# Partitions, polynomials and generating functions

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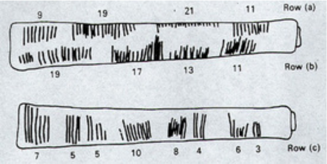
# Partitions (5 000 years ago)

MS 4647
Numbers 3+4. Sumer, ca. 3500-3200 BC

# The Ishango bone (20 000 years ago)




https://mathtimeline.weebly.com/early-human-counting-tools.html

# Counting partitions

Let $n$ be a positive integer.

In how many ways $p(n)$ can we write $n$ as a sum of positive integers?

$1 = 1$
$2 = 2 = 1 + 1$
$3 = 3 = 2 + 1 = 1 + 1 + 1$
$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$

$p(1) = 1$
$p(2) = 2$
$p(3) = 3$
$p(4) = 5$

A closed formula for $p(n)$?

# The partition function (Euler)

Expand the infinite product

$$(1+q+q^2+\cdots)(1+q^2+q^4+\cdots)\cdots(1+q^m+q^{2m}+\cdots)\cdots$$

The coefficient of $q^n$ is $p(n)$, the number of ways we can write $n = n_1 + n_2 + \ldots$, where $n_1 \geq n_2 \geq \cdots > 0$.

The *generating function*: $P(q) := \sum_{n\geq 0} p(n)q^n$ is

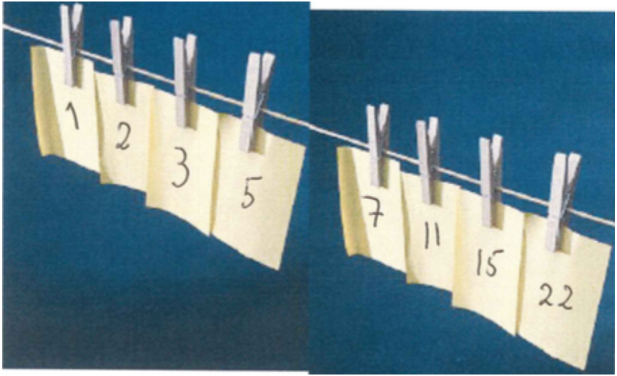
$$P(q) = \prod_{m\geq 1}(1+q^m+q^{2m}+q^{3m}+\cdots) = \prod_{m\geq 1}(1-q^m)^{-1}.$$

The function $P(q)$ displays the integer sequence

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, 77, 101, \ldots$$

# Generating functions

H.S.Wilf: “A generating function is a clothesline on which we hang up a sequence of numbers for display."



# Plane partitions

Consider next the number of *plane partitions* $\pi(n)$:

$$1 = 1 \Rightarrow \pi(1) = 1$$

$$2 = 2,\ \begin{array}{c|c} 1 & 1 \end{array},\ \begin{array}{c} 1 \\ \hline 1 \end{array} \Rightarrow \pi(2) = 3$$

$$3 = 3,\ \begin{array}{c|c} 2 & 1 \end{array},\ \begin{array}{c} 2 \\ \hline 1 \end{array},\ \begin{array}{c|c|c} 1 & 1 & 1 \end{array},\ \begin{array}{c|c} 1 & 1 \\ \hline 1 & \end{array},\ \begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array} \Rightarrow \pi(3) = 6$$

The generating function for $\pi(n)$ is the MacMahon function:

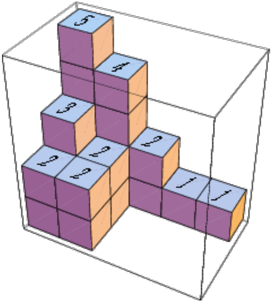
$$M(q) := \sum_{n \geq 0} \pi(n) q^n = \prod_{m \geq 1} (1 - q^m)^{-m}$$

$$
\begin{aligned}
M(q) &= (1+q+q^2+q^3+\cdots)(1+q^2+\cdots)^2(1+q^3+\cdots)^3\cdots \\
&= 1+q+q^2+2q^2+q^3+q\cdot 2q^2+3q^3+\cdots
\end{aligned}
$$

$$
\begin{aligned}
q &\leftrightarrow 1 \\
q^2 &\leftrightarrow 2 \\
2q^2 &\leftrightarrow \; 1 \mid 1 \;\text{ and } \frac{1}{1} \\
q^3 &\leftrightarrow 3 \\
q\cdot 2q^2 &\leftrightarrow \; 2 \mid 1 \;\text{ and } \frac{2}{1} \\
3q^3 &\leftrightarrow \; 1 \mid 1 \mid 1 \;\text{ and } \begin{array}{c|c} 1 & 1 \\ \hline 1 & \end{array} \;\text{ and } \begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array}
\end{aligned}
$$

# A plane partition of 22

| | | | | |
|---|---|---|---|---|
| 5 | 4 | 2 | 1 | 1 |
| 3 | 2 | | | |
| 2 | 2 | | | |



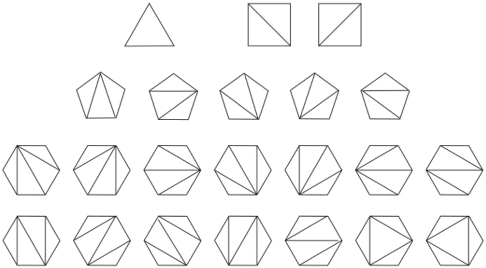
# We want to count:

The number of ways to join $2n$ points on a circle to form $n$ nonintersecting chords.

Minggatu (Mongolia, 1730's) obtained a recursion formula for these numbers.

# Euler (1750's) counted:

The number of ways to triangulate a regular $(n+2)$-gon.



# Euler had a guess for these numbers $C_n$

J. A. von Segner showed the convolution formula

$$C_{n+1} = \sum_{i=0}^{n} C_i \, C_{n-i}$$

By 1758, Euler, Goldbach, and Segner arrived at

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Eugène Charles Catalan (1814–1894) contributed

$$C_n = \binom{2n}{n} - \binom{2n}{n-1},$$

which eventually secured him the name *Catalan* numbers.

The generating function $C(q) := \sum_{n=0}^{\infty} C_n \, q^n$ satisfies the equation $q\,C(q)^2 - C(q) + 1 = 0$, hence $C(q) = \frac{1-\sqrt{1-4q}}{2q}$.

# Schubert calculus

- How many lines intersect four given lines in $\mathbb{P}^3$?

  Degenerate the four lines to two pairs of intersecting lines. Then the line through the two points of intersection, and the line which is the intersection of the two planes are the lines meeting all four lines – the answer is 2.
- How many lines meet six given planes in $\mathbb{P}^4$?

  The answer is 5.
- How many lines meet $2n$ given $(n-1)$-planes in $\mathbb{P}^{n+1}$?

  The answer is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The function $C(q) = \frac{1-\sqrt{1-4q}}{2q}$ displays the sequence

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

# Generalized Catalan numbers

- The *higher dimensional* Catalan numbers

$$C_n^{(m)} = C_m^{(n)} = (mn)! \prod_{i=0}^{m-1} \frac{i!}{(n+i)!} = (mn)! \prod_{j=0}^{n-1} \frac{j!}{(m+j)!}$$

  count the number of linear $(m-1)$-spaces in $\mathbb{P}^{n+m-1}$ meeting $mn$ linear $(n-1)$-spaces, and the number of standard $m \times n$ Young diagrams. Note that $C_n^{(2)} = C_n$.
- The *super* Catalan numbers (Ira Gessel)

$$C_{m,n} = \frac{(2m)!(2n)!}{(m+n)!m!n!}$$

  – but what do they count? *Open problem* for $m, n \geq 5$ and $m \neq n$. Note that $C_{1,n} = 2C_n$.

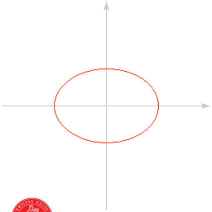
# Plane curves

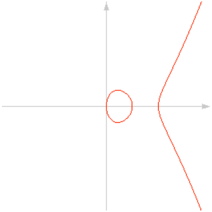
A plane curve of degree $d$

$$X = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | \, F(x_0, x_1, x_2) = 0\},$$

where $F$ is a homogeneous polynomial of degree $d$.

The curve on the left is a conic ($d = 2$). It is a *rational* curve (it has genus 0). The one to the right is a cubic ($d = 3$). It is *elliptic* (it has genus 1).

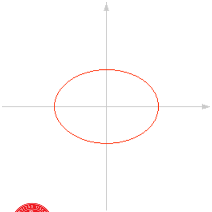


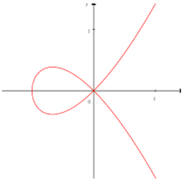


# Plane curves

A plane curve of degree $d$ is given as the set of zeros of a (homogeneous) polynomial $F(x_0, x_1, x_2)$ in three variables of degree $d$.

The curve on the left is a conic ($d = 2$). It is a *rational* curve (it has genus 0). The one to the right is a cubic ($d = 3$) with a node. It is *rational*.





# Counting rational plane curves ("Gromov–Witten")

The space of all curves of degree $d$ in $\mathbb{P}^2$ is a projective space of dimension $\binom{d+2}{2} - 1$.

The *rational* curves are those which have $\binom{d-1}{2}$ singular points. They form a subset (*Severi variety*) of dimension $3d - 1$.

Problem: Find the number $N_d$ of plane rational curves of degree $d$ passing through $3d - 1$ points.

Example

$N_1 = 1$, $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$, $\ldots$.

# Kontsevich's recursion formula

Set $n_d := \frac{N_d}{(3d-1)!}$.

$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} \frac{d_1 d_2((3d_1-2)(3d_2-2)(d+2)+8(d-1))}{6(3d-1)(3d-2)(3d-3)}$$

This gives a generating function (in two variables)

$$\Gamma(q_1, q_2) := \sum_d n_d e^{q_1 d} q_2^{3d-1},$$

which satisfies the differential equation

$$\partial^3\Gamma/\partial q_2^3 = (\partial^3\Gamma/\partial q_1^2 \partial q_2)^2 - (\partial^3\Gamma/\partial q_1^3)(\partial^3\Gamma/\partial q_1 \partial q_2^2).$$

# Rational curves on a quartic surface

Let $S \subset \mathbb{P}^3$ be a quartic surface. If $H \subset \mathbb{P}^3$ is a plane, then $C := S \cap H \subset H \cong \mathbb{P}^2$ is a plane quartic curve.

Problem: How many $H$ are such that $C$ has three nodes? Or, how many $H$ are triply tangent to $S$?

Answer 1: The number of tri-tangent planes to a surface of degree $d$ in $\mathbb{P}^3$ is

$$t(d) = \frac{1}{6}d(d-2)(d^7-4d^6+7d^5-45d^4+114d^3-111d^2+548d-960),$$

so the answer is $t(4) = 3200$.

# The Yau—Zaslow formula

Answer 2:

$$P(q)^{24} = 1 + 24q + 324q^2 + 3200q^3 + \cdots ,$$

where $P(q) = \sum p(n)q^n = \prod_{m\geq 1}(1-q^m)^{-1}$ is the partition function!

Hint:

Let $S$ be a K3 surface (e.g., a quartic surface $S \subset \mathbb{P}^3$) and $|D|$ a linear system of dimension $r$. Nonsingular curves in $|D|$ have genus $r$. Let $M_r$ denote the number of rational curves in $|D|$ (= the number of curves with $r$ singular points). Then

$$\sum_{r=0}^{\infty} M_r\, q^r = P(q)^{24}.$$

(Degenerate $S$ to an elliptic fibration over $\mathbb{P}^1$ with $c_2(S) = 24$ singular (hence rational) fibers. . . )

# Stirling and Bell numbers

The *Stirling* numbers $S_{n,k}$ count the number of partitions of a *set* with $n$ elements into $k$ blocks.

The *Bell* numbers count *all* partitions:

$$B_n := \sum_{k=1}^{n} S_{n,k}.$$

They satisfy a recursive relation: set $B_0 := 1$, then

$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i.$$

We get

$$B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, \ldots$$

# Block partitions

Set $\Pi_n := \{\text{partitions of } \{1, \ldots, n\}\}$.

Given $\mathbf{k} = (k_1, \ldots, k_n)$, $k_i \geq 0$, $\sum_{i=1}^n ik_i = n$, we say $\pi \in \Pi_n$ has *type* $\mathbf{k}$ if $\pi$ has $k_i$ blocks of size $i$.

A partition of type $\mathbf{k}$ has $k := \sum_{i=1}^n k_i$ blocks.

Let $\beta_{\mathbf{k}}$ denote the number of partitions of type $\mathbf{k}$. Then

$$\beta_{\mathbf{k}} := \frac{n!}{k_1! \cdots k_n!} \Big(\frac{1}{1!}\Big)^{k_1} \cdots \Big(\frac{1}{n!}\Big)^{k_n}.$$

We have

$$S_{n,k} = \sum_{\mathbf{k},k} \beta_{\mathbf{k}} \text{ and } B_n = \sum_{\mathbf{k}} \beta_{\mathbf{k}}.$$

## Example

$n = 4,\ k = 2$

$\mathbf{k} = (1,0,1,0)$:

$\{1\},\{2,3,4\};\{2\},\{1,3,4\};\{3\},\{1,2,4\};\{4\},\{1,2,3\}.$

$\mathbf{k} = (0,2,0,0)$:

$\{1,2\},\{3,4\};\{1,3\},\{2,4\};\{1,4\},\{2,3\}.$

There are $\beta_{(1,0,1,0)} = \frac{4!}{1!1!}(\frac{1}{1!})^1(\frac{1}{3!})^1 = 4$ of the first type,

and $\beta_{(0,2,0,0)} = \frac{4!}{2!}(\frac{1}{2!})^2 = 3$ of the second type.

$S_{4,2} = 4 + 3 = 7$

$\sum_{k=1}^{4} S_{4,k} = 1 + 7 + 6 + 1 = 15 = B_4$

# Polydiagonals

Let $X$ be a space, and consider

$$X^n := X \times \cdots \times X = \{(x_1, \ldots, x_n) \,|\, x_i \in X\}.$$

For $\pi \in \Pi_n$, set

$$\Delta_\pi^{(n)} := \{(x_1, \ldots, x_n) \in X^n | \, x_i = x_j \text{ if } i, j \text{ in same block of } \pi\}.$$

If $\pi$ has type $\mathbf{k}$, we say that $\Delta_\pi^{(n)}$ is a *polydiagonal* of type $\mathbf{k}$.

There are $\beta_{\mathbf{k}}$ polydiagonals of type $\mathbf{k}$, and $\sum_{\mathbf{k}} \beta_{\mathbf{k}} = B_n$ polydiagonals.

## Example

The *small* diagonal is $\Delta_{\{1,\ldots,n\}}^{(n)} = \{(x, \ldots, x) \in X^n | \, x \in X\}$.

# Bell polynomials

The *Bell polynomials* are

$$B_n(z_1, \dots, z_n) := \sum_{\mathbf{k}} \beta_{\mathbf{k}} z_1^{k_1} \cdots z_n^{k_n}.$$

Note that $B_n(1, \dots, 1) = B_n$.

$$\begin{aligned} B_1(z_1) &= z_1, \\ B_2(z_1, z_2) &= z_1^2 + z_2, \\ B_3(z_1, z_2, z_3) &= z_1^3 + 3z_1 z_2 + z_3 \\ B_4(z_1, z_2, z_3, z_4) &= z_1^4 + 6z_1^2 z_2 + 4z_1 z_3 + 3z_2^2 + z_4 \end{aligned}$$

# Bell polynomials – other definitions

Recursively defined by $B_0 = 1$ and

$$B_{n+1}(z_1, \ldots, z_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(z_1, \ldots, z_{n-i}) z_{i+1},$$

or by the formal identity for the (exponential) *generating function*

$$\sum_{n \geq 0} \frac{1}{n!} B_n(z_1, \ldots, z_n) q^n = \exp\left(\sum_{j \geq 1} \frac{1}{j!} z_j q^j\right),$$

Note binomiality:

$$B_n(z_1 + z_1', \ldots, z_n + z_n') = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(z_1, \ldots, z_{n-i}) B_i(z_1', \ldots, z_i').$$

# Nodal curves on families of surfaces

Given a family of curves on a family of surfaces $D \subset F \xrightarrow{f} Y$, find an expression $N_r$ for the class of curves that have $r$ nodes.

Conjecture (Kleiman–P.): There exist universal polynomials $b_i$ of weighted degree $i+2$ in the Chern classes $c_1(\mathcal{O}_F(D))$, $c_1(\Omega^1_f)$, and $c_2(\Omega^1_f)$ such that, setting $a_i := f_* b_i$,

$$N_r = \tfrac{1}{r!} B_r(a_1, \ldots, a_r),$$

where $B_r$ is the $r$th Bell polynomial.

Proved for $r \leq 8$, and gave an explicit algorithm for the computations, using the recursive property of the Bell polynomials.

# Existence and shape of the polynomials

For a *trivial* family $F = S \times Y$, where $Y = |D|$ is a linear system on the surface $S$, this follows from Göttsche's conjecture, which has been proved (Tzeng, Kool–Shende–Thomas).

T. Laarakker (2018) proved part of our conjecture: there exist universal polynomials $U_r$ such that $N_r$ is equal to $U_r$ evaluated on classes $f_* c_1(\mathcal{O}_F(D))^a c_1(\Omega^1_f)^b c_2(\Omega^1_f)^c$, with $a + b + 2c \leq r + 2$.

He also showed that the polynomials are multiplicative:

$$U_r(F \sqcup F') = \textstyle\sum_i U_i(F) U_{r-i}(F').$$

Given the polynomiality of the Bell polynomials:

$$\tfrac{1}{r!} B_r(a_1 + a'_1, \ldots) = \textstyle\sum_i \tfrac{1}{i!} B_i(a_1, \ldots, a_i) \tfrac{1}{(r-i)!} B_{r-i}(a'_1, \ldots, a'_{r-i}),$$

this gives evidence for our conjecture that $U_r = \frac{1}{r!} B_r$.

# Intersection theory

Let $D \subset F \xrightarrow{f} Y$ be a family of curves on surfaces, and set

$$X := \{x \in D | \, x \in D_{f(x)} \text{ is singular}\}.$$

Let $\Delta \subset X^r = X \times_Y \cdots \times_Y X$ be the union of all diagonals: $X^r \setminus \Delta$ is the $r$th *configuration space* of $X$. Set $f^r : F^r \to Y$. Then

$$f^r_*[\overline{X^r \setminus \Delta}] = r! N_r.$$

Let $p_j : F^r \to F$ be the projection maps. Then

$$N_r = \tfrac{1}{r!} f^r_*[\overline{X^r \setminus \Delta}] = \tfrac{1}{r!} f^r_*\big(\prod p_j^*[X] - (p_1^* X \cdots p_r^* X)^\Delta\big),$$

where the last term is the sum of the *equivalences of all distinguished irreducible varieties* in $\Delta$ (Fulton).

# Why Bell polynomials?

We have (N. Qviller)

$$\prod_{j=1}^{r} p_j^*[X] - (p_1^*X \cdots p_r^*X)^{\Delta} = \sum_{\pi \in \Pi_r} n_\pi^{(r)} (p_1^*X \cdots p_r^*X)^{\Delta_\pi^{(r)}},$$

where

$$n_\pi^{(r)} := \prod_{i=1}^{r} ((-1)^{i-1}(i-1)!)^{k_i}$$

and $\mathbf{k} = (k_1, \ldots, k_r)$ is the type of $\pi$.

Set $b_i := (-1)^i (i-1)! (p_1^*X \cdots p_i^*X)^{\Delta^{(i)}_{\{1,\ldots,i\}}}$ and $a_i := f_*^i b_i$.

Then

$$n_\pi^{(r)} f_*^r (p_1^*X \cdots p_r^*X)^{\Delta_\pi^{(r)}} = a_1^{k_1} \cdots a_r^{k_r}$$

and

$$N_r = \frac{1}{r!} \sum_\pi a_1^{k_1} \cdots a_r^{k_r} = \frac{1}{r!} \sum_{\mathbf{k}} \beta_{\mathbf{k}} a_1^{k_1} \cdots a_r^{k_r} = \frac{1}{r!} B_r(a_1, \ldots, a_r).$$

# Counting more than nodes

If we replace $b_i$ by the equivalence $\widetilde{b}_i$ of $\Delta^{(i)}_{\{1,\dots,i\}}$ without including other irreducible distinguished varieties contained in $\Delta^{(i)}_{\{1,\dots,i\}}$, then we get classes $\widetilde{N}_r$ for curves with $r$ nodes or other multi-singularities of codimension $r$.

For example,

$$\widetilde{N}_1 = \widetilde{a}_1 = a_1 = N_1,$$

$$\widetilde{N}_2 = \tfrac{1}{2}(\widetilde{a}_1^2 + \widetilde{a}_2) = N_2 + N_{A_2},$$

$$\widetilde{N}_3 = \tfrac{1}{3!}(\widetilde{a}_1^3 + 3\widetilde{a}_1\widetilde{a}_2 + \widetilde{a}_3) = N_3 + N_{A_1A_2} + N_{A_3}.$$

(Cf. Qviller, Kazarian's Thom polynomials.)

# Back to MacMahon

A *plane partition* of $n$ can be viewed as a set of $n$ monomials in three variables $x, y, z$ such that all the other monomials generate an ideal in $\mathbb{C}[x, y, z]$, of colength $n$.

## Example

The plane partition of 4:

$$\begin{array}{c|c} 2 & 1 \\ \hline 1 & \end{array}$$

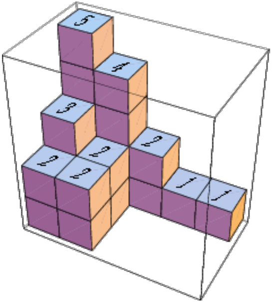
corresponds to the set of monomials $\{1, x, y, z\}$, which gives the ideal $\langle x^2, xy, xz, y^2, yz, z^2, \dots \rangle$ of colength 4.

# Recall the plane partition of 22:

| | | | | |
|---|---|---|---|---|
| 5 | 4 | 2 | 1 | 1 |
| 3 | 2 | | | |
| 2 | 2 | | | |

$\longleftrightarrow$

$$\{1, x, xz, xz^2, x^2, x^2z, xy, xyz, x^2y, x^2yz,\\ y, yz, yz^2, yz^3, y^2, y^2z, y^3, y^4, z, z^2, z^3, z^4\}$$



# Donaldson–Thomas and MacMahon

We have seen that the integer $\pi(n)$ counts the number of *monomial* ideals in $\mathbb{C}[x, y, z]$ of colength $n$.

These correspond to the colength $n$ ideals that are invariant under the action of the torus $(\mathbb{C}^*)^3$,

or to the length $n$ fixed points of $\mathbb{C}^3$ under the torus action.

These are (up to sign) the rank 1 Donaldson–Thomas invariants:

$$DT_1(\mathbb{C}^3)(q) = M(-q) = \prod_{m \geq 1} (1 - (-q)^m)^{-m}$$

(For more, see recent work of Fasola–Bonavari–Ricolfo.)

# Thank you for your attention!