Enriques diagrams and equisingular strata of families of curves

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Planar curve singularities

A planar curve singularity \((C, 0)\), given by \(f(x, y) = 0\), and its normalization \(n: C' \to C\).

Many numerical invariants associated to \((C, 0)\):

- **multiplicity** \(m\): \(f \in \mathfrak{m}^m, f \not\in \mathfrak{m}^{m+1}\)
- **delta invariant** \(\delta = \dim n_*\mathcal{O}_{C'}/\mathcal{O}_C\)
- **Milnor number** \(\mu = \dim k[[x, y]]/(f_x, f_y)\)
- **Tjurina number** \(\tau = \dim k[[x, y]]/(f, f_x, f_y)\)
- **number of branches** \(r = \#n^{-1}(0)\)

Milnor’s formula: \(\mu = 2\delta - r + 1\)
Resolution by blowing ups

A singular point on a curve on a smooth surface: \( 0 \in C \subset S \)

By a succession of blow-ups we get a \textit{good embedded resolution} \( \nu: S' \to S \): the strict transform \( C' \) is smooth and \( \nu^{-1}(0)_{\text{red}} = C' \cup \bigcup_i E_i \) is a normal crossing divisor.

The resolution can be encoded in a diagram, the \textit{Enriques diagram}.

An ordinary cusp
\[ f(x, y) = y^2 - x^3 \]
\( \nu^{-1}(0)_{\text{red}} = C' \cup E_1 \cup E_2 \cup E_3 \),
where \( E_3 \) intersects each of \( C' \), \( E_1 \) and \( E_2 \) in one point.
No other intersections.
Enriques diagrams

An *Enriques diagram* is a finite directed graph with no loops, and with assigned weights to the vertices.

There are three types of vertices: roots, free vertices and satellites.

![Enriques diagram with weights](image)
Figure: The Enriques diagram $\mathbf{M}_{m,p}$, with $m \geq p = 5$. For $m = 5$, $f(x,y) = y^5 - x^6$. 
Figure: The Enriques diagram corresponding to the Fibonacci singularity $f(x, y) = y^8 - x^{13}$.
Numerical characters

Let $\mathbf{D}$ be a diagram with one root $R$. Define

- $\delta(\mathbf{D}) := \sum_{V \in \mathbf{D}} \binom{m_V}{2}$
- $r(\mathbf{D}) := \sum_{V \in \mathbf{D}} (m_V - \sum_{W \text{ prox } V} m_W)$
- $\mu(\mathbf{D}) := 2\delta(\mathbf{D}) - r(\mathbf{D}) + 1$

Define for any $\mathbf{D}$

- $\text{dim}(\mathbf{D}) := \text{rts}(\mathbf{D}) + \text{frs}(\mathbf{D})$
- $\text{deg}(\mathbf{D}) := \sum_{V \in \mathbf{D}} \binom{m_V + 1}{2}$
- $\text{cod}(\mathbf{D}) := \text{deg}(\mathbf{D}) - \text{dim}(\mathbf{D})$
The $A_2$ singularity (ordinary cusp)

\[ \delta(D) = 1 \]
\[ r(D) = 2 - 1 - 1 + 1 - 1 + 1 = 1 \]
\[ \mu(D) = 2 \]
\[ \text{dim}(D) = 1 + 2 = 3 \]
\[ \text{deg}(D) = \binom{3}{2} + \binom{2}{2} + \binom{2}{2} = 5 \]
\[ \text{cod}(D) = 5 - 3 = 2. \]
Complete ideals

Let $\nu: S' \to S$ be a good embedded resolution of $0 \in C \subset S$. Let $D$ be the associated Enriques diagram. Each vertex $V \in D$ corresponds to an infinitely near point of 0. Set $E := \sum_{V \in D} m_V E_V$, where $E_V$ is the total transform of the exceptional divisor coming from blowing up the point corresponding to $V$. The ideal $I := \nu_* \mathcal{O}_{S'}(-E)$ is complete (integrally closed).

Enriques–Hoskin–Deligne–Casas: $\dim \mathcal{O}_S/I = \deg(D)$

The diagram $D$ can be recovered from $I$. 
Hilbert schemes

Let $D$ be a diagram, $d := \deg(D)$. Set

$$H(D) := \{ \mathcal{I} \subset \mathcal{O}_S \mid \mathcal{I} \text{ has diagram } D \} \subset \text{Hilb}_S^d$$

**Proposition**

$H(D)$ is locally closed, smooth and irreducible, of dimension $\dim(D)$.

**Example**

$\dim H(A_1) = 2$, in fact $H(A_1) \cong S$. 
Deformation space

Let \((C, 0)\) be a singularity, given by \(f(x, y) = 0\), with diagram \(\mathbf{D}\). The tangent space to the versal deformation space is \(B = \mathcal{O}_{C, 0}/(f, f_x, f_y)\).

\(\dim B = \tau\), the Tjurina number. Note: \(\tau \leq \mu\), and equality holds for quasi-homogeneous singularities.

Let \(B_{\text{es}} \subset B\) denote the (topological) equisingular locus. We have \(\text{cod}(B_{\text{es}}, B) = \text{cod}(\mathbf{D})\).

Example

If \(f(x, y) = xy(x^2 - y^2)\) is an ordinary quadruple point, then \(\tau = \mu = 9\) and \(\text{cod}(B_{\text{es}}, B) = \dim B - \dim B_{\text{es}} = 9 - 1 = 8 = \text{cod}(\mathbf{D}) = \deg(\mathbf{D}) - \dim(\mathbf{D}) = 10 - 2\).
Arbitrarily near points

Let $\pi : F \to Y$ be a family of surfaces, $F' \to F \times_Y F$ the blow up of the diagonal, and $\pi' : F' \to F$ the composition of the blow up with the second projection.

Define recursively $\pi^{(i)} : F^{(i)} \to F^{(i-1)}$ in the same way. Let $E^{(i)}$ denote the exceptional divisor.

Theorem

Let $D$ be an ordered (unweighted) Enriques diagram on $n + 1$ vertices. The functor of sequences of arbitrarily near $T$-points of $F/Y$ with diagram $D$ is represented by a subscheme $F(D) \subset F^{(n)}$, which is $Y$-smooth with irreducible geometric fibers of dimension $\dim D$. 
Relative Hilbert schemes

Let $D$ be an Enriques diagram with $\text{deg}(D) = d$. As in the case of a single surface $S$, define $H(D) \subset \text{Hilb}^d_{F/Y}$.

**Theorem**

*There is a natural bijective map*

$$
\Psi : Q(D) := F(D)/\text{Aut}(D) \rightarrow H(D),
$$

*which is an isomorphism in characteristic 0.*

The map $\Psi$ can be purely inseparable in characteristic $p > 0$, e.g. for $D = \mathcal{M}_{m,p}$ (Tyomkin).
The equisingular stratification

Let $D \subset F \to Y$ be a family of curves on a family of surfaces.

For a given Enriques diagram $\mathcal{D}$ with $\deg(\mathcal{D}) = d$, let $Y(\mathcal{D})$ parameterize the fibers of $D$ that have singularities of type $\mathcal{D}$. The expected codimension of $Y(\mathcal{D})$ is $\text{cod}(\mathcal{D})$.

**Problem:** Determine the class of the cycle $[Y(\mathcal{D})]$ in terms of the Chern classes of $F/Y$ and $D$.

To define the cycle, we set

$$G(\mathcal{D}) := \text{Hilb}_{D/Y}^d \times_{\text{Hilb}_{F/Y}^d} H(\mathcal{D})$$

and define $Y(\mathcal{D})$ as the image of the map $G(\mathcal{D}) \to Y$. 
Applications to curve counting

A family $D \subset F/Y$ is said to be $r$-generic if for every $D$ and every $y \in Y(D)$, we have

$$\text{cod}(Y(D), Y) \leq \min\{r + 1, \text{cod}(D)\}$$

Construct a derived family $D' \subset F'/Y'$, where $Y' = \text{the set of points of } D \text{ that are singular in a fibre of } D/Y$, a fiber of $F' \to Y'$ is the blowup of a fiber of $F/Y$ at a point of $Y'$, and $D' = D - 2E$.

Proposition

If $D \subset F/Y$ is $r$-generic, then $D' \subset F'/Y'$ is $(r - 1)$-generic.

This allows us to prove formulas e.g. for the classes $[Y(rA_1)]$. 

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Curves with singularities of given type

Given an Enriques diagram $D = D_1, \ldots, D_n$, how many curves in a given linear system $|\mathcal{L}|$ have singularities of these types (and pass through the required number of points)?

**Theorem**

(Li–Tzeng, Rennemo) *There exists a universal polynomial $P$ of degree $n$ in four variables such that this number is equal to $P$ evaluated in the Chern numbers of $S$ and $\mathcal{L}$.*

**Proof.**

(Rennemo (2013)) The number is given by the degree of the class $c_m(\mathcal{L}^d) \cap \overline{H(D)}$, where $m = \dim H(D)$, $d = \deg(D)$, and $\mathcal{L}^d = q_*p^*\mathcal{L}$ with $p: Z \to S$, $q: Z \to S^d$ are the projections from the incidence scheme $Z \subset S \times S^d$. 

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Cuspidal curves

For a cusp (unibranch singularity) the Enriques diagram can be replaced by the multiplicity sequence.

Many open questions:

- How many cusps can a plane cuspidal curve have? Tono: \( \leq \frac{21g + 17}{2} \). Conjecture: \( \leq 4 \) for \( g = 0 \).
- Coolidge–Nagata: Any rational plane cuspidal curve can be transformed to a line via a sequence of Cremona transformations.
- Orevkov–Chéniot: For a plane rational cuspidal curve, \( \sum \overline{M} \leq 3(\deg C - 3) + \dim \text{Stab}_{PGL(3)}(C) \).

T. K. Moe (2013): A cuspidal curve on a Hirzebruch surface can have at most \( \frac{21g + 29}{2} \) cusps.
Fibonacci cusps

Let \( \varphi_k \) denote \( k \)th Fibonacci number:

\[
\varphi_0 = 0, \varphi_1 = 1, \varphi_2 = 1, \varphi_3 = 2, \varphi_4 = 3, \varphi_5 = 5, \ldots
\]

The Fibonacci singularity \( F_k: y^{\varphi_k} - x^{\varphi_{k+1}} = 0 \) has the following numerical characters:

\begin{itemize}
  \item \( \dim(F_k) = 3 \)
  \item \( \deg(F_k) = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 1 \)
  \item \( \cod(F_k) = (\varphi_k + 1)(\varphi_{k+1} + 1)/2 - 4 \)
  \item \( \delta(F_k) = (\varphi_k - 1)(\varphi_{k+1} - 1)/2 \)
  \item \( \mu(F_k) = 2\delta(F_k) = (\varphi_k - 1)(\varphi_{k+1} - 1) \)
  \item \( \overline{M} := \cod(F_k) - \delta(F_k) = \varphi_{k+2} - 4 \)
\end{itemize}
Fibonacci curves

The rational plane curve $C_k : y^{\varphi_k} z^{\varphi_k-1} - x^{\varphi_k+1} = 0$ has two cusps: one Fibonacci cusp and one “semi”-Fibonacci cusp:

$z^{\varphi_k-1} - x^{\varphi_k+1} = 0$.

- $C_k$ is toric, and (hence) self-dual (the cusps are interchanged under the Gauss map).
- $C_k$ is “maximally inflected” (i.e., all cusps and flexes are real — here there are no flexes).
- The sum of the $\overline{M}$-numbers is (for $k \geq 4$)

$$\varphi_{k+2} - 4 + \varphi_{k+1} - 4 + \varphi_{k-1} = 3(\varphi_{k+1} - 3) + 1$$

cf. Orevkov–Chéniots conjecture:

$$\sum \overline{M} \leq 3(\deg C - 3) + \dim \text{Stab}_{PGL(3)}(C)$$
Thank you for your attention!