

Some formulas for a surface in $\mathbb{P}^3$

Ragni Piene *

Contents

1. Introduction
2. Review of general results
3. Surfaces with ordinary singularities
4. Surfaces with multiple curves
5. Surfaces with a double and a cuspidal curve

1. Introduction

In a previous article ([P1]) we defined polar classes for singular projective varieties and gave formulas for the polar classes of a hypersurface. The purpose of the present paper is to apply these results and methods to a surface in $\mathbb{P}^3$ in order to obtain some of the classical formulas of Salmon, Cayley, Zeuthen, and Noether.

In particular we obtain all the classical formulas in the case that the surface $X \subseteq \mathbb{P}^3$ has only ordinary singularities (i.e., a double curve, with a finite number of triple points and a finite number of pinch points), or has "almost ordinary" singularities (i.e., the "double curve" may be multiple of a higher order).

It is known ([L1],[R1]) that any smooth surface Z can be mapped finitely to a surface $X \subseteq \mathbb{P}^3$, where X has ordinary singularities, by a generic projection to $\mathbb{P}^3$ of an imbedding $X \subseteq \mathbb{P}^N$. In enumerative geometry, however, one is led to consider also surfaces which are not of this type. Firstly, the singular subvarieties of a variety with ordinary singularities do not themselves have ordinary

singularities. For example, the double surface of a threefold in $\mathbb{P}^4$ with ordinary singularities has a triple curve (see Appl.1), and the "surface of contact" on the threefold, with respect to a projection to $\mathbb{P}^3$, acquires a cuspidal curve (see Appl. 2).

Secondly, consider the dual surface $\check{X} \subseteq \check{\mathbb{P}}^3$ of $X \subseteq \mathbb{P}^3$, defined as the closure of the set of tangentplanes to X at smooth points. Then it is almost never true that X and $\check{X}$ both have ordinary singularities. This observation was made already by Salmon, who therefore allowed the surface to have both a double and a cuspidal curve in order to be able to apply the formulas also to the dual surface.

We shall consider a surface X with double and cuspidal curves in Section 5, but we restrict ourselves to the case that the normalization of X is smooth (in the next section we shall comment briefly on how the formulas have to be modified in the general case). Except for one relation (IV_0), that we conjecture but are unable to prove, we obtain also a complete set of formulas in this case. An important special case of the situation described in section 5 is the dual surface of a smooth surface (Appl. 3).

Some of the results below (in particular Thm.2 (I),(II),V), Thm.3 (I), Thm.4 (I)) are contained in my doctoral thesis (M.I.T., 1976). I would like to thank Steven Kleiman, my thesis advisor, for having suggested the topic, and for very helpful discussions particularly about the triple point formulas.

2. Review of general results ([P1]).

Let k be an algebraically closed field. By a surface we mean a purely 2-dimensional, reduced scheme, proper over k . An embedded surface $X \subseteq \mathbb{P}^3 = \mathbb{P}_k^3$ has polar classes $[M_1]$ and $[M_2]$ defined as follows. Let $P \in \mathbb{P}^3$ (resp. $L \subseteq \mathbb{P}^3$) be a point (resp. a line), and let T_x denote the tangent plane to X at a smooth point x . Set

* Supported by the Norwegian Research Council for Science and the Humanities.

$$M_1 = \text{closure of } \{x \in X \mid x \text{ smooth, } P \in T_x\}$$

$$M_2 = \{x \in X \mid x \text{ smooth, } L \subseteq T_x\}.$$

Then M_1 and M_2 have natural structures as subschemes of X (reduced if $\text{char } k = 0$). If P and L are chosen general, they have codimension 1 and 2 respectively, and the rational equivalence classes $[M_1]$ and $[M_2]$ in the Chow group $A.X$ (of cycles on X modulo rational equivalence) are independent of the particular P and L .

In ([P1]) we found formulas for the polar classes of a hypersurface. We shall state the results for the case of a surface, but first we need some notation.

Let $X \subseteq \mathbb{P}^3$ be a surface and let $\pi: \tilde{X} \rightarrow X$ denote the blowup of the jacobian ideal of X , i.e., of the 2nd Fitting ideal $F^2(\Omega_X^1)$ of the module of differentials on X (equal to the ideal generated by the partial derivatives of a polynomial defining X in $\mathbb{P}^3$).

Suppose $f: Z \rightarrow X$ is a desingularization of X . Then we let $\phi: \bar{Z} \rightarrow Z$ denote the blowup of the ramification ideal of f , i.e., of the 0th Fitting ideal $F^0(\Omega_{Z/X}^1)$ of the module of relative differentials. We have shown ([P1], 2.6) that $f \circ \phi$ factors through π ; let $g: \bar{Z} \rightarrow \tilde{X}$ denote the induced map.

We shall use Fulton's intersection theory for singular varieties ([F]). Let $A.Y$ and A^*Y denote the Chow group and Chow ring of a scheme Y , and $\cap: A^*Y \otimes A.Y \rightarrow A.Y$ the cap product.

Define the singular subscheme S of X (resp. the ramification subscheme R of Z) by the ideal $F^2(\Omega_X^1)$ (resp. $F^0(\Omega_{Z/X}^1)$). The 1th Segre covariant class $s_1 = s_1(S, X) \in A_1X$ (resp. $r_1 = r_1(R, Z) \in A_1Z$) is defined by:

$$s_1 = \pi_*(c_1(J^{-1}) \cap [\tilde{X}]), \quad s_0 = -\pi_*(c_1(J)^2 \cap [\tilde{X}]),$$

$$r_1 = \phi_*(c_1(I^{-1}) \cap [\bar{Z}]), \quad r_0 = -\phi_*(c_1(I)^2 \cap [\bar{Z}]).$$

Note that $s_i = 0$ if $i > \dim S$, and $r_i = 0$ if $i > \dim R$. Finally, set $h = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$, and let $H = h \cap [X]$ denote the class of a hyperplane section of X . From ([P1], 2.3 and 2.5 with $r = 2$) we obtain the following results:

Theorem 1: Let μ_0 denote the degree of the surface $X \subseteq \mathbb{P}^3$. The following equalities hold in $A.X$.

$$(i) \quad [M_1] = (\mu_0 - 1)H - s_1$$

$$[M_2] = (\mu_0 - 1)^2 h^2 \cap [X] - 2(\mu_0 - 1)h \cap s_1 - s_0$$

$$(ii) \quad [M_1] = f_*(c_1(\Omega_Z^1) \cap [Z]) + 3H - f_*r_1,$$

$$[M_2] = f_*(c_1(\Omega_Z^1)^2 \cap [Z]) + 6h \cap f_*(c_1(\Omega_Z^1) \cap [Z]) \\ + 9h^2 \cap [X] - 2f_*(c_1(\Omega_Z^1) \cap r_1) - 6h \cap f_*r_1 - r_0.$$

Now we define the rank μ_1 of X as the degree of $[M_1]$, and the class μ_2 of X as the degree of $[M_2]$. When M_2 is reduced (this is **always** true if $\text{char } k = 0$), μ_2 is indeed equal to the class of X as defined classically, i.e., μ_2 is the number of planes through a given line that are tangent to X at smooth points. (If $\text{char } k = p$, it might happen that each point of $\text{Supp } M_2$ occurs with multiplicity p^e , $e > 0$.)

The class μ_2 can also be interpreted as the degree of the dual surface. Define the dual variety $\check{X} \subseteq \check{\mathbb{P}}^3$ of X as the closure of the set of points of $\mathbb{P}^3$ corresponding to planes of $\mathbb{P}^3$ that are

tangent to X at a smooth point. The rational map $X \dashrightarrow \check{X}$ extends on $\check{X}$ to a morphism $\tilde{\pi}: \check{X} \rightarrow \check{X}$, called the dual map (or tangent map, or Gauss map) of $X \subseteq \mathbb{P}^3$ ([P1], § 3). We observe that $\check{X}$ need not be a surface, and that if it is, there are two cases :

a) $\tilde{\pi}$ is birational.

In this case, biduality holds (i.e., $\check{\check{X}} = X$) and $\tilde{\pi}$ is equal to the blowup of the jacobian ideal of $\check{X}$ ([P1], 3.3). Then $\nu_2 = \deg \check{X}$ holds.

b) $\deg \tilde{\pi} = p^e$, some $e > 0$, where $p = \text{char } k$.

In this case, $\nu_2 = p^e \cdot \deg \check{X}$ holds.

For certain kinds of ruled surfaces X , $\check{X}$ is a curve, and $\check{\check{X}}$ is a finite number of points iff X is a union of planes. Note that we in all cases have

$$\nu_2 = \deg \tilde{\pi}_* [\tilde{X}]$$

$$\nu_1 = \deg \pi_*(\check{h} \cap [\tilde{X}]),$$

where $\check{h} = c_1(Q_{\mathbb{P}^3}^\vee(1))$.

Remark 1. Let $S_1 \in A_1 X$ denote the codimension 1 part of the cycle defined by $S \subseteq X$, and write

$$S_1 = \sum m_i [S_1^i], \text{ where the } S_1^i \subseteq X \text{ are reduced. Then}$$

$$s_1 = \sum e_i [S_1^i]$$

holds, where e_i denotes the multiplicity of $F^2(\Omega_X^1)$ in the local ring of X at the generic point of S_1^i .

(ii) If $\dim S = 0$, then $s_1 = 0$, and

$$s_0 = \sum_{x \in S} e_x,$$

where $e_x = \text{multiplicity of } F^2(\Omega_X^1) \text{ in } \mathcal{O}_{X,x}([K], (II, 50))$.

(Similar computations are valid for r_1 and r_0 .)

Now recall the definition of the rank of a space curve, and of the class of immersion of a curve on a surface:

Let $C \subseteq \mathbb{P}^3$ be a (reduced, proper) curve. Then its rank is defined as the class (i.e., degree of the 1st polar class) of a generic projection $\bar{C} \subseteq \mathbb{P}^2$ of C . If C has degree r_0 , genus g , and if the ramification divisor of the normalization map $C' \rightarrow C$ has degree v , then the rank r_1 of C is given by

$$r_1 = 2r_0 + 2g - 2 - v.$$

Note that the rank of C is equal to the number (possibly counted with multiplicities) of lines that are tangent to C at smooth points and intersect a given line.

Suppose C is a curve on the surface $X \subseteq \mathbb{P}^3$. Then we define the class of immersion ρ of C in X to be

$$\rho = \int \check{h} \cap [C^\pi],$$

where C^π denotes the strict transform of C by $\pi: \check{X} \rightarrow X$, $\check{h} = c_1(Q_{\mathbb{P}^3}^\vee(1))$, and π is via $\tilde{\pi}: \check{X} \rightarrow \mathbb{P}^3$ - provided C is not contained in the singular locus of X . In that case, we define $\tilde{C} =$ the 1-dimensional part of $\pi^{-1}C$ that maps finitely onto C , and set

$$\rho = \int \check{h} \cap [\tilde{C}].$$

In the following three sections we shall consider surfaces $X \subseteq \mathbb{P}^3$ with singularities of three types. Common for these surfaces is that they all have smooth normalization. This restriction is mainly made in order to simplify the computations. It should be clear from the proofs, however, how the formulas would have to be

modified if one allows for example also isolated singularities (use Remark 1 (ii) locally). (See also [B], p.162.)

3. Surfaces with ordinary singularities.

Let $X \subseteq \mathbb{P}^3$ be a surface, and denote by $f: Z \rightarrow X$ its normalization. We say that X has ordinary singularities if the following conditions hold.

- (0) $\text{char } k \neq 2$
- (1) Z is smooth
- (2) the singular locus of X is a curve D , and D is smooth except for a finite number of triple points which are triple also for X . There are also a finite number of pinch points on X , situated on D ; these are the images of the points of ramification of f . If $x \in D$, then

$$\hat{O}_{X,x} \cong \begin{cases} k[t_1, t_2, t_3]/(t_1 t_2 t_3) & \text{if } x \text{ is triple} \\ k[t_1, t_2, t_3]/(t_2^2 - t_1^2 t_3) & \text{if } x \text{ is a pinch point} \\ k[t_1, t_2, t_3]/(t_1 t_2) & \text{otherwise} \end{cases}$$

- (3) Let $p: X \rightarrow \mathbb{P}^2$ be a generic projection. Then the ramification divisor $C \subseteq Z$ of $p \circ f: Z \rightarrow \mathbb{P}^2$, defined by $F^0(\Omega_{Z/\mathbb{P}^2}^1)$, is a smooth curve, and $p \circ f|_C: C \rightarrow \mathbb{P}^2$ is a generic immersion (i.e., is generically unramified).
- (4) Let $\pi: \tilde{X} \rightarrow X$ denote the blowup of the jacobian ideal $F^2(\Omega_X^1)$. The dual map $\pi': \tilde{X} \rightarrow \tilde{X} \subseteq \mathbb{P}^3$ is separable.

Remark 2. Given any smooth surface Z and an embedding $Z \subseteq \mathbb{P}^N$, then a generic projection $Z \rightarrow \mathbb{P}^3$ of $Z \subseteq \mathbb{P}^N$ (or of a twist of this embedding) satisfies (1), (2), and (3) if $\text{char } k \neq 2$ ([R1], Th.1,3; [R2], §11), and also (4) ([SGA7], Exp.XVII, (2.5) and [W], Th.8).

If $\text{char } k = 2$, we may assume (1) and (2) hold, except that the completion of the local ring at a pinch point looks like $k[[t_1, t_1 t_2, t_2^2 + t_2^3]]$. The curve C of (3) need not be smooth, and the map $p \circ f|_C: C \rightarrow \mathbb{P}^2$ is everywhere ramified ([R1], [R2]).

Note that a surface $X \subseteq \mathbb{P}^3$ might have ordinary singularities without being obtainable as a generic projection of any embedding of its normalization Z .

Assume now that $X \subseteq \mathbb{P}^3$ has ordinary singularities. Let $\underline{C} = \underline{\text{Hom}}_{O_X}(f_* O_Z, O_X)$ denote the conductor of Z in X ($\underline{C}$ is the largest sheaf of ideals of O_X which is also a sheaf of ideals of O_Z). From the local description it follows that $\underline{C}$ defines the curve D of double points (or double curve of X), with reduced structure. Set $\Gamma = f^{-1}D$. Then Γ is defined by $\underline{C}$ in O_Z , and the map $f|_\Gamma: \Gamma \rightarrow D$ has degree 2.

In addition to the degree μ_0 of X , its rank μ_1 , and its class μ_2 , we introduce the following numerical characters. Set

- $\epsilon_0 = \text{degree of } D$
- $\epsilon_1 = \text{rank of } D$
- $\rho = \text{class of immersion of } D \text{ in } X$
- $t = \# \text{ triple points of } D (= \text{ of } X)$
- $\gamma_2 = \# \text{ pinch points of } X.$

Finally, consider the curve of contact $C \subseteq Z$, defined in (4), and the map $\bar{p} = (p \circ f)|_C: C \rightarrow \mathbb{P}^2$. Define κ to be the degree of the ramification divisor of $\bar{p}$.

We are now going to prove the five fundamental relations ([B], p.159) that exist between the above 9 characters.

- Theorem 2:**
- (I) $\mu_1 = \mu_0(\mu_0 - 1) - 2\epsilon_0$
 - (II) $\mu_1(\mu_0 - 2) = \kappa + \rho$
 - (III) $\epsilon_0(\mu_0 - 2) = \rho + 3t$
 - (IV) $2\rho - 2\epsilon_1 = v_2$
 - (V) $\mu_2 + 2v_2 = \mu_1 + \kappa$

Proof: Our set-up is as follows:

$$\begin{array}{ccccccc}
 & & & \bar{Z} & & & \\
 & & \varphi & \swarrow & g & & \\
 \Gamma' & \longrightarrow & \Gamma & \hookrightarrow & Z & \xrightarrow{\pi} & \tilde{X} \xrightarrow{\tilde{f}} \tilde{X} \subseteq \tilde{\mathbb{P}}^3 \\
 \alpha \downarrow & & f \downarrow & & f \downarrow & & \\
 D' & \longrightarrow & D & \hookrightarrow & X & \subseteq & \mathbb{P}^3
 \end{array}$$

Here π denotes the blowup of the jacobian ideal $F^2(\Omega^1_X)$, φ the blowup of the ramification ideal $F^0(\Omega^1_{Z/X})$, and g is the map which exists since $F^2(\Omega^1_X)$ becomes invertible on Z ([P1], 2.6). We let D' (resp. Γ') denote the normalization of D (resp. Γ), and $\alpha: \Gamma' \rightarrow D'$ is the induced map.

Lemma 1: g is isomorphic to the normalization map of $\tilde{X}$.

Proof: The ramification ideal $F^0(\Omega^1_{Z/X})$ is equal to the maximal ideal of $\mathcal{O}_Z$ at a ramification point, and trivial otherwise (this is seen from the local description of X , see also [R3], p.162). Hence φ is the blowup of closed points on the smooth surface Z , so that $\bar{Z}$ is also smooth. Thus it suffices to show that g is finite.

We shall first show

$$F^2(\Omega^1_X)\mathcal{O}_Z = \underline{\mathbb{C}} \cdot F^0(\Omega^1_{Z/X}).$$

If $x \in D$ is a double point, $F^2(\Omega^1_X)_x = \underline{\mathbb{C}}_x$ holds, because both are equal, in $\hat{\mathcal{O}}_{X,x} \cong k[[t_1, t_2, t_3]]/(t_1 t_2)$, to (t_1, t_2) . Similarly, if x is a triple point, they are both equal to $(t_1 t_2, t_1 t_3, t_2 t_3)$. If $x \in D$ is a pinch point of X , we compute $F^2(\Omega^1_X)$ in

$\hat{\mathcal{O}}_{X,x} \cong k[[t_1, t_2, t_3]]/(t_2^2 - t_1^2 t_3) \cong k[[t_1, t_1 t_2, t_2^2]]$ to be $(t_1^2, t_1 t_2, t_1 t_2^2)$. Hence $F^2(\Omega^1_X)\mathcal{O}_Z$ is equal to $(t_1)(t_1, t_2)$ in $\hat{\mathcal{O}}_{Z,z} \cong k[[t_1, t_2]]$, which is just $\underline{\mathbb{C}} \cdot F^0(\Omega^1_{Z/X})$ in $\hat{\mathcal{O}}_{Z,z}$ (here $z \in Z$ denotes the point lying over x).

Since $\underline{\mathbb{C}}$ is invertible, it follows that $\bar{Z} \rightarrow Z$ is also isomorphic to the blowup of $F^2(\Omega^1_X)\mathcal{O}_Z$, and hence that there is an embedding $\bar{Z} \hookrightarrow Z \times_X \tilde{X}$. Since f is finite, so is $Z \times_X \tilde{X} \rightarrow \tilde{X}$, and hence so is g . Q.E.D.

Set $K_Z = c_1(\Omega^1_Z) \cap [Z]$ in $A_1 Z$, $K_{\bar{Z}} = c_1(\Omega^1_{\bar{Z}}) \cap [\bar{Z}]$ in $A_1 \bar{Z}$, $J = F^2(\Omega^1_X)\mathcal{O}_{\bar{X}}$, $I = F^0(\Omega^1_{Z/X})\mathcal{O}_{\bar{Z}}$, $E \subseteq \bar{Z}$ the exceptional divisor of φ (so that, in $A_1 \bar{Z}$, $[E] = c_1(I^{-1}) \cap [\bar{Z}]$), and, as in Section 2, we set $h = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$, $\tilde{h} = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$, $H = h \cap [X]$.

By $\int \alpha$ we denote the degree of an element $\alpha \in A_0 Y$, and we also write $\int \beta$ for $\int \beta \cap [Y]$, if $\beta \in A^r Y$ and the scheme Y has dimension r .

Lemma 2. The number v_2 of pinch points satisfies

- (i) $v_2 = - \int c_1(I^{-1})^2$
- (ii) $v_2 = \int (c_1(\Omega^1_Z)^2 - c_2(\Omega^1_Z) + 4c_1(\Omega^1_Z) \cap H) + 6\mu_0$.

Proof: (i) follows from the fact that the selfintersection of an exceptional curve (of the first kind) is equal to -1 .
(ii) Consider the exact sequence

$$f^* \Omega^1_{\mathbb{P}^3}|_X \rightarrow \Omega^1_Z \rightarrow \Omega^1_{Z/X} \rightarrow 0.$$

Then v_2 is equal to the degree of the cycle defined by the scheme of zeros of the map $\Lambda^2 f^* \Omega^1_{\mathbb{P}^3}|_X \rightarrow \Lambda^2 \Omega^1_Z$, i.e., the scheme defined by $F^0(\Omega^1_{Z/X})$. Applying the determinantal formula ([K-L], Cor.11) we obtain (ii). Q.E.D.

Lemma 3: The following equalities of rational equivalence classes of cycles hold.

$$(i) \quad \check{h} \cap [\tilde{X}] = (\mu_0 - 1)(h \cap [\tilde{X}]) - c_1(J^{-1}) \cap [\tilde{X}]$$

$$\check{h} \cap [\tilde{Z}] = \varphi^* K_Z + 3(h \cap [Z]) - [E]$$

$$(ii) \quad K_{\tilde{Z}} = \varphi^* K_Z + [E]$$

$$(iii) \quad g^* c_1(J^{-1}) \cap [\tilde{Z}] = [\varphi^{-1} \Gamma] + [E].$$

Proof: (i) follows from ([P1], 2.3 and 2.4). (ii) is well known - it suffices to consider the exact sequence $0 \rightarrow \varphi^* \Omega_Z^1 \rightarrow \Omega_{\tilde{Z}}^1 \rightarrow \Omega_{\tilde{Z}/Z}^1 \rightarrow 0$, and observe that E is defined by the ideal $F^0(\Omega_{\tilde{Z}/Z}^1)$. (iii) follows from the isomorphism $g^* J \cong \varphi^* C \cdot I$ (Lemma 1).

Now we shall return to the proof of Theorem 2. First we observe that (I) follows immediately from Theorem 1 (i), since, from what we have seen above, $s_1 = \pi_*(c_1(J^{-1}) \cap [\tilde{X}]) = f_*[\Gamma] = 2[D]$.

(V): Let $C \subseteq Z$ be the ramification curve of a generic projection $p \circ f : Z \rightarrow \mathbb{P}^2$, as in (3), so that C is smooth and $p \circ f|_C : C \rightarrow \mathbb{P}^2$ is a generic immersion. Let $\check{\mathbb{P}}^2 \subseteq \check{\mathbb{P}}^3$ be the hyperplane corresponding to the point of projection of p , and let $\bar{C} \subseteq \bar{Z}$ be the inverse image of $\check{\mathbb{P}}^2$ via $\check{\pi} \circ g : \bar{Z} \rightarrow \check{\mathbb{P}}^3$. Then it is easy to see that $\varphi|_{\bar{C}} : \bar{C} \simeq C$ and $f(C) = M_1$ hold. In particular, the curve $\bar{C}$ has degree μ_1 in $\mathbb{P}^2$.

In order to show (V) we shall compute the class of $\bar{C} \rightarrow \mathbb{P}^2$ in two ways. First we note that $\check{\pi}$ is separable, by (4). If $\check{X}$ is a surface, i.e., if $\mu_2 \neq 0$, we can apply ([P1], 4.3), which says that the class of $\bar{C} \rightarrow \mathbb{P}^2$ is equal to the degree of $\bar{C} \rightarrow \check{\mathbb{P}}^2$, hence to μ_2 .

Let g denote the genus of $\bar{C}$. By the adjunction formula ($\bar{C}$ and $\bar{Z}$ are smooth), we obtain

$$2g - 2 = \bar{C} \cdot (K_{\bar{Z}} + \bar{C}) = \int \check{h} \cap (\varphi^* K_Z + [E] + \check{h} \cap [\bar{Z}]) = \int \check{h} \cap (2\check{h} \cap [\bar{Z}] - 3h \cap [Z] + 2[E])$$

$$(\text{lemma 3}) = 2\mu_2 - 3\mu_1 + 2 \int (\check{h} \cap [E]) = 2\mu_2 - 3\mu_1 + 2v_2 \quad (\text{lemma 3, and the projection formula}).$$

Applying one of the Plücker formulas ([P1], 3.9 (II)) we find that the class is equal to

$$2\mu_1 + 2g - 2 - \kappa,$$

hence we get

$$\mu_2 = 2\mu_1 + 2\mu_2 - 3\mu_1 + 2v_2 - \kappa.$$

Suppose now $\mu_2 = 0$. Then either $\check{X}$ is a union of points, in which case X is a union of planes, and all the characters occurring in (V) are then 0. Or $\check{X}$ is a curve. Then ([W], Thm.3) $f(C)$ is a union of lines, hence the class of C (w.r.t. $C \rightarrow \mathbb{P}^2$) is 0, and the last part of the reasoning is still valid.

(II) We compute

$$\begin{aligned} \mu_2 &= \int \check{h}^2 = \int \check{h} \cap ((\mu_0 - 1)H - c_1(J^{-1}) \cap [\tilde{X}]) \\ &= \mu_1(\mu_0 - 1) - \int \check{h} \cap ([\varphi^{-1} \Gamma] + [E]) \quad (\text{lemma 3 (iii)}) \\ &= \mu_1(\mu_0 - 1) - \rho - 2v_2, \end{aligned}$$

because ρ is equal to $\int \check{h} \cap (\text{strict transform of } \Gamma \text{ via } \varphi)$, by definition, and this strict transform is equal to $[\varphi^{-1} \Gamma] - [E]$. Together with (V), this proves (II).

(IV) The rational map $D \dashrightarrow G = \text{Grass}_2(V)$ which sends a smooth point of D to its tangent line, extends to a morphism $\gamma : D' \rightarrow G$, called the first associated map of $D \subseteq \mathbb{P}^3$, and the degree of the curve $D' \rightarrow G$ (via the Plücker embedding $G \hookrightarrow \mathbb{P}(\wedge^2 V)$) is the same as the rank ϵ_1 of D ([P3], §3).

The normalization maps $\Gamma' \rightarrow \Gamma$ and $D' \rightarrow D$ are both unramified, since D has only t triple points and Γ has $3t$ (ordinary)

double points, as singularities. Therefore $\alpha: \Gamma' \rightarrow D'$ is a morphism of degree 2 with v_2 ramification points. We shall now show that the map $\gamma \circ \alpha: \Gamma' \rightarrow G$ can be defined in another way, using the fact that at a (general) point of D there are two distinct tangent planes to X , and their intersection is the tangent line to D at the point.

Consider the secant map

$$\check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3 \dashrightarrow G^* = \text{Grass}_2(V^\vee),$$

which is defined outside the diagonal and sends a pair of points to the line joining them (or, a pair of planes in $\mathbb{P}^3$ to their line of intersection). Let $b: B \rightarrow \check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3$ denote the blowup of the diagonal. Then ([L], 4.2; [K], (V, 41-42)) the morphism $B \rightarrow G^*$ is defined by a 2-quotient $V_B^\vee \rightarrow \mathcal{E}$, where $\mathcal{E}$ fits into an exact sequence

$$0 \rightarrow \alpha \otimes \mathcal{L}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{L}_2 \rightarrow 0$$

with $\mathcal{L}_1 = b^* \text{pr}_1^* \mathcal{O}_{\check{\mathbb{P}}^3}(1)$, and where $\alpha = \mathcal{O}_B(1)$ denotes the ideal of the exceptional divisor.

Set $\bar{\Gamma}$ = closure of $\Gamma' \times_{D, \Gamma'} - \Delta$, where Δ = diagonal. Then the projection on the first factor induces a map $\bar{\Gamma} \rightarrow \Gamma'$, which is necessarily an isomorphism.

Let $s: \Gamma' \rightarrow \Gamma' \times_{D, \Gamma'}$ denote the corresponding section (so that $s = (\text{id}, 1)$, where $1: \Gamma' \rightarrow \Gamma'$ corresponds to the involution on $\bar{\Gamma}$ induced by the switch of factors on $\Gamma' \times_{D, \Gamma'}$). Now the dual map of $X \subseteq \mathbb{P}^3$ is defined on Z , except at the ramification points of f . Therefore there is a rational map

$$Z \times Z \dashrightarrow \check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3,$$

hence also $\Gamma' \times_{D, \Gamma'} \dashrightarrow \check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3$, which composed with $s: \Gamma' \rightarrow \Gamma' \times_{D, \Gamma'}$ gives a morphism (since Γ' is a smooth curve the rational map is everywhere defined) $\Gamma' \rightarrow \check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3$. Since the preimage of the diagonal in $\check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3$ is equal to $s^{-1}(\Delta)$ and therefore is invertible on Γ' , this morphism lifts to a morphism $\sigma: \Gamma' \rightarrow B$. Identifying G^* and G

via the canonical isomorphism, we thus have obtained a factorization

$$\Gamma' \xrightarrow{\sigma} B \rightarrow G^* \simeq G$$

of $\gamma \circ \alpha$.

Let us compute the degree of $\Gamma' \rightarrow G$ (earlier seen to be $2\varepsilon_1$) using this factorization. The map is defined by the 2-quotient $V_{\Gamma'}^\vee \rightarrow \sigma^* \mathcal{E}$, hence we must compute $\int c_1(\sigma^* \mathcal{E})$. From the exact sequence above we get

$$c_1(\sigma^* \mathcal{E}) = c_1(\sigma^* \mathcal{L}_1) + c_1(\sigma^* \mathcal{L}_2) + c_1(\sigma^* \alpha).$$

We let $\mathcal{L}^*$ denote the pullback of $\mathcal{O}_{\check{\mathbb{P}}^3}(1)$ to Γ' via the map induced by the dual map of X ($\Gamma' \rightarrow \check{\mathbb{P}}^3$ is a priori a rational map, but extends since Γ' is a smooth curve). Then we observe: $\sigma^* \mathcal{L}_1 = \mathcal{L}^*$, $\sigma^* \mathcal{L}_2 = 1^* \mathcal{L}^*$. Since $\sigma^* \alpha = \text{ideal of } s^{-1}(\Delta) \text{ in } \Gamma'$, we get

$$c_1(\sigma^* \alpha) = c_1(F^0(\Omega_{\Gamma'/D}^1)), \text{ and hence}$$

$$\begin{aligned} \int c_1(\sigma^* \mathcal{E}) &= 2 \int c_1(\mathcal{L}^*) - c_1(F^0(\Omega_{\Gamma'/D}^1)^{-1}) \\ &= 2\rho - v_2. \end{aligned}$$

Remark 3: The formula just proved, $2\rho - 2\varepsilon_1 = v_2$, is a version of the correspondence principle ([S-R], p.64). Namely, we are considering a simply infinite system $(H, H'; 1)$ of unordered plane pairs together with a line contained in both. Then: $\rho = \#$ of forms such that H contains a given point, $\varepsilon_1 = \#$ of forms such that 1 meets a given line, and the "number of coincidences", i.e., of forms such that H and H' coincide, is equal to the number of pinch points and to $2(\rho - \varepsilon_1)$.

In section 4 we show that our proof of (V) generalizes to the case of a surface with a higher order multiple curve.

(III): Suppose we prove

$$(*) \quad 2\varepsilon_1 + v_2 = 2(\mu_0 - 2)\varepsilon_0 - 6t.$$

Then (III) follows, using (IV).

Let g_a and g denote the arithmetic and geometric genus of r . Since r has $3t$ ordinary double points, $g_a = g + 3t$. The adjunction formula (and lemma 3) gives

$$2g_a - 2 = r \cdot (K_Z + r) = \int (\mu_0 - 4) h \cap [r] = 2(\mu_0 - 4) \epsilon_0.$$

Let $g(D)$ denote the genus of D . The rank ϵ_1 of D is given by

$$\epsilon_1 = 2\epsilon_0 + 2g(D) - 2.$$

Finally, the Riemann-Hurwitz formula gives

$$2g - 2 - v_2 = 2(2g(D) - 2),$$

and putting the above together, we obtain (*).

QED

Most of the above formulas are valid also for a surface $X \subseteq \mathbb{P}^3$ which has ordinary singularities in the following weaker sense.

Assume $X \subseteq \mathbb{P}^3$ satisfies (1) and (2)', where (2)' is the condition (2) with the following addition:

If $\text{char } k = 2$, the completion of the local ring of X at a pinch point is isomorphic to

$$k[[t_1, t_1 t_2, t_2^2 + t_2^3]].$$

The numerical characters of X are still well defined, except possibly κ . Now we define v_2 as the degree of the ramification scheme of f (defined by the ideal $F^0(\Omega_{Z/X}^1)$), and we note (use the local description, see [R3], p.162) that v_2 is equal to the number of pinch points except if $\text{char } k = 2$, and then v_2 is equal to twice the number of pinchpoints.

Going through the proof of Theorem 2, one checks that everything holds for a surface X satisfying (1) and (2)' except possibly

formula (V) and the ones depending on it.

Up to now we have been concerned with the "extrinsic" characters of the surface, i.e., such that depend on the given map to projective space. A smooth surface Z has also "intrinsic" invariants. We set

$$c_1^2 = \int c_1(\Omega_Z^1)^2,$$

$$c_2 = \int c_2(\Omega_Z^1),$$

$$\chi(O_Z) = \sum_{i=0}^2 (-1)^i \dim_k H^i(Z, O_Z).$$

Hence c_1^2 is the selfintersection of a canonical divisor ("canonical grade"), c_2 is the Euler characteristic, and $\chi(O_Z)$ is the Euler-Poincaré characteristic, equal to $p_a + 1$, where p_a is the arithmetic genus of Z .

These three characters are related by Noether's formula

$$12\chi(O_Z) = c_1^2 + c_2.$$

In ([P2]) we gave a proof of this formula along classical lines, using the fact ([R1], Thm.1,3) that any smooth surface Z can be obtained as the normalization of a surface $X \subseteq \mathbb{P}^3$ which satisfies (1) and (2)'. The proof consisted in expressing the three intrinsic invariants in terms of the extrinsic ones, as follows.

Proposition 1.

- (i) $c_1^2 = \mu_0(\mu_0 - 4)^2 - (3\mu_0 - 16)\epsilon_0 + 3t - v_2$
- (ii) $c_2 = \mu_0(\mu_0^2 - 4\mu_0 + 6) - (3\mu_0 - 8)\epsilon_0 + 3t - 2v_2$
- (iii) $\chi(O_Z) - 1 = \binom{\mu_0 - 1}{3} - \frac{1}{2}(\mu_0 - 4)\epsilon_0 + \frac{1}{2}t - \frac{1}{4}v_2.$

In the proof of (i) and (ii), however, we used Kleiman's triple point formula ([K], (I,39)):

Proposition 2. $3t = (\Gamma^2) - \mu_0 \epsilon_0 + v_2.$

We shall now show that this formula is a corollary of (III) of Theorem 2.

Proof of Prop. 2. Compute (using lemma 3)

$$\begin{aligned} \rho + (\Gamma^2) &= \int \check{h} \cap ([\varphi^{-1}\Gamma] - [E]) + (\Gamma^2) \\ &= (\mu_0 - 1) \int h \cap [\Gamma] - (E, \varphi^{-1}\Gamma) - \int \check{h} \cap [E] \\ &= 2(\mu_0 - 1)\epsilon_0 - v_2. \end{aligned}$$

We also have the following classical formulas ([B], p.197):

Proposition 3. (i) $c_1^2 = \mu_2 - 6\mu_1 + 9\mu_0 + v_2$
(ii) $c_2 = \mu_2 - 2\mu_1 + 3\mu_0.$

Proof. This is a corollary of Proposition 1 and Theorem 2. A simpler proof is obtained directly, by using lemma 3 (i) to compute (i), and then lemma 2 (ii) together with (i) to compute (ii).

Q.E.D.

For other formulas that can be deduced from the ones above, we refer to the full discussion given in ([B], Ch. IV and V). Examples are also given there, in particular fifteen of Noether's examples.

We shall close this section with an application of another result of ([P1]) concerning the dual surface. In particular, it shows that the dual surface of a surface with ordinary singularities almost always has a cuspidal curve.

Proposition 4. Let $X \subseteq \mathbb{P}^3$ be a surface which satisfies (1), (2)', and (4), and assume also that $\check{X}$ is a surface. Then the map $\check{\pi} \circ g: \bar{Z} \rightarrow \check{\mathbb{P}}^3$ ramifies along a curve $\check{R}$, whose degree with respect to

$\mathbb{P}^3$ is

$$4(\mu_0(\mu_0 - 2) - 2\epsilon_0),$$

and with respect to $\check{\mathbb{P}}^3$ is

$$4\mu_0(\mu_0 - 1)(\mu_0 - 2) - 24(\mu_0 - 2)\epsilon_0 + 12\epsilon_1 + 48t$$

([B], p.162).

Proof. It follows from ([P1], 3.4) that

$$[\check{R}] - 2[E] = \int 4(\check{h} - h) \cap [\bar{Z}]$$

holds,

$$\deg_{\mathbb{P}^3} [\check{R}] = \int h \cap [\check{R}] = 4(\mu_1 - \mu_0)$$

and

$$\deg_{\check{\mathbb{P}}^3} [\check{R}] = \int \check{h} \cap [\check{R}] = 4(\mu_2 - \mu_1) + 2 \int \check{h} \cap [E].$$

Lemma 3 implies $\int \check{h} \cap [E] = v_2$, and the proposition now follows from Theorem 2, (IV), (I), and (III).

Q.E.D.

Finally, let us just remark that each of the exceptional lines of $\bar{Z}$ (with respect to the blowup φ) is mapped to a line in $\check{\mathbb{P}}^3$. In fact, let $E_1 \subseteq \bar{Z}$ be such a line. Then

$$\deg \check{\pi}_* [E_1] = \int \check{h} \cap [E_1] = - (E, E_1) = 1.$$

Hence the dual surface $\check{X}$ contains at least v_2 lines.

Example 1. The Steiner surface.

Let $V \subseteq \mathbb{P}^5$ denote the Veronese surface (i.e., V is isomorphic to $\mathbb{P}^2$ via the embedding $\mathbb{P}^2 \rightarrow \mathbb{P}^5$ by conics), and let $f: V \rightarrow \mathbb{P}^3$ be a generic projection. Then the image $S = f(V)$ has ordinary singularities; it is the so-called Steiner surface.

The intrinsic invariants of S are equal to those of $\mathbb{P}^2$, hence

$$c_1^2 = 9, \quad c_2 = 3, \quad p_a = 0.$$

Since V has degree 4 in $\mathbb{P}^5$, so does S in $\mathbb{P}^3$. We can compute the number of pinchpoints (Lemma 2 (ii))

$$v_2 = c_1^2 - c_2 + 4 \int h \cap K_{\mathbb{P}^2} + 6 \mu_0 = 6.$$

Then the other characters of S are deduced from Thm. 2 and Prop. 1:

$$\mu_1 = 6, \quad \mu_2 = 3, \quad \epsilon_0 = 3, \quad \kappa = 9, \quad \rho = 3, \quad \epsilon_1 = 0, \quad t = 1.$$

Moreover, the (reducible) double curve $D \subseteq S$ and its preimage $r \subseteq V$ have arithmetic genus 0 and 1 respectively, while both curves have geometric genus equal to -2.

Geometrically, this shows that D is equal to the union of three (not co-planar) lines intersecting in one point, which is of course well known ([S-R], p.134).

Example 2. Enriques surface (char $k \neq 2$).

Let Z be a (smooth) Enriques surface, i.e., Z satisfies

$$K_Z \neq 0, \quad 2K_Z = 0, \quad \dim H^1(Z, \mathcal{O}_Z) = 0.$$

Artin showed ([A], Thm.3.1.1) that Z can be mapped birationally onto a surface $X \subseteq \mathbb{P}^3$ of degree 6, and that in general X has the edges of a tetrahedron as singular locus.

Assuming X has ordinary singularities then the statement about the nature of the singular locus holds, by the formulas of this section:

Since K_Z is numerically trivial, and since $K_Z \neq 0$ implies $p_g = \dim H^0(Z, K_Z) = 0$, the intrinsic characters are:

$$c_1^2 = 0, \quad c_2 = 12, \quad p_a = 0.$$

As in Ex. 1 we compute v_2 :

$$v_2 = c_1^2 - c_2 + 4 \int h \cap K_Z + 6 \mu_0 = 24.$$

The remaining characters are:

$$\mu_1 = 18, \quad \mu_2 = 30, \quad \epsilon_0 = 6, \quad \kappa = 60, \quad \rho = 12, \quad \epsilon_1 = 0, \quad t = 4.$$

Since $\epsilon_1 = 0$, the double curve $D \subseteq S$ is a union of lines. Moreover, we find that D has arithmetic genus 3 and geometric genus -5, while the genera of $r = f^{-1}(D) \subseteq Z$ are 13 and 1 respectively.

4. Surfaces with multiple curves.

In this section we shall generalize the results of the preceding one to the case that the surface $X \subseteq \mathbb{P}^3$ is allowed to have, instead of a double curve, a j -multiple curve, for some integer j . This is a special case of a type of surface studied by Noether ([N]). Here we shall assume for simplicity that X has no isolated singularities, and that all the components of the singular curve have the same multiplicity j (but see Remark 4).

Let $X \subseteq \mathbb{P}^3$ be a surface satisfying (0), (1), (3), (4) of the preceding section, and also

(2) j : The singular locus of X consists of a curve D of points of multiplicity j . At a general point on this curve the completion of the local ring of X is isomorphic to

$$k[t_1, t_2, t_3] / (l_1 l_2 \cdots l_j),$$

where the l_i 's are linear and such that l_1 and l_j are linearly independent, for $i \neq j$.

There are a finite number of pinch points on X , situated on D ; these are the images of the ramification points of f .

We set $D \subseteq X$ equal to the singular curve of X with reduced structure, and set $\Gamma = f^{-1}D \subseteq Z$. The degree μ_0 , rank μ_1 , and class μ_2 are defined as before, and so are the degree ϵ_0 , rank ϵ_1 , class of immersion ρ of D . Again v_2 denotes the number of pinchpoints (defined as the degree of the ramification cycle of f), κ is the degree of the ramification divisor of the projection $p \cdot f|_C: C \rightarrow \mathbb{P}^2$ of a curve of contact $C \subseteq Z$ (see (4)).

The character "replacing" the number t of triple points of a surface with ordinary singularities, is "the number of double points of r ", δ , which we define as the difference $g_a(\Gamma) - g(\Gamma)$ between the arithmetic and geometric genera of Γ .

Theorem 3:

$$\begin{aligned} \text{(I)} \quad \mu_1 &= \mu_0(\mu_0 - 1) - j(j-1)\epsilon_0 \\ \text{(II)} \quad \mu_1(\mu_0 - 2) &= (j-1)\rho + \kappa + (j-2)v_2 \\ \text{(III)} \quad j\epsilon_0(\mu_0 - j) &= j\rho + 2(j-1)\delta \\ \text{(IV)} \quad 2(j-1)\rho - j(j-1)\epsilon_1 &= v_2 \\ \text{(V)} \quad \mu_2 + 2v_2 &= \mu_1 + \kappa. \end{aligned}$$

Proof: The set up and notations are as in the proof of Thm.2, and the proof is of course similar (partly identical) to that proof.

First we observe that $\Gamma = f^{-1}D$ is reduced, $\Gamma \rightarrow D$ has degree j and v_2 ramification points. The conductor $\underline{C}$ defines on Z the cycle $(j-1)\Gamma$. Since $g^*J \cong \varphi^*\underline{C} \cdot I$ holds, we get $s_1 = j(j-1)[D]$, thus (I) is proved (Thm.1 (i)).

Moreover, lemma 2 remains true, and so does lemma 3, provided we replace (iii) by

$$(iii) \quad g^*c_1(J^{-1}) \cap [\bar{Z}] = (j-1)[\varphi^{-1}\Gamma] + [E].$$

The proof of (V) is as before. To prove (II), consider

$$\begin{aligned} \mu_2 &= \int \check{h}^2 = \int \check{h} \cap ((\mu_0 - 1)H - (j-1)[\varphi^{-1}\Gamma] - [E]) \\ &= (\mu_0 - 1)\mu_1 - (j-1) \int \check{h} \cap ([\varphi^{-1}\Gamma] - [E]) - j \int \check{h} \cap [E] \\ &= \mu_1(\mu_0 - 1) - (j-1)\rho - jv_2. \end{aligned}$$

Now substitute for μ_2 using (V).

(IV) The proof of Thm.2 (IV) generalizes; we shall keep the notations. Let $n: \Gamma'' \rightarrow \bar{\Gamma}$ denote the normalization of the closure $\bar{\Gamma}$ of $\Gamma' \times_D \Gamma'$, $\Gamma' = \Delta$, and let $i: \bar{\Gamma} \rightarrow \bar{\Gamma}$ denote the involution, induced by switching the factors of the product. Let $q: \Gamma' \rightarrow \check{\mathbb{P}}^3$ denote the morphism induced from the dual map $\check{\pi}: Z \dashrightarrow \check{\mathbb{P}}^3$, and $p: \bar{\Gamma} \rightarrow \Gamma'$ the map induced by projecting onto the first factor. As in Thm. 2, the map $\Gamma'' \rightarrow \check{\mathbb{P}}^3 \times \check{\mathbb{P}}^3$ lifts to B , and we denote this map by $\sigma: \Gamma'' \rightarrow B$. Thus $\Gamma'' \rightarrow G^*$ is defined by $V_{\Gamma''}^{\vee} \rightarrow \sigma^*\mathcal{E}$, where $\mathcal{E}$ fits into an exact sequence $0 \rightarrow I \otimes \mathcal{L}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{L}_2 \rightarrow 0$, as before. We get:

$$\sigma^*\mathcal{L}_1 = \sigma^*b^*pr_1^*O_{\check{\mathbb{P}}^3}^{\vee}(1) = n^*p^*q^*O_{\check{\mathbb{P}}^3}^{\vee}(1),$$

$$\sigma^*\mathcal{L}_2 = \sigma^*b^*pr_2^*O_{\check{\mathbb{P}}^3}^{\vee}(1) = n^*i^*p^*q^*O_{\check{\mathbb{P}}^3}^{\vee}(1).$$

Since p has degree $j-1$, we obtain

$$\int c_1(\sigma^*\mathcal{L}_1) = \int c_1(\sigma^*\mathcal{L}_2) = (j-1)\rho.$$

Next we observe that the ideal I pulled back to $\Gamma' \times_D \Gamma'$ is equal to the ideal of the diagonal $\Delta \subseteq \Gamma' \times_D \Gamma'$. Therefore $\int c_1(\sigma^*I^{-1}) = (\Delta, \bar{\Gamma})$, as divisors on $\Gamma' \times \Gamma'$. Moreover, since $\bar{\Gamma} = \Gamma' \times_D \Gamma' - \Delta$ and $\Gamma' \times_D \Gamma' = (\alpha \times \alpha)^{-1}(\Delta_D \times_D \Delta_D)$, straightforward computation yields

$$\begin{aligned} \int c_1(\sigma^*I^{-1}) &= - \int \alpha^*c_1(\Omega_D^1) + c_1(\Omega_{\Gamma'}^1) \\ &= \int c_1(F^0(\Omega_{\Gamma'}^1/D)^{-1}) = v_2. \end{aligned}$$

Hence the degree of the curve $r'' \rightarrow G^*$ is equal to

$$2(j-1)\rho - v_2.$$

On the other hand, since this map is equal to the composition

$$r'' \xrightarrow{\pi} \overline{r} \xrightarrow{p} r' \xrightarrow{\alpha} D' \xrightarrow{\gamma} G \cong G^* ;$$

since the rank of D' (equal to the degree of the 1st associated curve γ of D') is equal to ϵ_1 (by definition), and α and p have degrees j and $j-1$ respectively, the degree of the above curve is also equal to $j(j-1)\epsilon_1$. This proves (IV).

(III) We shall first prove

$$(*)j \quad j\epsilon_1 + v_2 = j(\mu_0 - 2)\epsilon_0 - 2\delta - (j-2)(r^2)$$

$$(**)j \quad \rho = (\mu_0 - 1)j\epsilon_0 - (j-1)(r^2) - v_2.$$

Let g denote the genus of r . Then the adjunction formula gives

$$2g - 2 + 2\delta = (r.K_Z + r) = (r.K_Z + (j-1)r - (j-2)r),$$

$$j\epsilon_1 - 2j\epsilon_0 + v_2 + 2\delta = (\mu_0 - 4)j\epsilon_0 - (j-2)(r^2),$$

from lemma 3, and from the Riemann-Hurwitz formula applied to $\alpha: r' \rightarrow D'$, as well as the formula for the rank of D' .

To prove $(**)j$ consider

$$\begin{aligned} \rho &= \int h \cap ([\varphi^{-1}r] - [E]) \quad (\text{by definition}) \\ &= \int (\mu_0 - 1)h \cap [\varphi^{-1}r] - (j-1)(\varphi^{-1}r^2) + (E^2) \\ &\quad (\text{by lemma 3 and the projection formula}) \\ &= (\mu_0 - 1)j\epsilon_0 - (j-1)(r^2) - v_2. \end{aligned}$$

Eliminating (r^2) from $(*)j$ and $(**)j$ and using (IV) to eliminate ϵ_1 , we get (III).

Q.E.D.

Corollary 1:

$$(i) \quad jv_2 = j(j-1)\{2(\mu_0 - j)\epsilon_0 - j\epsilon_1\} - 4(j-1)^2\delta$$

(Noether [N], see [B], p.163).

$$(ii) \quad \mu_2 = \mu_0(\mu_0 - 1)^2 - (j-1)\{(3j+1)\mu_0 - 2j(j+1)\}\epsilon_0$$

$$+ j^2(j-1)\epsilon_1 + \frac{2}{j}(j-1)^2(2j+1)\delta$$

(Salmon [S], p.308; Noether [N], see [B], p.163).

Proof: (i) follows from (IV) and (III).

(ii) follows from (V), (II), (I) + (III), (i).

From (III) and $(**)j$ we deduce the following formula for the double points of r .

$$\text{Corollary 2:} \quad 2(j-1)\delta = j(j-1)(r^2) - j(j-1)\mu_0\epsilon_0 + jv_2.$$

This formula yields the triple point formula (Prop. 2) when X has ordinary singularities: then $j = 2$ and $\delta = 3t$.

Again, let us look at the intrinsic invariants. Note that the formulas of Prop.3 also hold in this case, so that we can use Thm.3 and Cor. 1 to obtain formulas for c_1^2 and c_2 . Then we use Noether's formula to obtain an expression for $\chi(O_Z) = p_a + 1$.

Proposition 5:

$$(i) \quad c_1^2 = \mu_0(\mu_0 - 4)^2 - (j-1)\{(3j-1)\mu_0 - 2j(j+3)\}\epsilon_0$$

$$+ j(j-1)^2\epsilon_1 + \frac{2}{j}(j-1)^2(2j-1)\delta.$$

$$(ii) \quad c_2 = \mu_0(\mu_0^2 - 4\mu_0 + 6) - (j-1)\{(3j+1)\mu_0 - 2j(j+2)\}\epsilon_0$$

$$+ j^2(j-1)\epsilon_1 + \frac{2}{j}(j-1)^2(2j+1)\delta.$$

$$(iii) \quad p_a = \binom{\mu_0-1}{3} - \frac{1}{6} j(j-1)(3\mu_0-2j-5)\varepsilon_0 \\ + \frac{1}{12} j(j-1)(2j-1)\varepsilon_1 + \frac{2}{3}(j-1)^2\delta \\ (\text{Noether [N], see [B], p.163,p.180}).$$

Remark 4: If X has curves of different multiplicities, the methods above still yield formulas.

As an example, suppose that X , in addition to a double curve of degree ε_0 , rank ε_1 , class of immersion ρ , with v_2 of the pinch-points of X on it, has a triple curve of degree t , rank r_t , class of immersion ρ_t , and with v_2^t of the pinchpoints of X on it, then, if r_t denotes the inverse image on Z of the triple curve, and if we set $i = (r, r_t)$, the following formulas are seen to hold (simply go through the proof of Thm. 3 and see where it has to be modified according to the intersections i of r with r_t).

$$(I) \quad \mu_1 = \mu_0(\mu_0-1) - 2\varepsilon_0 - 6t$$

$$(II) \quad \mu_1(\mu_0-2) = \rho + 2\rho_t + \kappa + v_2^t$$

$$(III) \quad \varepsilon_0(\mu_0-2) = \rho + \delta + i$$

$$(III_t) \quad 3t(\mu_0-3) = 3\rho_t + 4\delta_t + i$$

$$(IV) \quad 2\rho - 2\varepsilon_1 = v_2$$

$$(IV_t) \quad 4\rho_t - 6r_t = v_2^t$$

$$(V) \quad \mu_2 + 2v_2 + 2v_2^t = \mu_1 + \kappa.$$

Application 1. The double surface of a 3-fold with ordinary singularities in $\mathbb{P}^4$. (See Roth [R].)

Let $V \subseteq \mathbb{P}^4$ be a 3-fold with ordinary singularities ([R]), and let $B \subseteq V$ denote its double surface. Then B has a triple curve. Let $B_0 \subseteq \mathbb{P}^3$ be a generic projection of B - then B_0 has in addition to the triple curve a double curve. With the notations of Remark 4, the class μ_2 of B_0 is given by

$$\mu_2 = \mu_0(\mu_0-1)^2 - \varepsilon_0(3\mu_0-4) - 4t(2\mu_0-3) \\ + \delta + \frac{8}{3}\delta_t + \frac{5}{3}i - 3v_2^t - 2v_2.$$

The terms involving only the double curve (i.e., the 1st, 2nd, 4th, and last term) agree with the corresponding terms as found by Roth ([R], (42)):

$$\mu_2 = \mu_0(\mu_0-1)^2 - \varepsilon_0(3\mu_0-4) - 12t - 15r_t + \delta \\ - 60\xi - \frac{13}{2}v_2^t - 2v_2.$$

(here ξ denotes the number of quadruple points on the triple curve; these points are 6-tuple for B_0 and lie also on the double curve). The other terms, however, do not agree, even when we substitute for r_t and try to interpret i and δ_t in terms of ξ . This sad fact is less surprising when one notices that Roth's formula which corresponds to (IV_t) of Remark 4, reads ([R], (37)):

$$2\rho_t - 4r_t = v_2^t.$$

Roth obtains it by applying Chasles' correspondence theorem: If the points on a line are in (α, β) correspondence, then their number of united points is equal to $\alpha + \beta$.

The correspondence in question is the one defined by the tangent planes to B_0 at points of its triple curve on a given line in $\mathbb{P}^3$. This correspondence is clearly $(2\rho_t, 2\rho_t)$ - as Roth remarks - and the united points are due to the pinchpoints v_2^t and to the points where the tangent line to the triple curve meets the line. There are ρ_t such tangent lines. Roth

deduces the above formula from this, while it seems that, in this correspondence, since we're considering triples of points on the line, a coincidence of all the three points should count 6 times and thus the correct formula should indeed be (IV_t) (in addition to the fact that its proof given for a j -tuple curve in general, seems to be OK). Moreover, Thm. 3 (IV) is in agreement with Noether's formulas.

5. Surfaces with a double and a cuspidal curve

Let Z be a smooth surface and $f: Z \rightarrow \mathbb{P}^3$ a finite map. Now we shall not assume that the ramification locus of f is finite. Thus $X = f(Z)$ might have a "cuspidal curve" - equal to the image by f of the 1-dimensional part of the ramification locus - in addition to isolated pinchpoints. Such a situation occurs for example if f is a nongeneric projection of an embedding $Z \subseteq \mathbb{P}^N$, or if f is the dual map of a smooth surface $Z \subseteq \mathbb{P}^3$.

Here we shall consider the case that X has only an "ordinary" double curve and an "ordinary" cuspidal curve. To be precise, we shall assume $f: Z \rightarrow f(Z) = X \subseteq \mathbb{P}^3$ satisfies the conditions (1), (3) and (4) of Section 3, together with

(0)' $\text{char } k \neq 2, 3$

(2)_{b+c}: The singular locus of X is as follows.

There is a curve D_b of ordinary double points. At all but a finite number of points of D_b the condition (2) of Section 3 is satisfied. The remaining points of D_b lie also on the cuspidal curve D_c , D_c being the image of the 1-dimensional part of the ramification locus of f . At a general point $x \in D_c$, the completion of the local ring of X is isomorphic to

$$k[[t_1, t_2, t_3]] / (t_1^3 - t_2^2).$$

The ramification ideal $F^0(\Omega_{Z/X}^1)$ has no embedded components.

Let $\varphi: \bar{Z} \rightarrow Z$ denote the blowup of the isolated ramification points of f , and set $E \subseteq \bar{Z}$ equal to the exceptional divisor. If $R \subseteq Z$ denotes the ramification scheme of f , defined by $F^0(\Omega_{Z/X}^1)$, we let Γ_c be the 1-dimensional part of R (hence $\varphi^{-1}R = \varphi^{-1}\Gamma_c \cup E$). Let $\underline{C} = \text{Hom}_{O_X}(f_*O_Z, O_X)$ denote the conductor of Z in X , and D (resp. r) the subscheme of X (resp. Z) defined by $\underline{C}$. From the above assumptions it follows that

$$[D] = [D_b] + [D_c]$$

holds in $A_1 X$. Now let Γ_b denote the reduced curve on Z satisfying

$$[\Gamma_b] = [f^{-1}\Gamma_b] \quad \text{in } A_1 Z$$

(beware that $f^{-1}\Gamma_b$ might have embedded components).

Lemma 4.

$$(i) \quad [r] = [\Gamma_b] + 2[\Gamma_c]$$

$$(ii) \quad K_Z = (\mu_0 - 4)H - [\Gamma_b] - 2[\Gamma_c]$$

$$(iii) \quad h^0(\bar{Z}) = (\mu_0 - 1)H - [\varphi^{-1}\Gamma_b] - 3[\varphi^{-1}\Gamma_c] - [E].$$

Proof: Similar to the proofs of the lemmas of Section 3.

We shall now introduce the various numerical characters of the surface $X \subseteq \mathbb{P}^3$ (compare [S], p.314; [Z], p.450, but note that the situation here is simpler since we assume X has no isolated singularities).

The characters $\mu_0, \mu_1, \mu_2, \kappa$, and $\nu_2 = -(E^2)$ are defined as before. We set

b = degree of D_b

q = rank of D_b

ρ = class of immersion of D_b in X

ν_b = # ramification points of $\Gamma_b \rightarrow D_b$

$v'_b = \#$ ramification points of $\Gamma'_b \rightarrow D'_b$

$v''_b = \#$ ramification points of $D'_b \rightarrow D_b$

$$\delta_b = g_a(\Gamma_b) - g(\Gamma_b).$$

$c =$ degree of D_c

$r =$ rank of D_c

$\sigma =$ class of immersion of D_c in X

$v_c = \#$ ramification points of $\Gamma'_c \rightarrow D_c$

$$\delta_c = g_a(\Gamma_c) - g(\Gamma_c)$$

$$i = (\Gamma_b, \Gamma_c).$$

The relations among the above characters that we are going to prove are stated in the next theorem. Unfortunately there is one relation we are unable to prove at the present time. We give it in a conjectural form (IV_c) together with another relation (III_c) we deduce from it, so as to get a picture of the complete set of relations.

Theorem 4: (I) $\mu_1 = \mu_0(\mu_0 - 1) - 2b - 3c$

(II) $\mu_1(\mu_0 - 2) = \kappa + \rho + 2\sigma$

(III_b) $b(\mu_0 - 2) = \rho + \delta_b + v''_b + i$

(IV_b) $2\rho - 2q = v'_b$

(V) $\mu_2 + 2v_2 + \sigma = \mu_1 + \kappa$

Conjecture: (III_c) $c(\mu_0 - 2) = 2\sigma + 2\delta_c + 2v_c + i$

(IV_c) $2\sigma - r = (\Gamma_c^2) - v_c.$

Proof. (I) follows from lemma 4 (iii). The proof of (V) goes as in thm. 2. Note that

$$\sigma = \int^v h \cap [\Gamma_c] \quad (\text{by definition}), \text{ hence}$$

$$\sigma = (\mu_0 - 1)c - i - 3(\Gamma_c^2) \quad (\text{by lemma 4 (iii)}).$$

To prove (II) we compute $\mu_2 = \int^v h^2$, using again lemma 4 (iii). We obtain

$$\mu_2 = (\mu_0 - 1)\mu_1 - \rho - 3\sigma - 2v_2,$$

which together with (V) yields (II).

The proof of (IV_b) is just like the proof of Thm. 2 (IV). From this and (*_b) of the next lemma, (III_b) follows.

Lemma 5: (*_b) $2q + 2v''_b + v'_b + 2\delta_b = 2b(\mu_0 - 2) - 2i$

(*_c) $r + v_c + 2\delta_c = c(\mu_0 - 2) - (\Gamma_c^2) - i.$

Proof: The adjunction formula for $\Gamma_b \subseteq Z$ gives (with lemma 4 (ii))

$$\begin{aligned} 2g_a(\Gamma_b) - 2 &= \int (\mu_0 - 4)h \cap [\Gamma_b] - 2(\Gamma_b, \Gamma_c) \\ &= 2b(\mu_0 - 4) - 2i. \end{aligned}$$

By definition of δ_b , and by the Riemann-Hurwitz formula applied to the map $\Gamma'_b \rightarrow D'_b$ between the normalizations of Γ_b and D_b ,

$$2(2g(D_b) - 2) + v'_b + 2\delta_b = 2b(\mu_0 - 4) - 2i.$$

Then apply the formula for the rank, q , of D_b ,

$$q = 2b + 2g(D_b) - 2 - v''_b$$

to get (*_b).

Similarly,

$$\begin{aligned} 2g_a(\Gamma_c) - 2 &= c(\mu_0 - 4) - i - (\Gamma_c^2), \\ r - 2c + v_c + 2\delta_c &= c(\mu_0 - 4) - i - (\Gamma_c^2). \end{aligned}$$

Q.E.D.

Corollary 3: $(\Gamma_b^2) = \mu_0 b + \delta_b - v_2 - 2i.$

Proof: Compute $\rho + (\Gamma_b^2)$, using $\rho = \int^v h \cap ([\varphi^{-1}\Gamma_b] - [E])$ and lemma 4 (iii). Then substitute for ρ , using Thm. 4 (III_b).

Q.E.D.

Remark that Cor. 3 can be viewed as a formula for the singularities δ_b of Γ_b , and generalizes the triple point formula Prop. 2.

In the same manner we see that, given (*_c) of Lemma 5, the con-

jectured formula (IV_C) is equivalent to $(IV_C)'$ $6(\Gamma_C^2) = \mu_0 c + 2\delta_C + 2v_C - i$.

Again, $(IV_C)'$ can be viewed as formula for the singularities δ_C of Γ_C , and this suggests that there should exist a "triple point formula" for X , giving the "number of singularities" of the "double point scheme" of $f: Z \rightarrow X$ (recall that Γ was defined by the conductor of Z in X), and that we should be able to obtain $(IV_C)'$ from such a formula.

Now Kleiman has a higher order multiple point theory (see [K], Ch. V, D), which he has generalized to local complete intersection maps (unpublished). It is easy to show (by reduction to hyperplane sections of X) that the formula giving the number of triple points of the map $f: Z \rightarrow X$ does apply in the present case. However, how to compute the triple points of f in terms of the characters given here, is not so clear (this requires an analysis of the double point scheme of f , so as to understand how much it differs from Γ ; the problem is mainly how to "count" the intersections of Γ_D and Γ_C).

The formula would look like:

$$(T) \quad * \text{ triple points of } f = (\Gamma_D^2) + 4(\Gamma_C^2) + 4i - \mu_0(b+c) + c_2(v)$$

where $c_2(v)$ denotes the degree of the 2nd Chern class of the virtual conormal bundle $f^*\Omega_{\mathbb{P}^3} - \Omega_Z^1$. A purely formal computation shows:

$$c_2(v) = c_1^2 - c_2 + 4 \int h \cap K_Z + 6\mu_0.$$

Since we have an expression for K_Z , we can find one for c_1^2 , but it is not so clear how to find an expression for c_2 . We prefer not to make further comments on this matter, but hope to report on the question later (see Appl. 3 below, however).

We shall now apply a result of ([P1]), as we did in Section 3, to obtain an expression for the cuspidal curve of the dual surface ${}^v X$ of X .

Let ${}^v \pi: \bar{Z} \rightarrow {}^v X = {}^v(\bar{Z})$ denote the dual map. We let $R \subseteq \bar{Z}$ denote the ramification scheme of $f \circ \pi: \bar{Z} \rightarrow X$, so that R is defined by the ideal $F^0(\Omega_{\bar{Z}/X}^1)$, and $R^* \subseteq \bar{Z}$ denotes the ramification scheme of ${}^v \pi$, defined by $F^0(\Omega_{\bar{Z}/{}^v X}^1)$. Then ([P1], 3.4) yields a formula

$$[R^*] - [R] = 4(h-h) \cap [\bar{Z}] \text{ in } A_1 \bar{Z}.$$

Proposition 6: (i) $\deg(f \circ \pi)_*[R^*] = 4(\mu_0 - 2)\mu_0 - 8b - 11c$

$$(ii) \quad \deg {}^v \pi_*[R^*] = 4\mu_0(\mu_0 - 1)(\mu_0 - 2) - 8b(2\mu_0 - 3) - c(23\mu_0 - 35) + 4(\Gamma_D^2) + 33(\Gamma_C^2) + 23i + 2v_2.$$

Proof: (i) $(f \circ \pi)_*[R^*] = 4(h-h) \cap [X] + (f \circ \pi)_*[R]$
 $= 4(\mu_0 - 2)H - 4[\Gamma_D] - 12[\Gamma_C] + [\Gamma_C].$

(ii) From the proof of ([P1], 3.4) it follows:

$$[R] = (c_1(\Omega_{\bar{Z}}^1) - \pi^* c_1(\Omega_Z^1)) \cap [\bar{Z}] + [\Gamma_C] + [E] = [\Gamma_C] + 2[E].$$

Hence we get (use also lemma 4 (iii)):

$$[R^*] = 4(\mu_0 - 2)h \cap [\bar{Z}] - 4[\Gamma_D] - 11[\Gamma_C] + 2[E]$$

$$\deg {}^v \pi_*[R^*] = \int {}^v h \cap [R^*] = 4(\mu_0 - 2)(\mu_0 - 1)\mu_0 - 4(\mu_0 - 1) \cdot 2b - 11(\mu_0 - 1)c - 4(\mu_0 - 2) \cdot 2b + 4(\Gamma_D^2) + 11i - 12(\mu_0 - 2)c + 12i + 33(\Gamma_C^2) + 2v_2,$$

$$\deg {}^v \pi_*[R^*] = 4\mu_0(\mu_0 - 1)(\mu_0 - 2) - 8b(2\mu_0 - 3) - (23\mu_0 - 35)c + 4(\Gamma_D^2) + 33(\Gamma_C^2) + 23i + 2v_2.$$

Q.E.D.

Application 2 (see [R], p. 549): Surface of contact.

Suppose $Y \subseteq \mathbb{P}^N$ is a smooth threefold. In the same way that we defined the curve of contact $C \subseteq Z$ with respect to a generic projection $Z \rightarrow \mathbb{P}^2$ of a smooth surface Z , we define the surface of contact $A \subseteq Y$ to be the subscheme defined by the ramification ideal $F^0(\Omega_{Y/\mathbb{P}^3}^1)$ of a generic projection $p: Y \rightarrow \mathbb{P}^3$. Set $A_0 = p(A)$. One can show that

A_0 has a double and a cuspidal curve in the sense of this section. Let us use Roth's notation: A_0 has degree a , rank m , a double curve of degree δ and containing $3\mu_2$ triple points, a cuspidal curve of degree κ , meeting the double curve in μ_4 points which are stationary on the cuspidal curve and in μ_{3-2} points which are stationary on the double curve. In order to find an expression for the class, w , of A_0 , he quotes a formula of Cayley (see [S], p. 319):

$$w = 3\mu_2 - 3\binom{a}{3} + \frac{3}{2}am - 2m + 8\mu_4 + \frac{9}{2}\mu_{3-2}.$$

If we use Thm. 4(I), this formula is the same as

$$w = a(a-1)^2 - \frac{1}{2}(2\delta+3\kappa)(3a-4) + 3\mu_2 + 8\mu_4 + \frac{9}{2}\mu_{3-2}.$$

Assume the following:

- (a) the conjecture (III_c) holds
- (b) the intersection number i of the pullbacks to A of the double and the cuspidal curve of A_0 is equal to $2\mu_4 + \mu_{3-2}$.

Then the above formula for the class w is a corollary of Thm. 4.

Application 3: The dual of a smooth surface

The preceding results apply in particular to the case that $X \subseteq \mathbb{P}^3$ is the dual surface of a smooth surface in $\check{\mathbb{P}}^3$. So assume $Z \subseteq \check{\mathbb{P}}^3$ is a smooth (non-linear) surface and let $f: Z \rightarrow f(Z) = X \subseteq \mathbb{P}^3$ denote its dual map (which we assume to be separable).

Lemma 6: The dual map $f: Z \rightarrow \mathbb{P}^3$ of the smooth surface $Z \subseteq \check{\mathbb{P}}^3$ is finite.

Proof: Suppose $E \subseteq Z$ is a curve. Then $\int h n[E] > 0$. Since $h = (n-1)\check{h}$, where $n = \text{degree of } Z \text{ in } \check{\mathbb{P}}^3$, it follows that $\int h n[E] > 0$, i.e., $\deg f_*[E] > 0$, hence f does not collapse E . Q.E.D.

We shall now compute the characters of X in terms of the degree n of Z in $\check{\mathbb{P}}^3$; we keep the notation of this section.

Lemma 7. (i) $[r_c] = 4(n-2)\check{H}$
 (ii) $[r_b] = (n-2)(n^3 - n^2 + n - 12)\check{H}$.

Proof: (i) follows from the proof of Prop. 6.

To prove (ii), we use Lemma 4 (iii) together with (i), and the fact that $\mu_0 = n(n-1)^2$ holds (Thm. 1, (i)).

Q.E.D.

In order to show that we actually obtain Salmon-Cayley-Zeuthen's formulas for the characters of $X \subseteq \mathbb{P}^3$, we have to make two assumptions - one is that Conj. (IV_c) holds, and the other has to do with how the singularities of $r \subseteq Z$ are. Namely, we assume $\delta_c = 0$ and $v_c = v'_b = \beta$ say, and that if we set $\gamma = v''_b$, then we have $\delta_b = 3\beta + \gamma$ and $i = 2\beta + \gamma$. Note that these last assumptions are in agreement with the geometric description of the singularities of X given by Salmon-Cayley-Zeuthen ([S],[Z]). We shall not, however, try to justify them further here - remark only that they follow if one shows that the intersections of D_b and D_c are of two types: "type γ " (see Fig. 1) and "type β " (see Fig. 2).

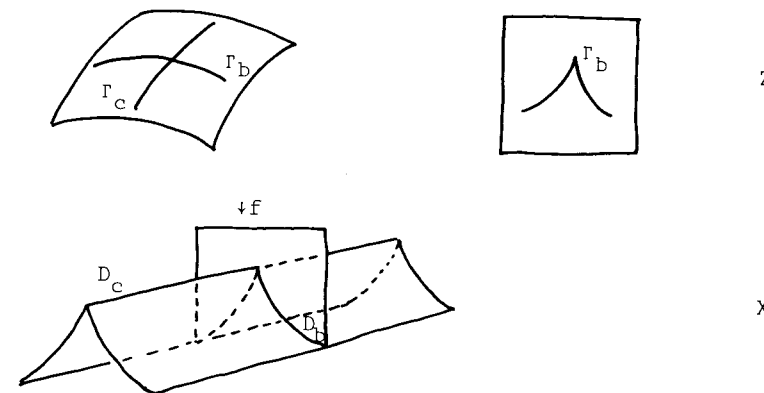


Fig. 1. The "real picture" of a point of "type γ ".

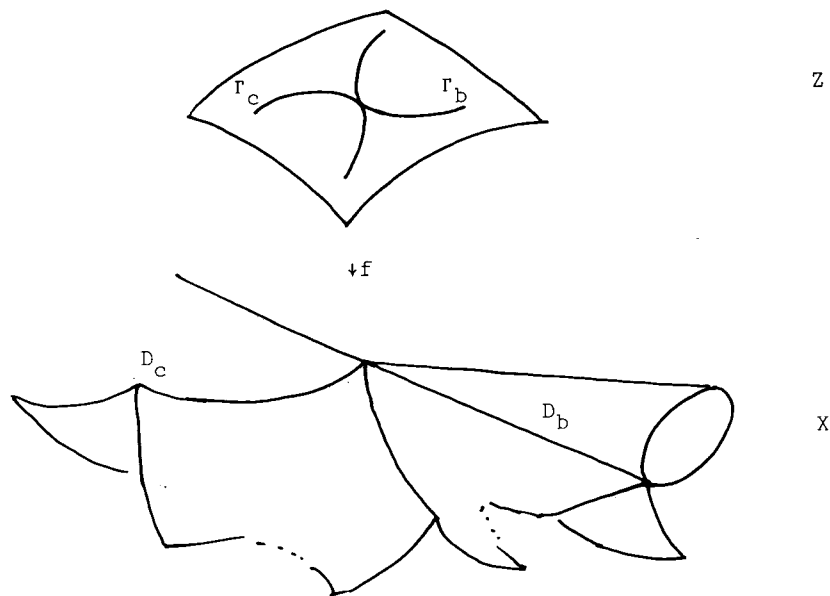


Fig. 2. The "real picture" of a point of "type β ".

Recall that n denotes the degree of Z in $\mathbb{P}^3$. From Thm. 2 it follows, since Z is smooth, that Z has rank $n(n-1)$, its class is $n(n-1)^2$, and if $C^* \subseteq Z$ is the curve of contact w.r.t. a generic projection $Z \rightarrow \mathbb{P}^2$ then C (which has degree $n(n-1)$ and class $n(n-1)^2$) acquires $n(n-1)(n-2)$ cusps in $\mathbb{P}^2$. From one of the Plücker formulas ([P1], 3.9(III)) it follows that the genus g of C^* satisfies

$$2g - 2 = n(n-1)(2n-5),$$

and from the other ([P1], 3.9(I)) that C^* acquires $\frac{1}{2}n(n-1)(n-2)(n-3)$ nodes. (Compare with [B], p. 150, p. 155.)

If $C \subseteq Z$ denotes the curve of contact w.r.t. a generic projection $Z \rightarrow \mathbb{P}^2$ of $f: Z \rightarrow \mathbb{P}^3$, we let δ denote the "number of apparent nodes" of C , i.e., the number of ordinary double points acquired by C in $\mathbb{P}^2$. (Recall that κ , by definition, is the

number of cusps acquired by C .) Likewise, we let k (resp. h) denote the number of apparent nodes of r_b (resp. r_c). From our assumptions it follows that r_c is smooth ($\delta_c = 0$), that $r_c \rightarrow D_c$ has β ramification points (whose images are cusps on D_c , see Fig. 2), that D_b and r_b have γ cusps, while D_b has t triple points and r_b has $3t$ double points, $r_b \rightarrow D_b$ has $\gamma + \beta$ ramification points, while $r'_b \rightarrow D'_b$ ramifies at β points and $D'_b \rightarrow D_b$ at γ points.

Applying Thm. 4, and the conjecture, as well as the Plücker formulas (as referred to above for C^*) to find δ , k , and h , we obtain the following expressions, which agree with Salmon-Cayley-Zeuthen's ([S], p. 602).

- (1) $\mu_0 = n(n-1)^2$
- (2) $\mu_1 = n(n-1)$
- (3) $\mu_2 = n$
- (4) $\delta = \frac{1}{2}n(n-2)(n^2-9)$
- (5) $\kappa = 3n(n-2)$
- (6) $b = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$
- (7) $k = \frac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464)$
- (8) $t = \frac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960)$
- (9) $q = n(n-2)(n-3)(n^2+2n-4)$
- (10) $\rho = n(n-2)(n^3-n^2+n-12)$
- (11) $c = 4n(n-1)(n-2)$
- (12) $h = n(n-2)(8n^4-32n^3+40n^2-54n+78)$
- (13) $r = 2n(n-2)(3n-4)$
- (14) $\sigma = 4n(n-2)$
- (15) $\beta = 2n(n-2)(11n-24)$
- (16) $\gamma = 4n(n-2)(n-3)(n^3+3n-16)$

Remark 5: The formulas (6), (8), and (16) are also proved in ([V], 8.5(ii),(iii), 9.5.1(ii)) - the approach there being the study of order of contact of Z with lines and planes in $\mathbb{P}^3$.

As remarked earlier, the triple point formula (T) applies in the present case. Moreover, one is able to compute the right hand side of (T):

$$(T)_n \text{ \#triple points of } X = \frac{1}{2}n(n^8 - 6n^7 + 15n^6 - 35n^5 + 84n^4 - 123n^3 + 158n^2 - 256n + 192)$$

In fact, we use Lemma 7 for the first two terms, then (1), (6), and (11) for the next, and finally Prop. 1 (i), (ii), and Lemma 4 (ii) to compute

$$\begin{aligned} c_2(v) &= n(n-4)^2 - n(n^2 - 4n + 6) + 4n(n-1)(n-4) + 6n(n-1)^2 \\ &= 2n(5n^2 - 18n + 16) \end{aligned}$$

Since the obtained expression for the number of triple points also turns out to be equal to

$$3t + 3\beta + 3\gamma,$$

using (8), (15), and (16), it seems reasonably clear that this is in fact the right way to count the triple points of X , though again we do not try to justify further the coefficients of β and γ here.

Example 3. A smooth cubic.

Suppose $n = 3$. The double curve D_b of X has degree $b = 27$, and its class of immersion (by definition equal to the degree of Γ_b in $\mathbb{P}^3$) is also 27 - in fact $\Gamma_b \subseteq Z$ is equal to the union of the 27 lines on Z . And D_b has 45 triple points while D_c has 54 cusps, so that there are 99 triple points altogether on X (there are no points of "type γ ").

The curve C^* has genus 4, while C has genus 1. The

cuspidal curve D_c has genus 19.

(See [S], pp. 184. Salmon was led to his theory of reciprocal surfaces by a study of the reciprocal of a cubic. He says "The first attempt to explain the effect of nodal and cuspidal lines on the degree of the reciprocal was made in the year 1847, in two papers which I contributed to the Cambridge and Dublin Mathematical Journal, II, p. 65, and IV, p. 188. It was not till the close of the year 1849, however, that the discovery of the twenty-seven right lines on a cubic by enabling me to form a clear conception of the nature of the reciprocal of a cubic, led me to the theory in the form here explained ..." ([S], Footnote on p. 301).)

Example 4. A smooth quartic.

Suppose $n = 4$ (as is well known, Z is then a $K3$ surface).

The double curve D_b has degree $b = 480$, $\gamma = 1920$ cusps, and $t = 3200$ triple points. It has genus 561.

The cuspidal curve has degree $c = 96$, $\beta = 320$ cusps, and genus 129.

In all, X has $t + \beta + \gamma = 5440$ triple points!

The curve of contact $C^* \subseteq Z$ with respect to a generic projection $Z \rightarrow \mathbb{P}^2$ has genus 19, while the curve of contact $C \subseteq Z$ w.r.t. a generic projection $Z \rightarrow \mathbb{P}^2$ has genus 9.

Bibliography

- [A] M. Artin, On Enriques Surfaces, Doctoral thesis, Harvard University, 1960.
- [B] H.F. Baker, Principles of geometry. Vol.VI. Introduction to the theory of algebraic surfaces and higher loci. Cambridge Univ.Press, 1933.
- [F] W. Fulton, "Rational equivalence on singular varieties". Publ.Math.IHES, 45 (1976), Paris.
- [K-L] G. Kempf, D.Laksov, "The determinantal formula of Schubert calculus". Acta Math., 132 (1974), 153-162.
- [K] S. Kleiman, "The enumerative theory of singularities". Proceedings of the Nordic Summer School, Oslo 1976(Noordhoff) .
- [La] D. Laksov, "Secant bundles and Todd's formula for the double points of maps into $\mathbb{P}^n$ ". Preprint, M.I.T., 1976.
- [Ll] E. Lluís, "De las singularidades que aparacen al proyectar variedades algebraicas". Bol.Soc.Mat. Mexicana, Ser. 2, 1 (1956), 1-9.
- [N] M. Noether, "Sulle curve multiple di superficie algebriche". Ann.d.Mat., 5 (1871-3), 163-177.
- [P1] R. Piene, "Polar classes of singular varieties". Ann. scient. Éc. Norm. Sup. t. 11, 1978, fasc. 2.
- [P2] R. Piene, "A proof of Noether's formula for the arithmetic genus of an algebraic surface". Preprint, M.I.T., 1977.
- [P3] R. Piene, "Numerical characters of a curve in projective n-space". Proceedings of the Nordic Summer School, Oslo 1976 (Noordhoff).
- [R1] J. Roberts, "Generic projections of algebraic varieties". Am.J.Math., 93 (1971), 191-215.
- [R2] J. Roberts, "Singularity subschemes and generic projections". Trans.AMS, 212 (1975), 229-268.

- [R3] J. Roberts, "Variations of singular cycles in an algebraic family of morphisms". Trans.AMS, 168 (1972), 153-164.
- [R] L. Roth, "Some formulae for primals in four dimensions". Proc. London Math.Soc., Ser. 2, 35 (1933), 540-550.
- [S] G. Salmon, A treatise on the analytic geometry of three dimensions. Vol. II, 5th ed., Dublin 1915.
- [SAG 7] N. Katz, "Pinceaux de Lefschetz: théorème d'existence". Exp. XVII in SGA 7, II, Springer L.N.M., 340.
- [S-R] J.G. Semple, S. Roth, Introduction to algebraic geometry. Oxford 1949.
- [V] I. Vainsencher, On the formula of de Jonquières for multiple contacts. Doctoral thesis, M.I.T., 1976.
- [W] A. Wallace, "Tangency and duality over arbitrary fields". Proc. London Math.Soc., Ser. 3, 6 (1956), 321-342.
- [Z] H.G. Zeuthen, "Révision et extension des formules numériques de la théorie des surfaces réciproques". Math.Ann., 10 (1876), 446-546.

Institute of Mathematics
University of Oslo
P.O.Box 1053 Blindern
Oslo 3, Norway