

# POLARS OF REAL SINGULAR PLANE CURVES

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**Abstract.** Polar varieties have in recent years been used by Bank, Giusti, Heintz, Mbakop, and Pardo, and by Safey El Din and Schost, to find efficient procedures for determining points on all real components of a given non-singular algebraic variety. In this note we review the classical notion of polars and polar varieties, as well as the construction of what we here call reciprocal polar varieties. In particular we consider the case of real affine plane curves, and we give conditions for when the polar varieties of singular curves contain points on all real components.

**Key words.** Polar varieties, Hypersurfaces, Plane curves, Tangent space, Flag.

**AMS(MOS) subject classifications.** Primary 14H50; Secondary 14J70, 14P05, 14Q05.

**1. Introduction.** The general study of polar varieties goes back to Severi, though polars of plane curves were introduced already by Poncelet. Polar varieties of complex, projective, non-singular varieties have later been studied by many, and the theory has been extended to singular varieties (see [6] and references therein). We shall refer to these polar varieties as *classical* polar varieties.

Another kind of polar varieties, which we here will call *reciprocal* polar varieties, were introduced in [3] under the name of dual polar varieties. The definition involves a quadric hypersurface and polarity of linear spaces with respect to this quadric. Classically, the *reciprocal curve* of a plane curve was defined to be the curve consisting of the polar points of the tangent lines of the curve with respect to a given conic. The reciprocal curve is isomorphic to the dual curve in  $(\mathbb{P}^2)^\vee$ , via the isomorphism of  $\mathbb{P}^2$  and  $(\mathbb{P}^2)^\vee$  given by the quadratic form defining the conic.

Bank, Giusti, Heintz, Mbakop, and Pardo have proved [1, 2, 3, 4] that polar varieties of *real, affine* non-singular varieties (with some requirements) contain points on each connected component of the variety, and this property is useful in CAGD for finding a point on each component. Related work has been done by Safey El Din and Schost [7, 8].

We will in this paper determine in which cases the polar varieties of a real affine *singular* curve contain at least one non-singular point of each component of the curve. In the next section we briefly review the definitions of classical and reciprocal polar varieties, and state and partly prove versions of some results from [3] adapted to our situation. The third section treats the case of plane curves; we show that the presence of ordinary multiple points does not affect these results, but that the presence of arbitrary singularities does.

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**2. Polar varieties.** Let  $V \subset \mathbb{P}^n$  be a complex projective variety. Given a hyperplane  $H$  we can consider the affine space  $\mathbb{A}^n := \mathbb{P}^n \setminus H$ , where  $H$  is called the hyperplane at infinity. We define the corresponding affine variety  $S$  to be the variety  $V \cap \mathbb{A}^n$ . We let  $V_{\mathbb{R}}$  and  $S_{\mathbb{R}}$  denote the corresponding real varieties.

If  $L_1$  and  $L_2$  are linear varieties in a projective space  $\mathbb{P}^n$ , we let  $\langle L_1, L_2 \rangle$  denote the linear variety spanned by them. We say that  $L_1$  and  $L_2$  intersect transversally if  $\langle L_1, L_2 \rangle = \mathbb{P}^n$ ; if they do not intersect transversally, we write  $L_1 \not\pitchfork L_2$ .

If  $I_1$  and  $I_2$  are sub-vector spaces of  $\mathbb{A}^n$  (considered as a vector space), we say that  $I_1$  and  $I_2$  intersect transversally if  $I_1 + I_2 = \mathbb{A}^n$ ; if they do not intersect transversally, we write  $I_1 \not\pitchfork I_2$ .

**2.1. Classical polar varieties.** Let us recall the definition of the classical polar varieties (or loci) of a possibly singular variety. Consider a flag of linear varieties in  $\mathbb{P}^n$ ,

$$\mathcal{L} : L_0 \subset L_1 \subset \dots \subset L_{n-1} \subset \mathbb{P}^n$$

and a (non degenerate) variety  $V \subset \mathbb{P}^n$  of codimension  $p$ . Let  $V_{\text{ns}}$  denote the set of nonsingular points of  $V$ . For each point  $P \in V_{\text{ns}}$ , let  $T_P V$  denote the *projective* tangent space to  $V$  at  $P$ . The  $i$ -th polar variety  $W_{L_{i+p-2}}(V)$ ,  $1 \leq i \leq n - p$ , of  $V$  with respect to  $\mathcal{L}$  is the closure of the set

$$\{P \in V_{\text{ns}} \setminus L_{i+p-2} \mid T_P V \not\pitchfork L_{i+p-2}\}.$$

Take  $H = L_{n-1}$  to be the hyperplane at infinity and let  $S = V \cap \mathbb{A}^n$  be the affine part of  $V$ . Then we can define the *affine* polar varieties of  $S$  with respect to the flag  $\mathcal{L}$  as follows: the  $i$ -th affine polar variety  $W_{L_{i+p-2}}(S)$  of  $S$  with respect to  $\mathcal{L}$  is the intersection  $W_{L_{i+p-2}}(V) \cap \mathbb{A}^n$ .

Since  $L_{n-1}$  is the hyperplane at infinity, and all the other elements of the flag  $\mathcal{L}$  are contained in  $L_{n-1}$ , we can look at the affine cone over each element  $L_j$  of the flag, considered as a  $(j+1)$ -dimensional linear sub-vector space  $I_{j+1}$  of  $\mathbb{A}^n$ . Hence we get the flag (of vector spaces)

$$\mathcal{I} : I_1 \subset I_2 \subset \dots \subset I_{n-1} \subset \mathbb{A}^n,$$

and the affine polar variety  $W_{L_{j-1}}(S)$  can also be defined as the closure of the set

$$\{P \in S_{\text{ns}} \mid t_P S \not\pitchfork I_j\},$$

where  $t_P S$  denotes the *affine* tangent space to  $S$  at  $P$  translated to the origin (hence considered as a sub-vector space of  $\mathbb{A}^n$ ). For more details on these two equivalent definitions of affine polar varieties, see [3].

If  $f_1, \dots, f_r$  are homogeneous polynomials in  $n+1$  variables, we let  $\mathcal{V}(f_1, \dots, f_r) \subset \mathbb{P}^n$  denote the corresponding algebraic variety (their common zero set). Similarly, if  $f_1, \dots, f_r$  are polynomials in  $n$  variables, we

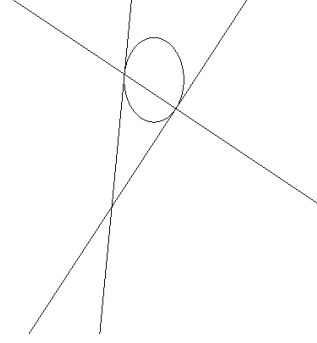


Fig. 1: A conic with two tangents. The polar locus of the conic (with respect to the point of intersection of the two tangents) consists of the two points of tangency. The *polar* of the point is the line through the two points of tangency.

denote by  $\mathcal{V}(f_1, \dots, f_r) \subset \mathbb{A}^n$  the corresponding affine variety. We shall often use the flag

$$\mathcal{L} : \mathcal{V}(X_0, X_2, \dots, X_n) \subset \dots \subset \mathcal{V}(X_0, X_n) \subset \mathcal{V}(X_0) \subset \mathbb{P}^n \quad (2.1)$$

and the corresponding affine flag (considered as vector spaces)

$$\mathcal{I} : \mathcal{V}(X_2, \dots, X_n) \subset \dots \subset \mathcal{V}(X_{n-1}, X_n) \subset \mathcal{V}(X_n) \subset \mathbb{A}^n. \quad (2.2)$$

The following proposition is part of a proposition stated and proved in [2, 2.4, p. 134]. Since this particular result is valid under fewer assumptions, we shall give a different and more intuitive proof. We let  $L$  denote any member of the flag  $\mathcal{L}$ .

**PROPOSITION 2.1.** *Let  $S \subset \mathbb{A}^n$  be a pure  $p$ -codimensional reduced variety. Suppose that  $S_{\mathbb{R}}$  is non-empty, pure  $p$ -codimensional, non-singular, and compact. Then  $W_L(S_{\mathbb{R}})$  contains at least one point of each connected component of  $S_{\mathbb{R}}$ .*

*Proof.* By an affine coordinate change, we may assume that the flag  $\mathcal{I}$  is the flag (2.2). The polar varieties form a sequence of inclusions

$$W_{L_{n-2}}(S_{\mathbb{R}}) \subset W_{L_{n-3}}(S_{\mathbb{R}}) \subset \dots \subset W_{L_{p-1}}(S_{\mathbb{R}}),$$

so it is sufficient to find a point of  $W_{L_{n-2}}(S_{\mathbb{R}})$  on each component of the variety  $S_{\mathbb{R}}$ . We will show that the maximum point for the last coordinate  $X_n$  of each component of  $S_{\mathbb{R}}$  is also a point on  $W_{L_{n-2}}(S_{\mathbb{R}})$ .

The variety  $W_{L_{n-2}}(S_{\mathbb{R}})$  is the closure of the set

$$\{P \in S_{\mathbb{R}} \mid t_P S \not\subset \mathcal{V}(X_n)\}.$$

Let  $C$  be any connected component of  $S_{\mathbb{R}}$ , and let  $A := (a_1, \dots, a_n) \in C$  be a local maximum point for the last coordinate  $X_n$  in  $C$ . Such a maximum point exists since  $C$  is compact. Since  $A$  is a local maximum point for the  $X_n$ -coordinate, the variety  $S_{\mathbb{R}}$  has to flatten out in the  $X_n$ -direction in the point  $A$ , which means that the tangent space  $t_A S$  is contained in the hyperplane  $\mathcal{V}(X_n)$ . To show this, consider a real local parameterization  $S_{\mathbb{R}}$  in a neighborhood of  $A$ . The neighborhood is chosen to be so small that it contains no other local maximum points for the  $X_n$ -coordinate.

Consider a local parameterization

$$(s_1, \dots, s_{n-p}) \longmapsto (X_1(s_1, \dots, s_{n-p}), \dots, X_n(s_1, \dots, s_{n-p})),$$

with

$$A = (X_1(0, \dots, 0), \dots, X_n(0, \dots, 0)).$$

The rows of the matrix

$$\begin{bmatrix} \frac{\partial X_1}{\partial s_1}(0, \dots, 0) & \cdots & \frac{\partial X_n}{\partial s_1}(0, \dots, 0) \\ \vdots & \ddots & \vdots \\ \frac{\partial X_1}{\partial s_{n-p}}(0, \dots, 0) & \cdots & \frac{\partial X_n}{\partial s_{n-p}}(0, \dots, 0) \end{bmatrix}$$

span the tangent space  $t_A S$  of  $S_{\mathbb{R}}$  at  $A$ . We will show that  $\frac{\partial X_n}{\partial s_i}(0, \dots, 0) = 0$  for  $i = 1, \dots, n-p$ . Then we know that  $t_A S$  is contained in the hyperplane  $\mathcal{V}(X_n)$ .

By the definition of derivatives we have that

$$\begin{aligned} \frac{\partial X_n}{\partial s_i}(0, \dots, 0) &= \lim_{s_i \rightarrow 0^-} \frac{X_n(0, \dots, 0, s_i, 0, \dots, 0) - X_n(0, \dots, 0)}{s_i - 0} \\ &= \lim_{s_i \rightarrow 0^+} \frac{X_n(0, \dots, 0, s_i, 0, \dots, 0) - X_n(0, \dots, 0)}{s_i - 0} \end{aligned}$$

If  $s_i$  goes to 0 from below, then  $s_i$  is negative, and  $X_n(0, \dots, 0, s_i, 0, \dots, 0) - X_n(0, \dots, 0) \leq 0$  since  $X_n(0, \dots, 0)$  is the local maximum, so we have

$$\lim_{s_i \rightarrow 0^-} \frac{X_n(0, \dots, 0, s_i, 0, \dots, 0) - X_n(0, \dots, 0)}{s_i - 0} \geq 0$$

When  $s_i$  goes to 0 from above, then  $s_i$  is positive, so

$$\lim_{s_i \rightarrow 0^+} \frac{X_n(0, \dots, 0, s_i, 0, \dots, 0) - X_n(0, \dots, 0)}{s_i - 0} \leq 0$$

Since these limits are equal, they both have to be zero, so  $\frac{\partial X_n}{\partial s_i}(0, \dots, 0) = 0$  for  $i = 1, \dots, n-p$ .

Since  $t_A S$  is contained in  $\mathcal{V}(X_n)$ , we have  $t_A S \not\subset \mathcal{V}(X_n)$ , and hence  $A$  is a point of  $W_{L_{n-2}}(S_{\mathbb{R}})$ .  $\square$

In the proof of this proposition we can replace the phrase “local maximum point of the  $X_n$ -coordinate” with “local minimum point ...”, since

the consideration we did for the tangent space is the same in both cases. So we have found two points of each connected component, unless the local maximum and the local minimum for the last coordinate coincide. The local maximum and the local minimum coincide if and only if the variety is contained in the hyperplane  $\mathcal{V}(X_n)$ , which implies that  $V$  is degenerate.

In the above proposition we did not assume that the coordinates were in generic position with respect to the polynomials generating the variety, so we allow situations where the polar variety can contain a piece of a component of dimension greater than zero.

Safey El Din and Schost [8] show that one can find points on each component of a smooth, not necessarily compact, affine real variety. They use all the polar varieties given by a flag, and intersect each of these varieties with other varieties. The union of these intersections will be zero-dimensional and it contains points from each connected component of the variety we started with.

**2.2. Reciprocal polar varieties.** Let  $Q = \mathcal{V}(q)$  be a non-degenerate hyperquadric defined in  $\mathbb{P}^n$  by a polynomial  $q$ . If  $A$  is a point, its polar hyperplane  $A^\perp$  with respect to  $Q$  is the linear span of the points on  $Q$  such that the tangent hyperplanes for  $Q$  at these point pass through  $A$ . This gives the hyperplane  $\sum \frac{\partial q}{\partial X_i}(A)X_i = 0$ .

If  $H$  is a hyperplane, then its polar point,  $H^\perp$ , is the intersection of the tangent hyperplanes for  $Q$  at the points on  $Q \cap H$ . Finally, if  $L$  is a linear space of dimension  $d$ , its polar space  $L^\perp$  is the intersection of the polar hyperplanes to points in  $L$ . Equivalently,  $L^\perp$  is the linear span of all points  $H^\perp$ , where  $H$  is a hyperplane containing  $L$ . The dimension of  $L^\perp$  is  $n - d - 1$ .

As an example, consider the case  $n = 3$ . If  $A$  is a point in  $\mathbb{P}^3$  then  $A^\perp$  is the plane in  $\mathbb{P}^3$  defined by the polynomial  $\frac{\partial q}{\partial X_0}(A)X_0 + \dots + \frac{\partial q}{\partial X_3}(A)X_3$ . If  $L$  is a hyperplane defined by the polynomial  $b_0X_0 + b_1X_1 + b_2X_2 + b_3X_3$ , then  $L^\perp$  is the point  $A$  such that

$$(b_0 : b_1 : b_2 : b_3) = (\frac{\partial q}{\partial X_0}(A) : \frac{\partial q}{\partial X_1}(A) : \frac{\partial q}{\partial X_2}(A) : \frac{\partial q}{\partial X_3}(A))$$

Finally, if  $L$  is the line spanned by the points  $A$  and  $B$ , then  $L^\perp$  is the intersection of the two hyperplanes  $A^\perp$  and  $B^\perp$ , i. e.,

$$L^\perp = \mathcal{V}(\sum_{i=0}^3 \frac{\partial q}{\partial X_i}(A)X_i) \cap \mathcal{V}(\sum_{i=0}^3 \frac{\partial q}{\partial X_i}(B)X_i)$$

Note that if  $Q$  is defined by the polynomial  $q = \sum_{i=0}^n X_i^2$ , then the polar variety of a point  $(a_0 : \dots : a_n) \in \mathbb{P}^n$  is the hyperplane  $H = \mathcal{V}(\sum_{i=0}^n a_i X_i) \subset \mathbb{P}^n$ .

Let  $\mathcal{L} : L_0 \subset L_1 \subset \dots \subset L_{n-1}$  be a flag in  $\mathbb{P}^n$ , where  $L_{n-1}$  is the plane  $H$  at infinity if we consider the affine space. We then get the polar flag with respect to  $Q$ :

$$\mathcal{L}^\perp : L_{n-1}^\perp \subset L_{n-2}^\perp \subset \dots \subset L_1^\perp \subset L_0^\perp$$

DEFINITION 2.1 (cf. [3, p. 527]). *The  $i$ -th reciprocal polar variety  $W_{L_{i+p-1}}^\perp(V)$ ,  $1 \leq i \leq n-p$ , of a variety  $V$  with respect to the flag  $\mathcal{L}$ , is defined to be the Zariski closure of the set*

$$\{P \in V_{\text{ns}} \setminus L_{i+p-1}^\perp \mid T_P V \not\subset \langle P, L_{i+p-1}^\perp \rangle^\perp\}$$

When  $V$  is a hypersurface in  $\mathbb{P}^n$ , the reciprocal polar variety  $W_{L_{n-1}}^\perp(V)$  is the set  $\text{Cl}\{P \in V_{\text{ns}} \setminus L_{n-1}^\perp \mid T_P V \supset \langle P, L_{n-1}^\perp \rangle^\perp\}$ . In this case,  $\langle P, L_{n-1}^\perp \rangle$  is the line spanned by  $P$  and the point  $L_{n-1}^\perp$ . Since  $A^{\perp\perp} = A$ , and  $A \subseteq B$  implies  $A^\perp \supseteq B^\perp$ , it follows that

$$T_P V \supseteq \langle P, L_{n-1}^\perp \rangle^\perp \Leftrightarrow T_P V^\perp \subseteq \langle P, L_{n-1}^\perp \rangle.$$

The point  $T_P V^\perp$  is on the line  $\langle P, L_{n-1}^\perp \rangle$  if and only if the point  $L_{n-1}^\perp$  is on the line  $\langle P, T_P V^\perp \rangle$ . So, when  $V$  is a hypersurface, the  $(n-1)$ -th reciprocal variety is

$$W_{L_{n-1}}^\perp(V) = \text{Cl}\{P \in V_{\text{ns}} \setminus (\{L_{n-1}^\perp\} \cup H) \mid L_{n-1}^\perp \in \langle P, T_P V^\perp \rangle\}.$$

This way of writing the reciprocal polar variety can sometimes be useful, and it gives a better geometric understanding of the reciprocal polar variety.

Let  $S \subset \mathbb{A}^n = \mathbb{P}^n \setminus H$  denote the affine part of the variety  $V$ , where  $H = L_{n-1}$  is the hyperplane at infinity, and let  $L \in \mathcal{L}$ . We define the *affine* reciprocal polar variety  $W_L^\perp(S)$  to be the affine part  $W_L^\perp(V) \cap \mathbb{A}^n$  of  $W_L^\perp(V)$ . The linear variety  $\langle P, L^\perp \rangle^\perp$  is contained in the hyperplane at infinity, so we can consider the affine cone of  $\langle P, L^\perp \rangle^\perp$  as a linear variety  $I_{P, L^\perp}$  in the affine space. Then the affine reciprocal polar variety can be written as

$$W_L^\perp(S) = \text{Cl}\{P \in S_{\text{ns}} \setminus L^\perp \mid t_P S \not\subset I_{P, L^\perp}\}$$

where  $t_P S$  is the tangent space at  $P$ , moved to the origin.

The following result, formulated slightly differently, is proved in [3].

PROPOSITION 2.2 ([3, p. 529]). *Let  $S_{\mathbb{R}} \subset \mathbb{A}_{\mathbb{R}}^n$  be a non-empty, non-singular real variety of pure codimension  $p$ . Let  $\mathcal{L}$  be a flag in  $\mathbb{P}^n$ , where  $L_{n-1} = H = \mathbb{P}^n \setminus \mathbb{A}^n$  is the hyperplane at infinity. Assume  $Q = \mathcal{V}(q)$ , where  $q$  restricts to a positive definite quadratic form on  $\mathbb{A}_{\mathbb{R}}^n$ . Assume  $H^\perp \notin S_{\mathbb{R}}$ . Let  $L$  be any member of the flag  $\mathcal{L}$  with  $\dim L \geq p$ . Then the real affine reciprocal polar variety  $W_L^\perp(S_{\mathbb{R}})$  contains at least one point from each connected component of  $S_{\mathbb{R}}$ .*

In Proposition 2.2 we had to assume that  $H^\perp \notin S_{\mathbb{R}}$  in order to prove that  $W_H^\perp(S_{\mathbb{R}})$  contains a point from each component of  $S_{\mathbb{R}}$ . The following

proposition states that if the variety is a hypersurface and contains  $H^\perp$ , then we can choose another quadric  $Q'$ , so that the polar point  $H^{\perp'}$  with respect to this quadric is not on  $S_{\mathbb{R}}$ , and we will thus still be able to find points on each component of the hypersurface.

**PROPOSITION 2.3.** *Let  $V \subset \mathbb{P}^n$  be a hypersurface and  $H$  a hyperplane, and set  $\mathbb{A}^n = \mathbb{P}^n \setminus H$  and  $S = V \cap \mathbb{A}^n$ . Assume  $S_{\mathbb{R}}$  is non-empty and non-singular. Given any point  $A \in \mathbb{A}_{\mathbb{R}}^n \setminus S_{\mathbb{R}}$ , there exists a quadric  $Q'$  such that the polar point  $H^{\perp'}$  with respect to  $Q'$  is equal to  $A$ , and such that the real affine reciprocal polar variety  $W_H^{\perp'}(S_{\mathbb{R}})$  contains at least one point from each connected component of  $S_{\mathbb{R}}$ .*

*Proof.* By Proposition 2.2 it suffices to show that we can find  $Q' = \mathcal{V}(q')$  such that  $H^{\perp'} = A = (1 : a_1 : \dots : a_n)$  and such that  $q'$  restricts to a positive definite form on  $\mathbb{A}^n$ . Take  $q' = (1 + \sum_{i=1}^n a_i^2)X_0^2 - 2 \sum_{i=1}^n a_i X_0 X_i + \sum_{i=1}^n X_i^2$ . Then  $\frac{\partial q'}{\partial X_0}(A) = 2$ , and  $\frac{\partial q'}{\partial X_i}(A) = 0$  for  $i = 1, \dots, n$ . The restriction of  $q'$  to  $\mathbb{A}^n$  is  $\sum_{i=1}^n X_i^2$ .  $\square$

The next proposition provides an explicit description of the affine reciprocal polar variety of a hypersurface  $S \subset \mathbb{A}^n$ . A similar result is proved in the more general case of a complete intersection variety in [3, 3.1, p. 531].

**PROPOSITION 2.4.** *Let  $V = \mathcal{V}(f) \subset \mathbb{P}^n$  be a real hypersurface, and let  $Q = \mathcal{V}(q)$  be defined by an irreducible quadratic polynomial  $q$ . We consider the affine space  $\mathbb{A}^n = \mathbb{P}^n \setminus H$ , where  $H = \mathcal{V}(X_0)$ . Then the affine reciprocal polar variety  $W_H^\perp(S)$  of  $S = V \cap \mathbb{A}^n$  with respect to  $Q$  is equal to the closure of the intersection of  $S_{\text{ns}} \setminus \{H^\perp\}$  with the variety defined by the 2-minors of the matrix*

$$\begin{pmatrix} \frac{\partial f}{\partial X_1} & \cdots & \frac{\partial f}{\partial X_n} \\ \frac{\partial q}{\partial X_1} & \cdots & \frac{\partial q}{\partial X_n} \end{pmatrix}.$$

*Proof.* The reciprocal variety  $W_H^\perp(S)$  is the closure of the set

$$\{P \in S_{\text{ns}} \setminus H^\perp \mid I_{P, H^\perp} \not\subset t_P S\}.$$

We want to show that for  $P = (1 : p_1 : \dots : p_n) \in S$  the affine cone  $I_{P, H^\perp}$  over  $\langle P, H^\perp \rangle^\perp$  intersects the tangent space  $t_P S$  non-transversally if and only if

$$\text{rank} \begin{pmatrix} \frac{\partial f}{\partial X_1}(P) & \cdots & \frac{\partial f}{\partial X_n}(P) \\ \frac{\partial q}{\partial X_1}(P) & \cdots & \frac{\partial q}{\partial X_n}(P) \end{pmatrix} \leq 1.$$

Note that we have

$$\langle P, H^\perp \rangle^\perp = P^\perp \cap H,$$

so, since  $P^\perp = \mathcal{V}(\sum_{i=0}^n \frac{\partial q}{\partial X_i}(P)X_i)$  and  $H = \mathcal{V}(X_0)$ , we find

$$\langle P, H^\perp \rangle^\perp = \{(0 : X_1 : \dots : X_n) \mid \sum_{i=1}^n \frac{\partial q}{\partial X_i}(P)X_i = 0\}$$

The affine cone  $I_{P, H^\perp}$  is the hyperplane  $\mathcal{V}(\sum_{i=1}^n \frac{\partial q}{\partial X_i}(P)X_i)$ . The affine tangent space  $t_P S$  is the hyperplane given by  $\sum_{i=1}^n \frac{\partial f}{\partial X_i}(P)X_i = 0$ , which implies what we wanted to prove.  $\square$

Note that the square of the distance function in the affine space  $\mathbb{A}^n$  is given by a quadratic polynomial, so if we let  $Q = \mathcal{V}(q)$  be a quadric such that  $q$  restricts to this polynomial, we see that the variety defined in Theorem 2 in [7] is nothing but the affine reciprocal polar variety with respect to  $Q$ . (This is because elimination of the extra variable gives the 2-minors of the matrix in Proposition 2.4.) Hence [7, Thm. 2] follows from Proposition 2.2.

**3. Polar varieties of real singular curves.** In this section our varieties will be curves in  $\mathbb{P}^2$ , so the flags will in this case be of the following form

$$\mathcal{L} : L_0 \subset L_1 \subset \mathbb{P}^2 \text{ and } \mathcal{L}^\perp : L_1^\perp \subset L_0^\perp \subset \mathbb{P}^2$$

where  $L_0$  is a point, and  $L_1$  is the line at infinity, when we look at the affine case. This gives only one interesting classical polar variety and reciprocal polar variety for each choice of flag, namely  $W_{L_0}(V)$  and  $W_{L_1}^\perp(V)$ .

Let  $V = \mathcal{V}(f) \subset \mathbb{P}^2$  be a plane curve. The *polar* of  $V$  with respect to a point  $A = (a_0 : a_1 : a_2)$  is the curve  $V' = \mathcal{V}(\sum_{i=0}^2 a_i \frac{\partial f}{\partial X_i})$ . It follows from the definition that the polar variety of  $V$  with respect to  $A$  is equal to

$$W_A(V) = V_{\text{ns}} \cap V'.$$

Given a conic  $Q = \mathcal{V}(q)$  and a line  $L$ , the *reciprocal polar* of the affine curve  $S = V \cap \mathbb{A}^2 \subset \mathbb{A}^2 = \mathbb{P}^2 \setminus L$ , is the curve  $S'' = \mathcal{V}(\frac{\partial f}{\partial X_1} \frac{\partial q}{\partial X_2} - \frac{\partial f}{\partial X_2} \frac{\partial q}{\partial X_1})$ , where  $f$  and  $q$  are dehomogenized by setting  $X_0 = 1$ . The affine reciprocal polar variety of  $S$  is equal to

$$W_L^\perp(S) = S_{\text{ns}} \cap S''.$$

More generally, we define the *reciprocal polar* of the curve  $V \subset \mathbb{P}^2$  with respect to  $Q$  and a point  $A = (a_0 : a_1 : a_2)$  to be the curve  $V'' = \mathcal{V}(\det(f, q, A))$ , where  $\det(f, q, A)$  is the determinant of the matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ \frac{\partial f}{\partial X_0} & \frac{\partial f}{\partial X_1} & \frac{\partial f}{\partial X_2} \\ \frac{\partial q}{\partial X_0} & \frac{\partial q}{\partial X_1} & \frac{\partial q}{\partial X_2} \end{bmatrix}.$$

The reciprocal polar of the affine curve  $S$  is then the affine part of the reciprocal polar with respect to the origin  $(1 : 0 : 0)$ .

Note that the reciprocal polars form a linear system on  $\mathbb{P}^2$  of degree  $d$ , whereas the classical polars form a linear system of degree  $d-1$ . A classical polar variety consists of at most  $d(d-1)$  points, whereas a reciprocal polar variety can have as many as  $d^2$  points.

In the following sections we will consider affine curves with singularities, and we determine in which cases the classical polar variety or the reciprocal polar variety will contain non-singular points of each connected component of the real part of the curve.

**3.1. Classical polar varieties of real singular affine curves.** The classical polar variety of an affine curve  $S \subset \mathbb{A}^2$  associated to a given flag  $\mathcal{L}$  is the set

$$W_{L_0}(S) = \{P \in S_{\text{ns}} \mid I_1 = t_P S\},$$

where  $I_1$  is the affine line equal to the cone over the point  $L_0$ . By an *ordinary real multiple point* we shall mean a singular point with at least two real branches, such that the tangent lines intersect pairwise transversally. If the curve  $S_{\mathbb{R}}$  only has ordinary real multiple points as singularities, we have the following proposition.

**PROPOSITION 3.1.** *Suppose  $S_{\mathbb{R}}$  is non-empty and compact and has only ordinary real multiple points as singularities. Then  $W_{L_0}(S_{\mathbb{R}})$  contains at least one non-singular point of each connected component.*

*Proof.* We may assume that  $I_1 = \mathcal{V}(X_2)$ . Let  $C$  be a connected component of  $S_{\mathbb{R}}$ . The component  $C$  has a local maximum point for the coordinate  $X_2$ , since  $C$  is compact. If this point is a non-singular point, then we know by the proof of Proposition 2.1 that it is contained in the real polar variety. Assume on the contrary that the maximum point  $P$  is a singular point, hence an ordinary real multiple point. If  $P = (p_1, p_2)$  is a local maximum point for the  $X_2$ -coordinate then each of the real branches through  $P$  has  $p_2$  as a local maximum for the  $X_2$ -coordinate, so the line  $\mathcal{V}(X_2 - p_2)$  is a tangent line for each branch. This means that the branches have a common tangent line, hence they do not intersect transversally.  $\square$

When it comes to singularities other than ordinary real multiple points, we cannot say whether the singularity can be a maximum point for the  $X_2$ -coordinate. But we know that the local maximum and the local minimum points are on the real polar  $S'_{\mathbb{R}}$ . One point on a component cannot be both a minimum and a maximum unless the component is a line, which is not compact, and therefore excluded. If the singularity is a local maximum point, we know that the minimum point is also on the polar. So we can allow each real component of the curve to have one additional singularity which is not an ordinary real multiple point, and the above result will still remain valid.

For curves with arbitrary singularities, clearly we can have a situation where the points with the maximum and the minimum values for the last coordinate are both singular, with the result that the conclusion of Proposition 3.1 will not be valid for the given choice of flag  $I_1$ . One could ask whether it is possible to choose a different affine flag (different affine coordinates) so that the results still holds.

**PROPOSITION 3.2.** *There exists an affine singular plane curve  $S_{\mathbb{R}}$  such that for no choice of flag  $I_1$  does the polar variety contain a non-singular point from each component.*

*Proof.* We prove this by giving an example of such a curve. Since cusps disturb the continuity of the curvature of a curve, it is natural to look for examples among curves with cusps. So we want to find a curve with at least two components, where each component has cusps. One way to construct such a curve is to consider two, not necessarily irreducible, affine curves  $\mathcal{V}(f)$  and  $\mathcal{V}(g)$ , and then look at the curve  $\mathcal{V}(h)$ , where  $h = f^2 + \epsilon g^3$ , which will have cusps at the points of intersection between  $\mathcal{V}(f)$  and  $\mathcal{V}(g)$ . We let  $\mathcal{V}(f)$  be the union of two disjoint circles, and  $\mathcal{V}(g)$  the union of four lines, where two of the lines intersect one of the circles twice, and the other two lines intersect the other circle twice. We can take

$$f = (X_1^2 + X_2^2 - 1)((X_1 - 4)^2 + (X_2 - 2)^2 - 1)$$

and

$$g = (X_2 - \frac{1}{2})(X_2 + \frac{1}{2})(X_1 - \frac{7}{2})(X_1 - \frac{9}{2}).$$

The curve  $\mathcal{V}(h) = \mathcal{V}(f^2 + \frac{1}{100}g^3)$  has then four compact components with two cusps on each component.

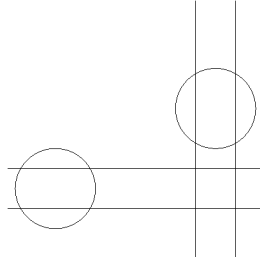


Fig. 2: The curves  $\mathcal{V}(f)$  and  $\mathcal{V}(g)$ .

A non-singular point  $P$  is on the affine polar variety with respect to a line  $I = \mathcal{V}(aX_1 + bX_2)$  if the tangent line  $\mathcal{V}(\frac{\partial f}{\partial X_1}(P)X_1 + \frac{\partial f}{\partial X_2}(P)X_2)$  is equal to  $I$ . Consider the Gauss map  $\gamma: S_{\mathbb{R}} \rightarrow \mathbb{P}_{\mathbb{R}}^1$  given by

$$P \mapsto (\frac{\partial f}{\partial X_1}(P) : \frac{\partial f}{\partial X_2}(P)).$$

Let  $C_i$ ,  $i = 1, \dots, 4$ , denote the connected components of  $S_{\mathbb{R}}$ . Let us show that  $\cap_{i=1}^4 \gamma(C_i) = \emptyset$ .



Fig. 3: The curve  $\mathcal{V}(f^2 + \frac{1}{100}g^3)$ .

The two lower components have cusps at the points  $(-\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$  and  $(\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$ , and the map  $\gamma$  sends these points to the lines  $\mathcal{V}(3X_1 \mp \sqrt{3}X_2)$  and  $\mathcal{V}(3X_1 \pm \sqrt{3}X_2)$  respectively. Each of these components have points that are mapped to the line  $\mathcal{V}(X_1)$  by  $\gamma$ . The components do not have any inflection points, so the image of the map  $\gamma$  varies continuously from  $\mathcal{V}(X_1)$  to  $\mathcal{V}(3X_1 + \sqrt{3}X_2)$  and  $\mathcal{V}(3X_1 - \sqrt{3}X_2)$ , hence the other tangent lines lie in the sector between the two lines  $\mathcal{V}(3X_1 + \sqrt{3}X_2)$  and  $\mathcal{V}(3X_1 - \sqrt{3}X_2)$ .

The same happens if we calculate the tangent lines at the cusps for the two upper components. The tangent lines of these components have to be in the sector between  $\mathcal{V}(3X_2 + \sqrt{3}X_1)$  and  $\mathcal{V}(3X_2 - \sqrt{3}X_1)$ , which contains the line  $\mathcal{V}(X_2)$ . These two sectors do not intersect except at the origin, so for any choice of line  $I = \mathcal{V}(aX_1 + bX_2)$  the set  $(\gamma^{-1}(I)) \cap C$  is empty or consists of points of the two lower components or points of the two upper components.

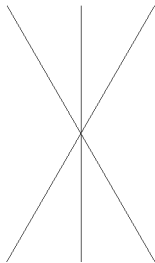


Fig. 4: The lines  $\mathcal{V}(3X_1 + \sqrt{3}X_2)$ ,  $\mathcal{V}(3X_1 - \sqrt{3}X_2)$ , and  $\mathcal{V}(X_1)$ .

□

### 3.2. Reciprocal polar varieties of affine real singular curves.

As for the classical polar varieties we will here determine whether Proposition 2.2 is valid for affine curves with singularities. We can easily prove the following for curves with ordinary multiple points.

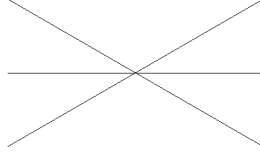


Fig. 5: The lines  $\mathcal{V}(3X_2 + \sqrt{3}X_1)$ ,  $\mathcal{V}(3X_2 - \sqrt{3}X_1)$ , and  $\mathcal{V}(X_2)$ .

**PROPOSITION 3.3.** *Let  $S \subset \mathbb{A}^2 = \mathbb{P}^2 \setminus L$  be an affine plane curve. Suppose that  $S_{\mathbb{R}}$  is non-empty and has no other singularities than ordinary real multiple points. Further, let  $Q$  be defined by a polynomial  $q$  which restricts to a positive definite quadratic form on  $\mathbb{A}_{\mathbb{R}}^2$ . Assume that  $L^{\perp}$  is not contained in  $S_{\mathbb{R}}$ . Then the real affine reciprocal polar variety  $W_L^{\perp}(S_{\mathbb{R}})$  contains at least one non-singular point from each connected component of  $S_{\mathbb{R}}$ .*

*Proof.* Note that the hypotheses imply that  $L^{\perp} \in \mathbb{A}^2 = \mathbb{P}^2 \setminus L$ . Moreover, the restriction of the quadratic polynomial  $q$  to  $\mathbb{A}^2$  defines a distance function on  $\mathbb{A}_{\mathbb{R}}^2$ . The proof of Proposition 2.2 given in [3] consists, in this case, of showing that for each component, a point with the shortest distance to the point  $L^{\perp}$  is a point on the reciprocal polar variety. So, we must show that if the component contains ordinary real multiple points, then these points can not be among the points with the shortest distance to the point  $L^{\perp}$ .

Assume on the contrary that, for a given component  $C$  of the curve  $S_{\mathbb{R}}$ , there is an ordinary real multiple point  $P$  which has the shortest distance to the point  $L^{\perp}$ . The conic with centre in  $L^{\perp}$  and radius  $\text{dist}(L^{\perp}, P)$  will by our assumption be tangent to the component at the point  $P$ . Since  $P$  is an ordinary real multiple point, there is at least one other real branch, and this branch intersects the conic transversally. Hence this branch contains points inside the conic, and these points are closer to  $L^{\perp}$  than  $P$ .  $\square$

As in the case of classical polar varieties we cannot prove the above proposition for curves with arbitrary singularities, since the situation will depend on the type of singularities and how the singularities are placed on the component.

**4. Examples.** In this section we shall look at some examples of singular plane affine curves and their polars and reciprocal polars, thus illustrating the propositions in the previous sections. We use SURF [5] to draw the curves.

EXAMPLE 1. Consider the real affine curve  $S_1$  of degree 6 defined by the polynomial

$$f_1 := X_1^6 + 3X_1^4X_2^2 - 12X_1^4X_2 + 7X_1^4 + 3X_1^2X_2^4 - 24X_1^2X_2^3 + 66X_1^2X_2^2 - 132X_1^2X_2 + 136X_1^2 + X_2^6 - 12X_2^5 + 59X_2^4 - 132X_2^3 + 84X_2^2 + 144X_2 - 143.$$

This curve is compact and smooth, and it has three connected components.

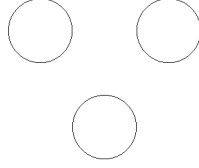


Fig. 6: The curve  $S_1$ .

We let  $\mathcal{L}$  be the flag

$$L_0 = \mathcal{V}(X_0, X_1) \subset L_1 = \mathcal{V}(X_0).$$

The real polar variety is the set

$$W_{L_0}((S_1)_{\mathbb{R}}) = \{P \in (S_1)_{\mathbb{R}} \mid \frac{\partial f_1}{\partial X_2}(P) = 0\},$$

which is equal to intersection of  $(S_1)_{\mathbb{R}}$  and its polar  $(S'_1)_{\mathbb{R}} = \mathcal{V}(\frac{\partial f_1}{\partial X_2})_{\mathbb{R}}$ . We see that the polar variety contains points from each connected component and that the points of the affine polar variety are exactly those points on each component which give maximal and minimal values for the  $X_1$ -coordinate.

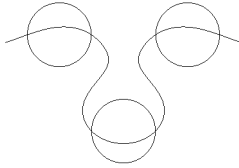


Fig. 7: The curve  $S_1$  and its polar  $S'_1$ .

Let  $Q$  be the standard quadric, given by  $q = \sum_{i=0}^2 X_i = 0$ . The real affine reciprocal polar variety consists of the real points of the intersection between  $S_1$  and its reciprocal polar  $S''_1 = \mathcal{V}(X_2 \frac{\partial f_1}{\partial X_1} - X_1 \frac{\partial f_1}{\partial X_2})$ , and we

know from Proposition 2.2 that also the real affine reciprocal polar variety contains points from each connected component. The reciprocal polar variety consists of the points on each component with the locally shortest or longest Euclidean distance to the origin.

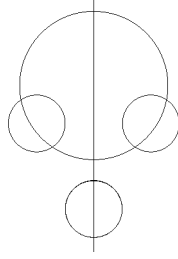


Fig. 8: The curve  $S_1$  and its reciprocal polar  $S_1''$ .

EXAMPLE 2. To illustrate Proposition 3.1 we consider an affine irreducible curve  $S_2$  with two compact components and with one ordinary double point on each of the components. The curve is given by the polynomial

$$f_2 = ((X_1 + 2)X_2 - (X_1 + 2)^6 - X_2^6)(X_1X_2 - X_1^6 - X_2^6) + \frac{1}{100}X_2^6.$$

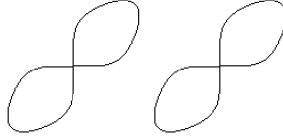


Fig. 9: The curve  $S_2$ .

We let  $\mathcal{L}$  be as in the example above, and we see that the affine polar variety contains non-singular points from each of the components of the curve.

EXAMPLE 3. Consider the irreducible curve  $S_3 = \mathcal{V}(f_3)$ , where

$$f_3 = \mathcal{V}(144 - 24X_2^2 - 88X_1^2 + X_2^4 - X_1^6 + 17X_1^4 - 14X_2^2X_1^2 + \frac{1}{100}X_2^6).$$

The real part of the curve consists of two non-compact components with one ordinary double point on each component.

The reciprocal polar, with respect to the standard flag and standard quadric,  $S_3'' = \mathcal{V}(X_2 \frac{\partial f_3}{\partial X_1} - X_1 \frac{\partial f_3}{\partial X_2})$ , intersects each component in non-singular points, since the double points are not among the points on each component with the locally shortest or longest distance from the origin.

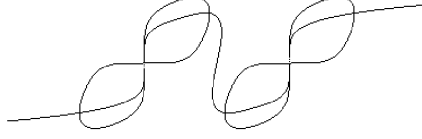


Fig. 10: The curve  $S_2$  and its polar  $S'_2$ .

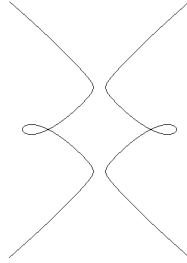


Fig. 11: The curve  $S_3$ .

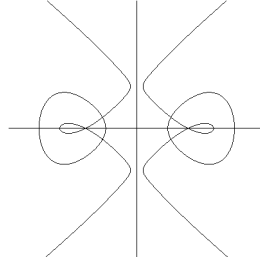


Fig. 12: The curve  $S_3$  and its reciprocal polar  $S''_3$ .

EXAMPLE 4. This example illustrates Proposition 2.3. We start with the irreducible affine curve  $S_4$  defined by the polynomial

$$f_4 := X_1^2 - X_2(X_2 + 1)(X_2 + 2).$$

This curve passes through the origin, so the reciprocal polar variety, when  $Q$  is the standard quadric and  $L = \mathcal{V}(X_0)$  is the line at infinity, will not contain any points from the connected component containing the origin, since we are not counting points which are on both the variety and the flag. Instead we choose the point  $(1, 0)$  (or  $(1 : 1 : 0)$  in projective coordinates), and we must find a polynomial  $q'$  such that  $(1 : 1 : 0)^{\perp'}$  is the line  $\mathcal{V}(X_0)$ . We see that the polynomial  $2X_0^2 - 2X_0X_1 + X_1^2 + X_2^2$  will do, and we will now find the reciprocal polar variety  $W_L^{\perp'}(S_4)$ . If  $P = (1 : p_1 : p_2)$ , the point  $\langle P, L^{\perp'} \rangle^{\perp'} = P^{\perp'} \cap L$  is the point  $(0 : p_2 : 1 - p_1)$ , so the affine

cone over it,  $I_{P,L^\perp'}$ , is the line  $\mathcal{V}((p_1 - 1)X_1 + p_2X_2)$ . The reciprocal polar variety  $W_L^{\perp'}(S_4)$  is the set

$$\{P \in (S_4)_{\mathbb{R}} \mid p_2 \frac{\partial f_4}{\partial X_1}(P) + (1 - p_1) \frac{\partial f_4}{\partial X_2}(P) = 0\};$$

this set consists of the points on  $(S_4)_{\mathbb{R}}$  with the locally shortest or longest Euclidean distance to the point  $(1, 0)$ , and it contains points from each component of  $(S_4)_{\mathbb{R}}$ .

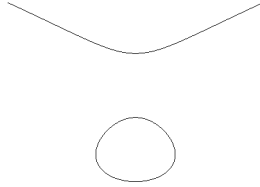


Fig. 13: The curve  $S_4$ .

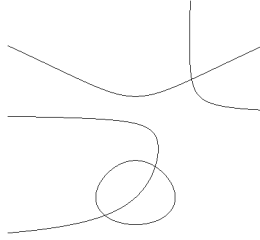


Fig. 14: The curve  $S_4$  and its reciprocal polar  $S_4''$ .

EXAMPLE 5. This is an example showing that Proposition 3.3 does not hold for curves with arbitrary singularities. Consider the curve  $S_5$  given by the polynomial

$$f_5 = ((X_1 - 4)^2 + (X_2 - 2)^2 - 1)^2 + \frac{1}{100}((X_1 - \frac{7}{2})(X_1 - \frac{9}{2}))^3,$$

The real components of this curve do not contain points on the reciprocal polar variety other than the four cusps. This can be seen by calculating the intersection points between  $S_5$  and its reciprocal polar  $S_5'' = \mathcal{V}(X_1 \frac{\partial f_5}{\partial X_2} - X_2 \frac{\partial f_5}{\partial X_1})$ .

## REFERENCES



Fig. 15: The curve  $S_5$ .

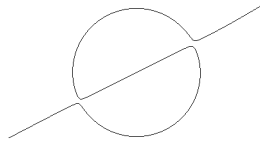


Fig. 16: The reciprocal polar  $S''_5$ .

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