}^{\infty} g_j q^j / j!\right) \\ &= \prod_{j=1}^{\infty} (1 + g_j q^j / j! + 1/2! (g_j q^j / j!)^2 + \dots).\end{aligned}$$

In other words,

$$P_n(g_1, \dots, g_n) = \sum_{(k) \vdash n} \frac{n!}{k_1! \dots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \dots \left(\frac{g_n}{n!}\right)^{k_n},$$

where $(k) \vdash n$ means $k_1 + 2k_2 + \dots + nk_n = n$.

The P_n are called the (complete, exponential) Bell polynomials.



Faà di Bruno's formula

Let h be a composed function, $h(t) = f(g(t))$.

Differentiating once: $h'(t) = f'(g(t))g'(t)$,

and twice: $h''(t) = f''(g(t))g'(t)^2 + f'(g(t))g''(t)$.

$$h_1 = f_1 \cdot g_1$$

$$h_2 = f_2 \cdot g_1^2 + f_1 \cdot g_2$$

$$h_3 = f_3 \cdot g_1^3 + 3f_2 \cdot g_1 \cdot g_2 + f_1 \cdot g_3$$

Faà di Bruno's formula:

$$h_n = \sum_{(k) \vdash n, m} \frac{n!}{k_1! \cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_n}{n!}\right)^{k_n} f_m,$$

with $m = 1, \dots, n$ and $k_1 + k_2 + \cdots + k_n = m$.





Node polynomials

Let $\pi: F \rightarrow Y$ be a family of smooth projective surfaces and $D \subset F$ a relative divisor.

Set $Y_r := \{y \in Y \mid D_y \text{ has } r \text{ nodes}\}$.

The expected codimension of Y_r in Y is r .

- ▶ Problem: determine the class of the cycle $[\overline{Y}_r]$ in the Chow group $A^r Y$.
- ▶ $\overline{Y}_1$ = the discriminant of the family of curves $D \rightarrow Y$, and $[\overline{Y}_1] = \pi_*(\text{polynomial in the Chern classes of the family})$.
- ▶ To find a formula for general r , proceed by recursion: resolve one node at a time, and reduce a family of r -nodal curves to a family of $(r-1)$ -nodal curves with “known” Chern classes.



Theorem (Kleiman–P.)

If the family of curves $D \subset F \rightarrow Y$ is dimensionally general, then $\overline{Y}_r$ is the support of a natural nonnegative cycle U_r .

Moreover, for $r \leq 8$, the class $u_r := [U_r]$ is given by

$$u_r = P_r(a_1, \dots, a_r)/r!$$

where $a_i = \pi_* b_i$, and the b_i are polynomials in the Chern classes of D and F/Y and are output by a certain algorithm.

Why do the Bell polynomials appear?

- ▶ Let $\tilde{F} \rightarrow F \times_Y F$ be the blowup of the diagonal, with E the exceptional divisor. Compose the blowup with projection to the second factor, and let $\pi': F' \rightarrow X$ denote the restriction to $X \subset F$ consisting of the singular points of the fibers of D .
- ▶ Set $D' := \pi'^*D - 2E$. If $x \in X$ is a node of $D_{\pi(x)}$, then D'_x has one node less than $D_{\pi(x)}$.
- ▶ Hence, if we let u'_i denote the i -nodal classes on $D' \subset F' \rightarrow X$, we expect to get some sort of recursion formula

$$ru_r = \pi_*(u'_{r-1} \cdot [X]).$$

- ▶ We have $u_1 = \pi_*[X]$, and $u'_{r-1} = \pi^*u_{r-1} + z_{r-1}$, where z_{r-1} is some “correction term.”
- ▶ If we set $\partial(u_{r-1}) := \pi_*(z_{r-1} \cdot [X])$, we get a recursive relation, for $r \geq 2$, of the form

$$ru_r = u_{r-1}u_1 + \partial(u_{r-1}).$$

- ▶ Pretend that ∂ behaves like a derivation:

$$u_1 = u_1$$

$$2u_2 = u_1^2 + \partial(u_1)$$

$$3!u_3 = (u_1^2 + \partial(u_1))u_1 + \partial(u_1^2 + \partial(u_1)) = u_1^3 + 3u_1\partial(u_1) + \partial^2(u_1)$$

...

$$r!u_r = P_r(u_1, \dots, \partial^r(u_1)).$$



The real difficulties start for $r \geq 8$. Then the family $F' \rightarrow X$ has non-reduced fibres (coming from points of multiplicity ≥ 4 in the fibers of D) in “too high” dimension. Here we need to use the theory of residual, or excess, intersection — Method 2!



Thom polynomials

There is an alternative approach to determining the classes u_r , based on using multiple point theory for the map $\pi|_X: X \rightarrow Y$. The problem with this approach is that this map is not generic.

- ▶ However, Kazarian observed that it is (generically) a Legendre map, and that therefore it is possible to find formulas based on the knowledge of the Thom polynomials of all relevant singularities (so this can only be done explicitly only “up to” the codimension for which all singularities are classified and their Thom polynomials known).

Diagonals

Consider the fibered r -fold product $X \times_Y \dots \times_Y X$.

- Problem: how many diagonals of each type exists?

Use the following shorthand notation

$$d_i = [i \text{ points are equal}]$$

- For example, $d_1^{r-5} d_2 d_3$ is the class of a diagonal where, for fixed distinct integers i, j, k, l, m , $x_i = x_j$ and $x_k = x_l = x_m$.
- The number of diagonals of class $d_1^{k_1} \dots d_r^{k_r}$ is equal to

$$\frac{r!}{k_1! \dots k_r!} \left(\frac{1}{1!}\right)^{k_1} \dots \left(\frac{1}{r!}\right)^{k_r},$$

the coefficient appearing in the Bell polynomial.



- Kazarian's formula for $r!u_r$ is given as a linear combination of terms of the form

$$S_{A_1^1}^{k_1} \cdots S_{A_r^1}^{k_r}.$$

with $k_1 + 2k_2 + \dots + rk_r = r$.

- The class $S_{A_1^i}$ corresponds to an “excess” or “residual” intersection coming from a diagonal of type d_i . The coefficients in the linear combination are the same as the ones appearing in Faà di Bruno's formula and in the Bell polynomials.



- Kazarian's formula translates to

$$r!u_r = P_r(d_1, -d_2, \dots, (-1)^r d_r),$$

but, unfortunately, the “diagonal approach” does not make the computations any easier.

- For example, d_2 is equal to the equivalence of the diagonal plus twice the class of fibers with A_2 singularities, so we have to determine the latter also. Similarly, for higher d_i , all singularities of codimension i will have to be taken into account.



Tropical geometry

A new approach to the enumeration of (complex) curves on toric surfaces has been given by Mikhalkin, using tropical geometry. The number of curves is given by a count of certain lattice paths in the (Newton) polygon defining the tropical curves. (Also recent work by Gathmann and Markwig.)

- ▶ In the case of plane curves, the number $N_{g,d}$ is equal to the (weighted) number of so-called λ -increasing lattice paths of length $3d - 1 + g$ in the triangle with corners $(0, 0)$, $(d, 0)$, $(0, d)$.
- ▶ This is again an example of a specialization argument reducing the original problem to a combinatorial one!