

Discriminants, polytopes and toric geometry

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Resultants

Eliminate, eliminate, eliminate

Eliminate the eliminators of elimination theory.

S. S. Abhyankar (1970)

Let $f(x)$ and $g(x)$ be polynomials in one variable, of degrees m, n , with $n \geq m$.

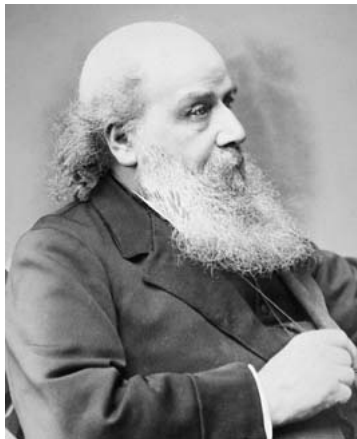
$$f(x) = a_m x^m + \cdots + a_0$$

$$g(x) = b_n x^n + \cdots + b_0$$

The *resultant* (or “eliminant”) of f and g is a polynomial in the coefficients of f and g such the vanishing of this polynomial ensures that f and g have a common solution.



James Joseph Sylvester (1814–1897)



The *Sylvester resultant* is a polynomial of degree $m + n$ in the coefficients of f and g .

It is the determinant of the $((m + n) \times (m + n))$ -matrix:

$$\text{Res}(f, g) = \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{vmatrix}$$



A student of Sylvester

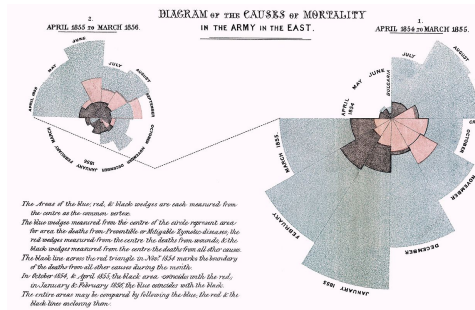


Figure: Diagram of the Causes of Mortality in the Army in the East

Florence Nightingale (1820-1910)



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Étienne Bézout (1730–1783)



The *Bézoutian* is the determinant

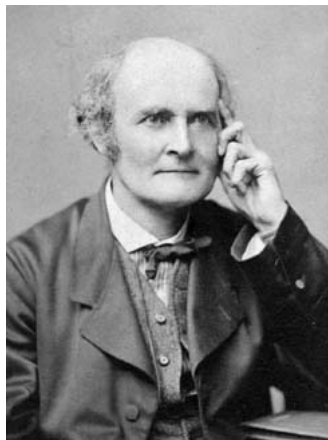
$$B(f, g) = \det(a_{r,s})$$

where $(a_{r,s})$ is the symmetric $(n \times n)$ -matrix with $a_{r,s}$ polynomials in the coefficients of f and g given by

$$\begin{aligned} & \frac{f(x)g(y) - f(y)g(x)}{x - y} \\ &= \sum_{r,s \geq 0} a_{r,s} x^r y^s \end{aligned}$$



Arthur Cayley (1821–1895)



Instead of the resultant $\text{Res}(f, g)$ we can consider the *discriminant* $\Delta(h)$ of

$$h(x, y) := f(x) + yg(x)$$

Indeed:

$$\partial h / \partial x = \partial f / \partial x + y \partial g / \partial x,$$

$$\partial h / \partial y = g,$$

so $f = g = 0$ at x_0 if and only

if $h = \partial h / \partial x = \partial h / \partial y = 0$

at (x_0, y_0) , where

$$y_0 = -\frac{\partial f / \partial x}{\partial g / \partial x}(x_0).$$



The Cayley trick

Consider $k + 1$ polynomials $f_i(x_1, \dots, x_k)$, and set

$$h(x_1, \dots, x_k, y_1, \dots, y_k) := \\ f_0(x_1, \dots, x_k) + y_1 f_1(x_1, \dots, x_k) + y_k f_k(x_1, \dots, x_k).$$

The *discriminant* $\Delta(h)$ of h is obtained by eliminating the x_i 's and y_i 's from the $2k + 1$ equations

$$h = 0, \partial h / \partial x_i = 0, \partial h / \partial y_i = 0.$$

Again, to find $\Delta(h)$ is essentially equivalent to finding $\text{Res}(f_0, \dots, f_k)$.

Geometric interpretation

Consider the embedding

$$\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^{m+n+1}$$

given by

$$(x, y) \mapsto (x, \dots, x^m, y, xy, \dots, x^n y)$$

The pullback of a hyperplane is equal to the set $\{h(x, y) := f(x) + yg(x) = 0\}$, where the coefficients of f and g are determined by the coefficients of the hyperplane equation.

The discriminant $\Delta(h)$ gives the relation on the coefficients of h for the hyperplane section to be singular.

Observe that this is an affine part of the rational normal scroll

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \rightarrow X \subset \mathbb{P}^{m+n+1}.$$

The toric interpretation

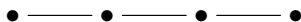
A convex lattice polytope gives a toric embedding.

The polytope P_0 :



corresponds to the toric embedding $\mathbb{C}^* \rightarrow \mathbb{P}^2$ given by $x \mapsto (1 : x : x^2)$; its closure X_{P_0} is a conic.

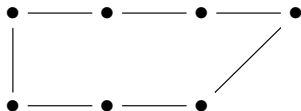
The polytope P_1 :



corresponds to the toric embedding $\mathbb{C}^* \rightarrow \mathbb{P}^3$ given by $x \mapsto (1 : x : x^2 : x^3)$; its closure X_{P_1} is a twisted cubic curve.

The Cayley sum

The polytope P



corresponds to the embedding

$$(\mathbb{C}^*)^2 \rightarrow \mathbb{P}^6$$

given by

$$(x, y) \mapsto (1 : x : x^2 : y : xy : x^2y : x^3y);$$

its closure X_P is a rational normal scroll of type $(2, 3)$.

Cayley polytopes

Let $P_0, \dots, P_k \subset \mathbb{R}^{n-k}$, $k \geq 1$, be lattice polytopes, all having the same normal fan.

The polytope

$$P = \text{Conv}\{P_0 \times e_0, \dots, P_k \times e_k\} \subset \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n,$$

where $e_0, \dots, e_k$ are the vertices of $\Delta_k \subset \mathbb{R}^k$, is called a *Cayley polytope*.

We write

$$P = P_0 \star \dots \star P_k.$$

A Cayley polytope is “hollow”: it has no interior lattice points.

Hyperplane sections and discriminants

Consider the toric embedding $X_P \subseteq \mathbb{P}^N$ defined by the polytope $P = P_0 \star \cdots \star P_k$.

A hyperplane section of X_P is given by

$$h(x_1, \dots, x_{n-k}, y_1, \dots, y_k) := f_0 + y_1 f_1 + \cdots + y_k f_k = 0,$$

where $f_i = 0$ is a hyperplane section of the toric embedding given by P_i .

Generalize the Cayley trick: determine the relation between

$$\text{Res}(f_0(x_1, \dots, x_{n-k}), \dots, f_k(x_1, \dots, x_{n-k})) \text{ and } \Delta(h).$$



Discriminants and dual varieties

If $k \leq n - k$, then the discriminant is a polynomial in the coefficients of h , and its zeros define a hypersurface in the space of coefficients: the *dual variety* $X_P^\vee \subseteq (\mathbb{P}^N)^\vee$ of X_P .

When $k > n - k$, the system $f_0 = \cdots = f_k = 0$ has more equations than variables. This implies that the discriminant of h consists of more than one polynomial and so the dual variety is not a hypersurface.

A variety X is called *defective* if its dual variety $X^\vee$ is not a hypersurface. A polytope P is defective if X_P is defective.

The *defect* of a defective variety X is the positive integer $\text{codim } X^\vee - 1$.

We conclude: If $k > n - k$, then the Cayley polytope $P = P_0 \star \cdots \star P_k$ is defective.

The degree of the dual variety

Assume P is regular, so that X_P is a smooth variety. Recall that the degree of the dual variety can be expressed in terms of the Chern classes of X_P and the line bundle L_P giving the embedding.

The Chern classes of X_P and L_P can be interpreted in terms of the faces of P . For example,

$$c_n(T_{X_P}) = \# \text{ vertices of } P,$$

$$c_1(L_P)^n = \text{Vol}_{\mathbb{Z}}(P),$$

$$c_i(T_{X_P})c_1(L_P)^{n-i} = \sum_{F_i} \text{Vol}_{\mathbb{Z}}(F_i),$$

where the sum is taken over all faces of P of codimension i .

Gelfand–Kapranov–Zelevinski:

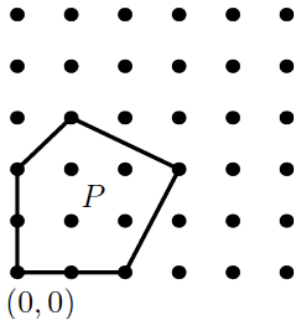
$$\deg X_P^{\vee} = \sum_{F \subseteq P} (-1)^{\text{codim} F} (\dim F + 1) \text{Vol}_{\mathbb{Z}}(F).$$



Lattice points

Let P be a lattice polytope. Count its lattice points:

$$\#(P \cap \mathbb{Z}^2) = 11$$

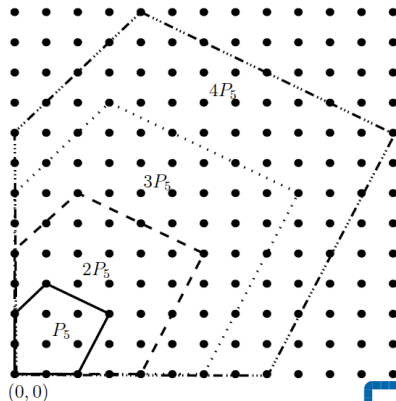


Dilated polytopes

Count the lattice points of the dilated polytopes mP .

$$\#(mP \cap \mathbb{Z}^2) = ?$$

(Thanks to Benjamin J. Braun for the figure.)



Ehrhart polynomials



The number of lattice points in the m th dilation of P ,

$$e_P(m) := \#(mP \cap \mathbb{Z}^n)$$

is a polynomial in m of degree $n = \dim P$.

Reciprocity: $e_P(-m) = (-1)^n \#(\text{int}(mP) \cap \mathbb{Z}^n)$

Why? If (X, L) is the toric embedding determined by P , then

$$e_P(m) = \chi(mL) = H^0(X, mL)$$

$$e_P(-m) = \chi(-mL) = (-1)^n \chi(mL - K_X) = (-1)^n H^0(X, mL - K_X),$$

and $(X, mL - K_X)$ is the toric embedding defined by $\text{int}(P)$.



Ehrhart series (R. Stanley)

The generating function of the Ehrhart polynomials is

$$E_P(t) := \sum e_P(m)t^m = \frac{\sum_{i=0}^n h_i t^i}{(1-t)^{n+1}},$$

where the h_i are non-negative integers such that $h_0 = 1$ and

- ▶ $\sum_{i=0}^n h_i = \text{Vol}_{\mathbb{Z}}(P),$
- ▶ $h_1 = \#(P \cap \mathbb{Z}^n) - (n+1),$
- ▶ $h_n = \#(\text{int}(P) \cap \mathbb{Z}^n).$

In the example, we get

$$E_P(t) = \frac{1 + 8t + 4t^2}{(1-t)^3}$$

which is equivalent to (Pick's formula!)

$$e_P(m) = \#(mP \cap \mathbb{Z}^2) = \frac{13}{2}m^2 + \frac{7}{2}m + 1.$$

The codegree of a polytope

The *codegree* of P is defined by

$$\text{codeg}(P) = \min_{\mathbb{N}} \{m \mid mP \text{ has interior lattice points}\}.$$

The *degree* of P is $\deg(P) = n + 1 - \text{codeg}(P)$.

Example

Δ_n = the standard n -simplex. Then $\text{codeg}(\Delta_n) = n + 1$ and $\text{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil$.

Example

$P = P_0 \star \cdots \star P_k$ a Cayley polytope. Then $\text{codeg}(P) \geq k + 1$.

The Cayley polytope conjecture

Batyrev–Nill: Given d , does there exist an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Haase, Nill, and Payne showed that for general polytopes, there is such a $N(d)$, and

$$N(d) \leq (d^2 + 19d - 4)/2.$$

Dickenstein–Di Rocco–P.: $N(d) = 2d + 1$, if P is regular and $\mathbb{Q}$ -normal.

Note that $n \geq 2d + 1$ is equivalent to $\text{codeg}(P) \geq \frac{n+3}{2}$.

The assumption of $\mathbb{Q}$ -normality was later shown to be unnecessary (Dickenstein–Nill).

Theorem (Dickenstein, Di Rocco, Nill, Piene)

Let P be a regular lattice polytope of dimension n . The following are equivalent

- (1) $\text{codeg}(P) \geq \frac{n+3}{2}$
- (2) $P = P_0 \star \cdots \star P_k$ is a regular Cayley polytope with $k + 1 = \text{codeg}(P)$ and $k > \frac{n}{2}$.
- (3) P is defective, with defect $2k - n$.

With the additional assumption in (1) that P is $\mathbb{Q}$ -normal, i.e., $\tau(P) = \text{codeg}_{\mathbb{Q}}(P)$, our proof (Dickenstein–Di Rocco–P.) is algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

The proof (Dickenstein–Nill) that $\text{codeg}(P) \geq \frac{n+3}{2}$ implies $\tau(P) = \text{codeg}_{\mathbb{Q}}(P)$, is purely combinatorial and uses Ehrhart polynomials and Ehrhart reciprocity.



Higher order dual varieties (higher discriminants)

The k th dual variety $X^{(k)}$ of $X \subset \mathbb{P}^m$ is defined as:

$$X^{(k)} = \overline{\{H \in \mathbb{P}^{m^\vee} \mid H \text{ is tangent to } X \text{ to the order } k\}}.$$

Here H is tangent to the order k means $H \supseteq \mathbb{T}_{X,x}^k$ (the k th osculating space) for some smooth point $x \in X$.

Note: $X^{(1)} = X^\vee$, $X^{(k-1)} \supseteq X^{(k)}$, $X^{(k)} \subseteq \text{Sing} X^\vee$, $k \geq 2$.

The dimension of $\mathbb{T}_{X,x}^k$ is $\leq \binom{n+k}{k} - 1$, $n = \dim X$. If equality holds for (almost) all x , we say that X is (generically) k -regular.

The *expected dimension* of $X^{(k)}$ is $n + m - \binom{n+k}{k}$. We say that X is k -defective if $\dim X^{(k)} < n + m - \binom{n+k}{k}$.

Toric threefolds

Theorem (Dickenstein–Di Rocco–P.)

Let $(X, \mathcal{L})$ be a smooth, 2-regular toric threefold embedding corresponding to a lattice polytope P . Then X is 2-defective if and only if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}(2))$. Moreover:

- (1) $\deg X^{(2)} = 120$ if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$
- (2) $\deg X^{(2)} = 6(8(a + b + c) - 7)$ if $(X, \mathcal{L}) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi)$, where ξ denotes the tautological line bundle,
- (3) In all other cases,
 $\deg X^{(2)} = 62V - 57F + 28E - 8v + 58V_1 + 51F_1 + 20E_1$,
where V, F, E (resp. V_1, F_1, E_1) denote the (lattice) volume, area of facets, length of edges of P (resp. the adjoint polytope $\text{Conv}(P^\circ)$), and $v = \#\{\text{vertices of } P\}$.



Example

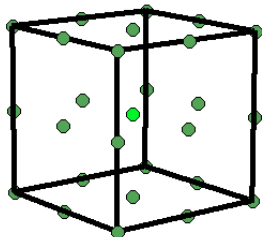
If P is a cube with edge lengths 2, then

$$(X_P, \mathcal{L}_P) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 2, 2)).$$

$$V = 3!8 = 48, F = 6 \cdot 2 \cdot 4 = 48, E = 12 \cdot 2 = 24, v = 8.$$

Since $\text{int}(P) = \{(1, 1, 1)\}$ is a point, $V_1 = F_1 = E_1 = 0$. Hence

$$\deg X^{(2)} = 62V - 57F + 28E - 8v = 848.$$



Non- k -regular varieties

The typical examples are scrolls (varieties ruled in linear spaces). They are not k -regular for $k \geq 2$.

Rational normal scrolls are toric varieties: they correspond to Cayley polytopes:

$$P = \text{Cayley}(d_1\Delta_1, \dots, d_n\Delta_1).$$

The higher order dual varieties of rational normal scrolls were completely described in [P.–Sacchiero].

k -selfdual toric varieties (joint with A. Dickenstein)

Let $\mathcal{A} = \{a_0, \dots, a_N\} \subset \mathbb{Z}^n$ be a lattice point configuration, and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric embedding.

Form the matrix A by adding a row of 1's to the matrix $(a_0 | \dots | a_N)$, and denote by $\mathbf{v}_0 = (1, \dots, 1)$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^{N+1}$ the row vectors of A . For any $\alpha \in \mathbb{N}^{n+1}$, denote by $\mathbf{v}_{\alpha} \in \mathbb{Z}^{N+1}$ the vector obtained as the coordinatewise product of α_0 times the row vector $\mathbf{v}_0$ times $\dots$ times α_n times the row vector $\mathbf{v}_n$.

For any k , we define the matrix $A^{(k)}$ as follows. Order the vectors $\{\mathbf{v}_{\alpha} : |\alpha| \leq k\}$ (for instance, by degree and then by lexicographic order with $0 > 1 > \dots > n$), and let $A^{(k)}$ be the $\binom{n+k}{k} \times (N+1)$ integer matrix with these rows.

$$k = 2$$

Set $k = 2$ and denote by $\mathbf{v}_i * \mathbf{v}_j \in \mathbb{Z}^{m+1}$ the vector given by the coordinatewise product of these vectors. We define the following matrix $A^{(2)} \in \mathbb{Z}^{\binom{n+2}{2} \times (m+1)}$

$$A^{(2)} = \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{v}_1 * \mathbf{v}_1 \\ \mathbf{v}_1 * \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} * \mathbf{v}_n \\ \mathbf{v}_n * \mathbf{v}_n \end{pmatrix},$$

where the last rows are given by the products $\mathbf{v}_i * \mathbf{v}_j$, $1 \leq i \leq j \leq n$. Then, $\mathbb{P}(\text{Rowspan}(A^{(2)})) = \mathbb{T}_{X_{\mathcal{A}},1}^2$ describes the second osculating space of $X_{\mathcal{A}}$ at the point $\mathbf{1}$.

Rational normal curve

Take $\mathcal{A} = \{0, \dots, d\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \end{pmatrix},$$

and

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 1 & 4 & 9 & \cdots & d^2 \\ 0 & 1 & 8 & 27 & \cdots & d^3 \end{pmatrix}.$$

Note that

$$A^{(3)} \cong \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 3 & \cdots & \binom{d}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{d}{3} \end{pmatrix}.$$

Non-pyramidal configurations

The configuration $\mathcal{A}$ is *non-pyramidal* (nap) if the configuration of columns in A is not a pyramid (i.e., no basis vector e_i of $\mathbb{R}^{N+1}$ lies in the rowspan of the matrix).

The configuration $\mathcal{A}$ is *knap* if the configuration of columns in $A^{(k)}$ is not a pyramid.

Example

$\mathcal{A}$ is a pyramid:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$



Characterization of k -self dual configurations

We say that $X_{\mathcal{A}}$ is k -selfdual if there exists a linear isomorphism $\phi: \mathbb{P}^N \rightarrow (\mathbb{P}^N)^\vee$ such that $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$.

Theorem (Dickenstein–P.)

- (1) $X_{\mathcal{A}}$ is k -selfdual if and only if $\dim X_{\mathcal{A}} = \dim X_{\mathcal{A}}^{(k)}$ and $\mathcal{A}$ is knap.
- (2) If $\mathcal{A}$ is knap and $\dim \operatorname{Ker} A^{(k)} = 1$, then $X_{\mathcal{A}}$ is k -selfdual.
- (3) If $\mathcal{A}$ is knap and k -selfdual, and $\dim \operatorname{Ker} A^{(k)} = r > 1$, then $\mathcal{A}$ is r -Cayley.

The proof generalizes [Bourel–Dickenstein–Rittatore] ($k = 1$).

Surfaces in $\mathbb{P}^3$

The lattice configuration

$$\mathcal{A} = \{(0, 0), (1, 0), (1, 1), (0, 2)\}$$

gives a toric variety parameterized as

$$(x, y) \mapsto (1 : x : xy : y^2).$$

Its dual variety is given by the parameterization

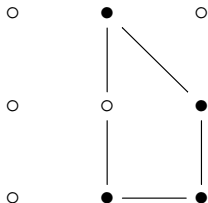
$$(x, y) \mapsto (-y^2 : 2x^{-1}y^2 : -2x^{-1}y : 1),$$

hence it corresponds to the lattice configuration

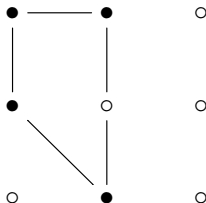
$$\mathcal{A}^\vee = \{(0, 2), (-1, 2), (-1, 1), (0, 0)\}.$$

The dual variety is projectively equivalent to the original variety, hence the variety is self dual.

The corresponding polytopes



$\mathcal{A}$



$\mathcal{A}^v$



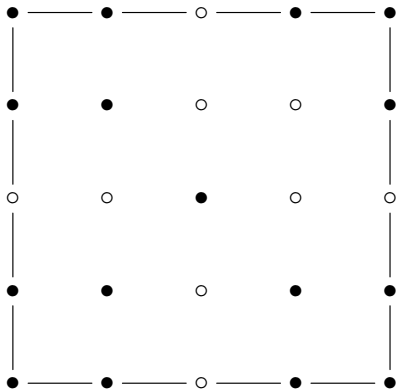
Mulliken's work

Perkinson's student Mulliken gave a definition of (higher) selfduality which is stronger than ours: he required that the toric embedding giving the k th dual variety should be *equal* to the original toric embedding. Therefore only *centrally symmetric* lattice configurations could possibly give selfdual varieties.

For example, in his terminology, the previous example is not self dual.

Example

This square is an example of a 4-selfdual smooth surface which is not centrally symmetric.



Connections with number theory

As observed by Perkinson, elements in the kernel of $A^{(k)}$ correspond to polynomials of degree k that vanish on all the lattice points of $\mathcal{A}$.

One can use this to construct examples of k -selfdual varieties.

Example

Consider three quadrics $Q_1, Q_2, Q_3 \in \mathbb{Z}[x_1, x_2, x_3]$ such that their intersection is equal to 8 lattice points $a_0, \dots, a_7 \in \mathbb{Z}^3 \subset \mathbb{R}^3$. Then $\mathcal{A} = \{a_0, \dots, a_7\}$ gives a 2-selfdual threefold:

The kernel of the (10×8) -matrix $A^{(2)}$ has rank 3. So the rank of $A^{(2)}$ is $10 - 3 = 7$, which is one less than the maximal rank.

This construction can be generalized, and we see an interesting connection to diophantine theory: polynomials with many integer solutions.

Togliatti revisited

Togliatti's surface is the surface $X_{\mathcal{A}} \subset \mathbb{P}^5$, where

$$\mathcal{A} = \{(0, 0), (1, 0), (0, 1), (2, 1), (1, 2), (2, 2)\}.$$

(The interior lattice point $(1, 1)$ of the hexagon is omitted – the surface is the (toric) projection of a projectively normal surface in $\mathbb{P}^6$). All 2nd order osculating spaces have dimension 4 (instead of 5).

Then $\mathcal{A}$ is 2nap and $\dim \operatorname{Ker} A^{(2)} = 1$, so $X_{\mathcal{A}}$ is 2-selfdual.

Consider $\mathcal{A}' := \mathcal{A} \setminus \{(2, 2)\}$. The unique quadric through these five points also go through the points $(4, 3)$ and $(4, 2)$. Thus,

$$\mathcal{A}' \cup \{(4, 3)\} \text{ and } \mathcal{A}' \cup \{(4, 2)\}$$

give (non-smooth, non centrally symmetric) 2-selfdual surfaces.

THANK YOU FOR YOUR ATTENTION!



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