Long paper's journey into arXiv: "Node polynomials for curves on surfaces"

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IMPA Virtual
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The timeline

- CAS, Oslo 1994
- Di Francesco–Itzykson 1994, Vainsencher 1995
- Institut Mittag-Leffler 1996
- Göttsche 1998
- Bologna 1997: the B-paper became the H-paper 1999
- The methods and applications paper 2004
- The lynx paper 2011
- arXiv:2202.11611
- and in between there was MIT, Oslo, Rio, Belo Horizonte, ...

CAS 1994, Mittag-Leffler 08.10.1996

The associativity equation $A(1,2,3,3)$ yields the recursion formula,

$$N_\beta = \sum N_{\beta_1} N_{\beta_2} \langle \beta_1 \cdot \beta_2 \rangle \langle E \cdot \beta_1 \rangle \left[\langle F \cdot \beta_2 \rangle \binom{k(\beta)-4}{k(\beta_1)-2} - \langle F \cdot \beta_1 \rangle \binom{k(\beta)-4}{k(\beta_2)-1} \right].$$

(This formula was worked out for the first time by Ragni Piene and the lecturer in March of 1994.) It turns out experimentally that, on writing $N(a,b;e)$ for N_β where $\beta = a[E] + b[F]$ on $\mathbb{F}_e$, we find the relation,

$$N(a,b;e) = N(a, b+a; e+2).$$

A conceptual explanation for it (explained to the lecturer by Sheldon Katz in September 1994) is this: $\mathbb{F}_e$ degenerates into $\mathbb{F}_{e+2}$, transforming F to F and E to $E+F$, while leaving the quantum cohomology invariant.

Steve 17.12.1997 Bologna

Let S be a smooth surface, $\mathcal{L}$ an invertible sheaf.
Set $N_r(\mathcal{L}) := \#$ of r-nodal curves in a general codimension r linear subsystem of $|\mathcal{L}|$.

Conjecture

If $\mathcal{L}$ ample enough, then $N_r(\mathcal{L})$ is a polynomial in the four Chern numbers of $(S, \mathcal{L})$.

Vainsencher: computed the polynomials for $r \leq 7$.

Kleiman–P: for $r = 7, 8$, but with "For $r = 8$, remains residual intersection theory in proof".

"*Göttsche*: marvelous conjecture for $\sum_r N_r(\mathcal{L})t^r$."

Hirzebruch 70

Enumerating singular curves on surfaces

Steven Kleiman and Ragni Piene

ABSTRACT. We enumerate the singular algebraic curves in a complete linear system on a smooth projective surface. The system must be suitably ample in a rather precise sense. The curves may have up to eight nodes, or a triple point of a given type and up to three nodes. The curves must also pass through appropriately many general points. The number of curves is given by a universal polynomial in four basic Chern numbers.

To justify the enumeration, we make a rudimentary classification of the types of singularities using Enriques diagrams, obtaining results like Arnold's. We show that the curves in question do appear with multiplicity 1 using the versal deformation space, Shustin's codimension formula, and Gotzmann's regularity theorem. Finally, we relate our work to Vainsencher's work with up to seven nodes.

Methods and applications 2004

Math. Nachr. **271**, 69 – 90 (2004) / **DOI** 10.1002/mana.200310182

Node polynomials for families: methods and applications

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Key words Enumerative geometry, nodal curve, node polynomial, Bell polynomial, Enriques diagram, Hilbert scheme, Göttsche's conjecture, quintic threefold, Abelian surface
MSC (2000) Primary: 14N10; Secondary: 14C20, 14H40, 14K05

We continue the development of methods for enumerating nodal curves on smooth complex surfaces, extending the range of validity. We apply the new methods in three important cases. First, for up to eight nodes, we prove Göttsche's conjecture about plane curves of low degree. Second, we prove Vainsencher's conjectural enumeration of irreducible six-nodal plane curves on a general quintic threefold in four-space, which is important for Clemens' conjecture and mirror symmetry. Third, we supplement Bryan and Leung's enumeration of nodal curves in a given homology class on an Abelian surface of Picard number 1.

The lynx paper 2011

Rend. Lincei Mat. Appl. 22 (2011), 411–451
DOI 10.4171/RLM/608

Algebraic Geometry — *Enriques diagrams, arbitrarily near points, and Hilbert schemes*, by STEVEN KLEIMAN, RAGNI PIENE, and with Appendix B by ILYA TYOMKIN, communicated on 11 February 2011.

ABSTRACT. — Given a smooth family F/Y of geometrically irreducible surfaces, we study sequences of *arbitrarily near* T-points of F/Y; they generalize the traditional sequences of infinitely near points of a single smooth surface. We distinguish a special sort of these new sequences, the *strict* sequences. To each strict sequence, we associate an ordered unweighted Enriques diagram. We prove that the various sequences with a fixed diagram form a functor, and we represent it by a smooth Y-scheme.

arXiv:2202.11611

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arXiv:2202.11611
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Title: Node polynomials for curves on surfaces
Authors: Steven Kleiman and Ragni Piene
Categories: math.AG
Comments: 24 pages
MSC-class: 14N10 (Primary) 14C20, 14H40, 14K05 (Secondary)
\\
We complete the proof of a theorem we announced and partly proved in [Math. Nachr., vol. 271 (2004), Thm. 2.5, p. 74]. The theorem concerns a family of curves on a family of surfaces. It has two parts. The first was proved in [Math. Nachr.]. It describes a natural cycle that enumerates the curves in the family with precisely r ordinary nodes. The second part is proved here. It asserts that, for $r\le 8$, the class of this cycle is given by a computable universal polynomial in the pushdowns to the parameter space of products of the Chern classes of the family.
\\ (https://arxiv.org/abs/2202.11611 , 34kb)

The main theorem

Let $\pi\colon F \to Y$ be a smooth projective family of surfaces, and D a relative effective divisor. Assume Y is Cohen–Macaulay and equidimensional. Fix an integer $r \geq 0$. Under certain precise genericity conditions, either $Y(r) := \{y \in Y \mid D_y \text{ has } r \text{ nodes}\}$ is empty, or it has pure codimension r; in either case, its closure $\overline{Y(r)}$ is the support of a natural nonnegative cycle $U(r)$, and if $r \leq 8$,

$$[U(r)] = \tfrac{1}{r!} P_r(a_1, \ldots, a_r) \cap [Y],$$

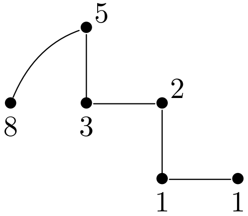
where the a_i are pushdowns of universal polynomials in the Chern classes of the family and P_r is the rth Bell polynomial.

Conjecture: True for all r.

Ingredients of the proof

- Enriques diagrams
- Arbitrarily near points
- Zariski clusters, complete ideals
- Relative Hilbert schemes
- Recursive relation: from $< r$ nodes to r nodes
- Correcting for non-reduced curves
- Intersection theory (generalized Fulton)
- Versal deformation space of a quadruple point

Enriques diagrams



The Enriques diagram corresponding to the Fibonacci singularity $f(x, y) = y^8 - x^{13}$.

Arbitrarily near points

$$
\begin{tikzcd}
 & & F^{(8)} \arrow[d, "\pi^{(8)}"] \\
W^{(7)} \subset X_2^{(7)} \subset F_2^{(7)} \arrow[d] \arrow[rr, hook] & & F^{(7)} \arrow[d, "\pi^{(7)}"] \\
W^{(6)} \arrow[r, hook] \arrow[d] & X_2^{(6)} \subset F_2^{(6)} \arrow[r, hook] \arrow[d] & F^{(6)} \arrow[d, "\pi^{(6)}"] \\
W^{(5)} \arrow[r, hook] \arrow[d] & X_2^{(5)} \subset F_2^{(5)} \arrow[r, hook] \arrow[d] & F^{(5)} \arrow[d, "\pi^{(5)}"] \\
W^{(4)} \arrow[r, hook] \arrow[d] & X_2^{(4)} \subset F_2^{(4)} \arrow[r, hook] \arrow[d] & F^{(4)} \arrow[d, "\pi^{(4)}"] \\
W^{(3)} \arrow[r, hook] \arrow[d] & X_2^{(3)} \subset F_2^{(3)} \arrow[r, hook] \arrow[d] & F^{(3)} \arrow[d, "\pi^{(3)}"] \\
W^{(2)} \arrow[r, hook] \arrow[d] & X_2^{(2)} \subset F_2^{(2)} \arrow[r, hook] \arrow[d] & F^{(2)} \arrow[d, "\pi^{(2)}"] \\
W^{(1)} \arrow[r, hook] \arrow[d] & X_2^{(1)} \subset F_2^{(1)} \arrow[r, hook] \arrow[d] & F^{(1)} \arrow[d, "\pi^{(1)}"] \\
X_4 \arrow[r, hook] & X_2 \arrow[r, phantom, "\subset"] & F \arrow[d, "\pi"] \\
 & & Y
\end{tikzcd}
$$

The non-reduced fibers

Assume $X_\varphi = \{x\}$.

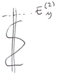
$(D_2^{(1)})_x$:

$W^{(1)}$:

Take $y \in W^{(1)}$.

If $y \neq p_i$:

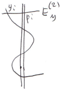
$(D_2^{(2)})_y$:



$W_y^{(2)}$:

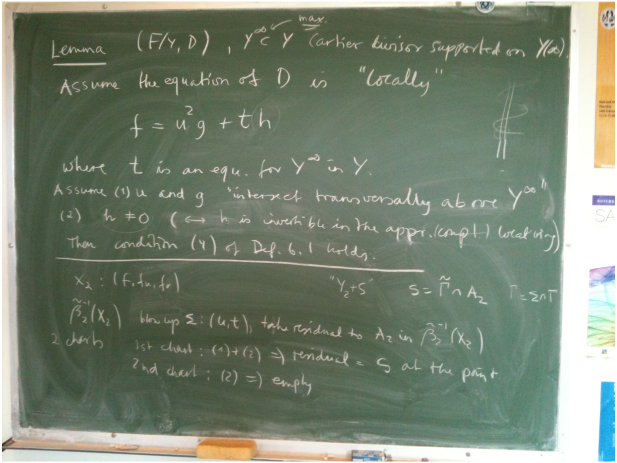
If $y = p_i$:

$(D_2^{(2)})_y$:



$W_y^{(2)}$:

Blackboard work



Related work and a conjecture

Laarakker (2018) showed part of our conjecture: he proved that for *all* r, $[U(r)]$ can be expressed as a universal polynomial in pushdowns of monomials in the Chern classes of the family, but did not prove that the node polynomials are Bell polynomials.

He applied this result to the enumeration of nodal plane curves in $\mathbb{P}^3$. Similar applications were done by *Mukherjee–Paul–Singh* (2019), and (allowing an additional non-nodal singularity) by *Basu–Mukherjee* (2019) and *Das–Mukherjee* (2021).

Conjecture

For any Enriques diagram $\mathbf{D}$, $[Y(\mathbf{D})]$ can be expressed as a universal polynomial in pushdowns of monomials in the Chern classes of the family.

Further results

Li–Tzeng (2014) and *Rennemo* (2017) showed the existence of universal polynomials for curves with arbitrary singularities in a linear system on a *fixed* surface.

Kazarian (2003) computed many explicit formulas for curves with a given multisingularity in a linear system on a fixed surface.

Bérczi–Szenes (arXiv:2112.15502) developed "a new approach to the study of the multipoint loci of holomorphic maps between complex manifolds", motivated by ideas of Kazarian and Rimanyi on Thom polynomials and residual polynomials.

Happy birthday, Steve!