Projective geometry from a toric point of view

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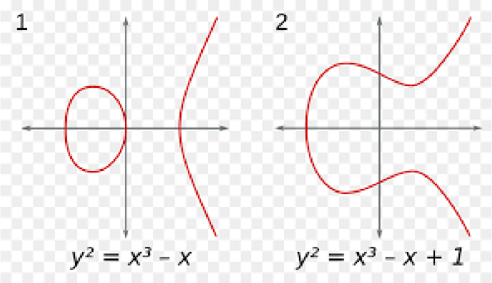
Plane curves

$$C_1 = \{(x,y) | y^2 - x^3 + x = 0\},$$
$$C_2 = \{(x,y) | y^2 - x^3 + x - 1 = 0\} \subset \mathbb{A}^2_{\mathbb{R}}.$$

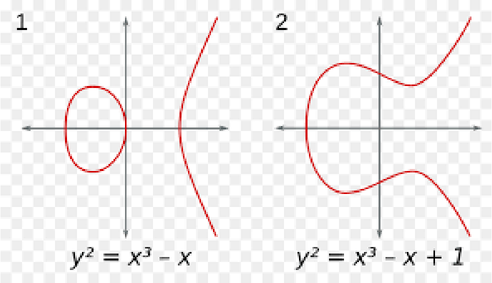


Plane curves

$$C_1 = \{(x:y:z) | y^2z - x^3 + xz^2 = 0\},$$
$$C_2 = \{(x:y:z) | y^2z - x^3 + xz^2 - z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Let $H = \{z = 0\}$ be the line at infinity. Then

$$C_1 \cap H = C_2 \cap H = \{(x:y:z) | z = x^3 = 0\} = \{(0:1:0)\}.$$



Algebraic geometry = study of zeros of polynomials

Let $\mathbb{K}$ be a field ($\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{F}_p$ or ...).

Affine algebraic n-space: $\mathbb{A}^n_{\mathbb{K}} \cong \mathbb{K}^n$
Affine algebraic variety:

$$X = \{(c_1, \ldots, c_n) | f_\alpha(c_1, \ldots, c_n) = 0, \alpha \in I\} \subseteq \mathbb{A}^n_{\mathbb{K}},$$

where the $f_\alpha \in \mathbb{K}[x_1, \ldots, x_n]$ are polynomials.

Projective algebraic n-space: $\mathbb{P}^n_{\mathbb{K}} = \mathbb{A}^n_{\mathbb{K}} \cup H$, where H is the hyperplane "at infinity".

Projective algebraic variety:

$$X = \{(c_0 : c_1 : \cdots : c_n) | F_\alpha(c_0, c_1, \ldots, c_n) = 0, \alpha \in I\} \subseteq \mathbb{P}^n_{\mathbb{K}},$$

where the $F_\alpha \in \mathbb{K}[x_0, \ldots, x_n]$ are *homogeneous* polynomials.

Differential geometry

Let $s : \mathbb{R} \to \mathbb{R}^N$ be a (parameterized) curve:

$$t \mapsto s(t) = (s_1(t), s_2(t), \ldots, s_N(t)),$$

where the s_i are differentiable functions.

The tangent to the curve at the point $s(t)$ is the line $\langle s(t), s'(t) \rangle$, the osculating plane is $\langle s(t), s'(t), s''(t) \rangle$, and so on.

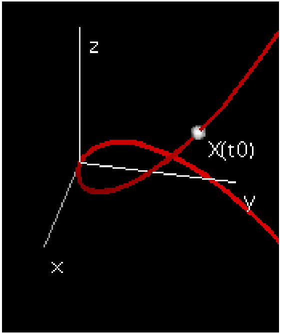
Example (The twisted cubic)

Let $s : \mathbb{R} \to \mathbb{R}^3$ be given by

$$s(t) = (t, t^2, t^3).$$

Then $s'(t) = (1, 2t, 3t^2)$ and $s''(t) = (0, 2, 6t)$.

At $s(0) = (0, 0, 0)$, the tangent line is the x-axis and the osculating plane is the xy-plane.



Tangent spaces, osculating spaces, ...

Consider a parameterization $s : \mathbb{C}^n \to \mathbb{P}^N_{\mathbb{C}}$,

$$t = (t_1, \dots, t_n) \mapsto (s_0(t) : \cdots : s_N(t)).$$

The rows of

$$A_s^k(t) := \begin{pmatrix} s_0(t) & s_1(t) & \cdots & \cdots & s_N(t) \\ \partial s_0(t)/\partial t_1 & \partial s_1(t)/\partial t_1 & \cdots & \cdots & \partial s_N(t)/\partial t_1 \\ \partial s_0(t)/\partial t_2 & \cdots & \cdots & \cdots & \partial s_N(t)/\partial t_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial^k s_0(t)/\partial t_n^k & \cdots & \cdots & \cdots & \partial^k s_N(t)/\partial t_n^k \end{pmatrix}.$$

considered as points in $\mathbb{P}^N_{\mathbb{C}}$ spans the kth osculating space $\mathrm{Osc}_s^k(t)$ to $s(\mathbb{C}^n)$ at $s(t)$.

We have

$$\dim \mathrm{Osc}_s^k(t) = \mathrm{rk}\, A_s^k(t) - 1.$$

We say that $s(t)$ is a kth *inflection point* if $\mathrm{rk}\, A_s^k(t) < \binom{n+k}{k}$.

Example

The curve $s : \mathbb{C} \to \mathbb{P}^2_{\mathbb{C}}$, given by

$$s(t) = (1 : t : t^3),$$

has

$$A_s^2(t) := \begin{pmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{pmatrix}.$$

Since $\mathrm{rk}\, A_s^2(0) = 2 < 3$, $s(0)$ is an inflection point.

The Veronese

Set $N := \binom{n+k}{k}$. The kth Veronese embedding $\nu_{n,k} \colon \mathbb{C}^n \to \mathbb{P}^N_{\mathbb{C}}$ is given by

$$(t_1, \ldots, t_n) \mapsto (1 : t_1 : \cdots : t_{n-1}t_n^{k-1} : t_n^k)$$

(all monomials of degree $\leq k$).

Theorem (Fulton–Kleiman–P.–Tai)

Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a (nonsingular) variety of dimension n, with $N = \binom{n+k}{k} - 1$. If X has no inflections, then $X = \overline{\nu_{n,k}(\mathbb{C}^n)}$ is the kth Veronese variety.

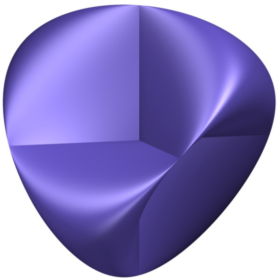
Example

The only curve $C \subset \mathbb{P}^N_{\mathbb{C}}$ with no inflection points is the rational normal curve $t \mapsto (1 : t : t^2 : \cdots : t^N)$.

Steiner's Roman surface

A linear projection of the 2nd Veronese surface:

$$X := \{(x_1, x_2, x_3) | x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 - x_0 x_1 x_2 x_3 = 0\} \subset \mathbb{P}^3_{\mathbb{R}}$$



Toric varieties

Take a set of lattice points

$$\mathcal{A} = \{a_0, \dots, a_N\} \subset \mathbb{Z}^n$$

and define $\varphi : (\mathbb{C}^*)^n \to \mathbb{P}^N$ by

$$t = (t_1, \dots, t_n) \mapsto (t^{a_0} : \cdots : t^{a_N}).$$

The associated projective *toric variety* is

$$X_{\mathcal{A}} := \overline{\varphi((\mathbb{C}^*)^n)}.$$

The torus $T^n := (\mathbb{C}^*)^n$ acts on $X_{\mathcal{A}}$, with open dense orbit $\varphi((\mathbb{C}^*)^n)$.

E.g., $\mathcal{A} = P \cap \mathbb{Z}^n$, for a convex lattice polytope $P \subset \mathbb{R}^n$.

Osculating spaces

Set $\mathbf{1} := (1, \ldots, 1) \in (\mathbb{C}^*)^n$. The rows of the matrix $A^k_\varphi(\mathbf{1})$ span the osculating space $\mathrm{Osc}^k_\varphi(\mathbf{1})$, and $\dim \mathrm{Osc}^k_\varphi(\mathbf{1}) = \operatorname{rk} A^k_\varphi(\mathbf{1}) - 1$.

Example

$$\nu_{2,2}(t_1, t_2) = (1 : t_1 : t_2 : t_1^2 : t_1 t_2 : t_2^2) \in \mathbb{P}^5$$

$$A^2_{\nu_{2,2}}(\mathbf{1}) := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Togliatti's surface

The lattice point configuration

$$\mathcal{A} = \{(1,0),(0,1),(2,0),(0,2),(2,1),(1,2)\} \subset \mathbb{Z}^2$$

defines the toric embedding

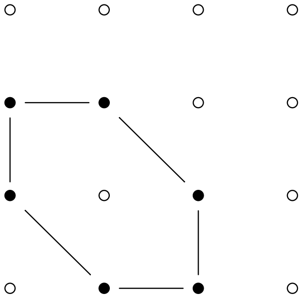
$$\varphi : (\mathbb{C}^*)^2 \to \mathbb{P}^5$$

given by

$$(t_1, t_2) \mapsto (t_1 : t_2 : t_1^2 : t_2^2 : t_1^2 t_2 : t_1 t_2^2).$$

Then $X_{\mathcal{A}} = \overline{\varphi((\mathbb{C}^*)^2)}$ is a toric surface.

Togliatti lattice point configuration



Projections and sections

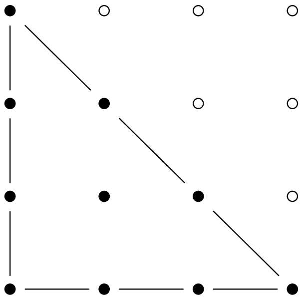
Let $\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ be a lattice point configuration and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric variety.

Let $\mathcal{A}' = \mathcal{A} \smallsetminus \{m \text{ points}\}$.

Then the toric variety $X_{\mathcal{A}'} \subset \mathbb{P}^{N'}$, where $N' = N - m$, is the (toric) linear projection of $X_{\mathcal{A}}$ with center equal to the linear span of the "removed points".

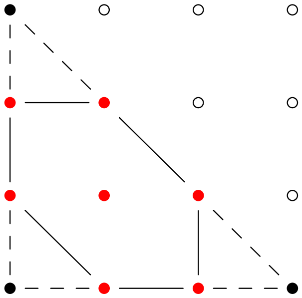
A *toric hyperplane section* $X_{\mathcal{B}}$ of $X_{\mathcal{A}}$ is given by $\mathcal{B} \subset \mathbb{Z}^{n-1}$ obtained by taking a hyperplane in $\mathbb{Z}^n$ and "collapsing" the point configuration $\mathcal{A}$ into this lattice hyperplane in such a way that one point is "lost": two points map to the same point.

Third Veronese surface $\nu_{2,3} : \mathbb{P}^2 \to \mathbb{P}^9$

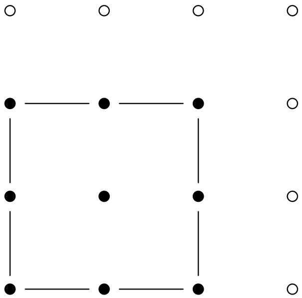


X_P, where $P = 3\Delta_2$

Del Pezzo: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^{9-3} = \mathbb{P}^6$

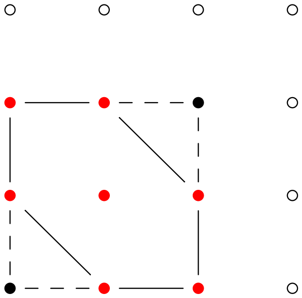


Segre–Veronese: $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^8$ via $\mathcal{O}(2,2)$

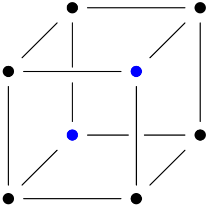


X_P, where $P = 2\square_2$

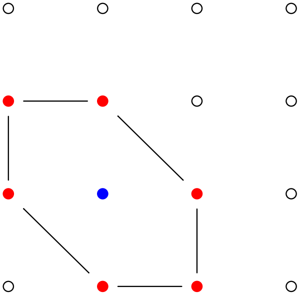
Del Pezzo: $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{8-2} = \mathbb{P}^6$



Del Pezzo as a section of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$



X_P, where $P = \square_3$



Polytopes and toric varieties: dictionary

$P \subset \mathbb{R}^n$ convex lattice polytope

$X_P := X_{\mathcal{A}} \subset \mathbb{P}^N$, where $\mathcal{A} = P \cap \mathbb{Z}^n$.

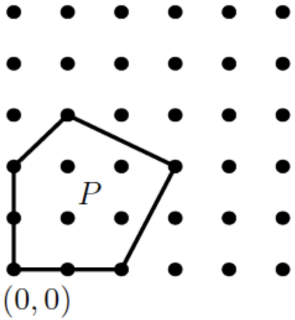
$L_P := \mathcal{O}_{\mathbb{P}^N}(1)|_{X_P}$

- T^n-orbits of $X_P \leftrightarrow$ faces of P
- $\deg X_P = \mathrm{Vol}_{\mathbb{Z}}(P)$
- X_P nonsingular iff P smooth (Delzant)
- topological Euler characteristic $e(X_P) = \#$ vertices of P
- $\dim H^0(X_P, mL_P) = \#(mP \cap \mathbb{Z}^n)$

Lattice points

Let P be a lattice polygon. Count its lattice points:

$$\#(P \cap \mathbb{Z}^2) = 11$$



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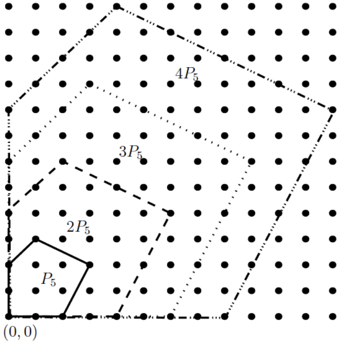
Dilated polytopes

Count the lattice points of the dilated polygons mP.

$$\#(mP \cap \mathbb{Z}^2) = ?$$



Ehrhart polynomials

The number of lattice points in the mth dilation of P,

$$e_P(m) := \#(mP \cap \mathbb{Z}^n)$$

is a polynomial in m of degree $n = \dim P$.

Reciprocity: $e_P(-m) = (-1)^n \#\mathrm{int}(mP \cap \mathbb{Z}^n)$

Why? If $(X, L) = (X_P, L_P)$, then

$$e_P(m) = \dim H^0(X, mL) = \chi(mL),$$

$$e_P(-m) = \chi(-mL) = (-1)^n \chi(mL - K_X) = (-1)^n \dim H^0(X, mL - K_X),$$

and $(X, mL - K_X)$ is the toric variety defined by $\mathrm{int}(mP \cap \mathbb{Z}^n)$.

Ehrhart series (R. Stanley)

The generating function of the Ehrhart polynomials is

$$E_P(t) := \sum e_P(m)t^m = \frac{\sum_{i=0}^{n} h_i t^i}{(1-t)^{n+1}},$$

where the h_i are non-negative integers such that $h_0 = 1$ and

- $\sum_{i=0}^{n} h_i = \mathrm{Vol}_{\mathbb{Z}}(P)$,
- $h_1 = \#(P \cap \mathbb{Z}^n) - (n+1)$,
- $h_n = \#\mathrm{int}(P \cap \mathbb{Z}^n)$.

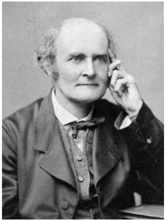
In the example, we get

$$E_P(t) = \frac{1+8t+4t^2}{(1-t)^3}$$

which is equivalent to (Pick's formula!)

$$e_P(m) = \#(mP \cap \mathbb{Z}^2) = \tfrac{13}{2}m^2 + \tfrac{7}{2}m + 1.$$

Cayley polytopes



Convex lattice polytopes

$$P_0, \ldots, P_r \subset \mathbb{R}^{n-r}$$

Vertices of $\Delta_r \subset \mathbb{R}^r$

$$e_0, \ldots, e_r$$

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The polytope

$$P = \mathrm{Conv}\{e_0 \times P_0, \ldots, e_r \times P_r\} \subset \mathbb{R}^n$$

is called a *Cayley polytope*.

We write

$$P = \mathrm{Cayley}(P_0, \ldots, P_r).$$

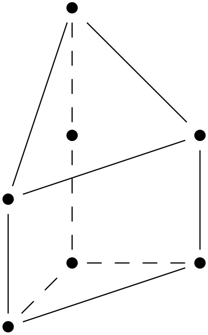
A Cayley polytope is "hollow" – it has no interior lattice points.

Hollow polytopes

A Cayley polytope is "hollow": it has no interior lattice points.

Example

$n = 3$, $r = 2$, $P_0 = [0, 2]$, $P_1 = P_2 = [0, 1]$



The codegree and degree of a polytope

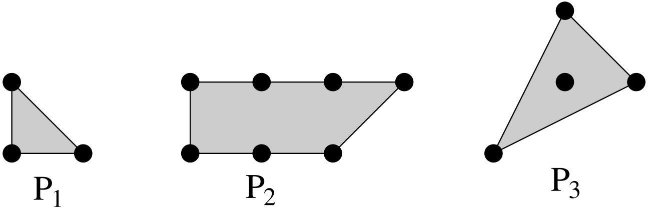
$$\operatorname{codeg}(P) := \min\{m \mid mP \text{ has interior lattice points}\}.$$

$$\deg(P) := n + 1 - \operatorname{codeg}(P)$$

Example

- $\operatorname{codeg}(\Delta_n) = n + 1$
- $\operatorname{codeg}(\square_n) = 2$
- $P = \operatorname{Cayley}(P_0, \ldots, P_r)$ implies $\operatorname{codeg}(P) \geq r + 1$.

Examples



$\mathrm{codeg}(P_1) = 3$ $\mathrm{codeg}(P_2) = 2$ $\mathrm{codeg}(P_3) = 1$

The Cayley polytope conjecture

Question (Batyrev–Nill): Is there an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Answer (Haase–Nill–Payne): Yes, and $N(d) \leq (d^2 + 19d - 4)/2$

Question: Is $N(d)$ linear in d?

Answer (Dickenstein–Di Rocco–P.): Yes, $N(d) = 2d + 1$ (if P is smooth and $\mathbb{Q}$-normal).

Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.

Theorem (Dickenstein, Di Rocco, P., Nill)

Let P be a smooth lattice polytope of dimension n. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = \operatorname{Cayley}(P_0, \dots, P_r)$ *is a smooth Cayley polytope with* $r + 1 = \operatorname{codeg}(P)$ *and* $r > \frac{n}{2}$.

The proof of this combinatorial result is algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

Resultants and discriminants

Il faut éliminer la théorie de l'élimination.

J. Dieudonné (1969)

Eliminate, eliminate, eliminate
Eliminate the eliminators of elimination theory.

S. S. Abhyankar (1970)

Résultant, discriminant

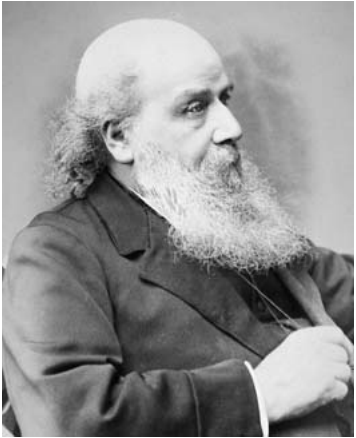
M. Demazure (2011) – à J.-P. Serre pour son 85-ième anniversaire

Question: For which $a_0, \ldots, a_m$ and $b_0, \ldots, b_n$ do

$$f(x) = a_m x^m + \cdots + a_0 \text{ and } g(x) = b_n x^n + \cdots + b_0$$

have a common root?

James Joseph Sylvester (1814–1897)



The *Sylvester matrix* is the $(m+n) \times (m+n)$-matrix

$$\begin{pmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & \cdots \\ 0 & a_m & a_{m-1} & a_{m-2} & \cdots \\ \vdots & & & \vdots & \\ b_n & b_{n-1} & b_{n-2} & \cdots & \cdots \\ 0 & b_n & b_{n-1} & b_{n-2} & \cdots \\ \vdots & & & \vdots & \end{pmatrix}$$

The *resultant* $\mathrm{Res}(f,g)$ is the determinant of this matrix.

A student of Sylvester: Florence Nightingale (1820-1910)

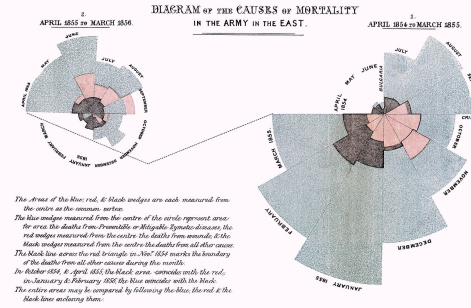
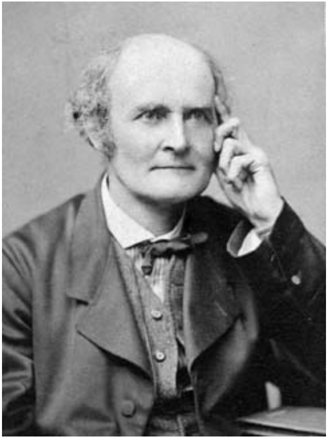


Figure: Diagram of the Causes of Mortality in the Army in the East

Arthur Cayley (1821–1895)



Set

$$h(x, y) := f(x) + yg(x).$$

If α is a common root of f and g, then

$$(\alpha, -\frac{f_x(\alpha)}{g_x(\alpha)})$$

is a common zero of h, h_x, h_y.

The Cayley trick

Consider

$$h(x_1, \ldots, x_{n-r}, y_1, \ldots, y_r) :=$$
$$f_0(x_1, \ldots, x_{n-r}) + y_1 f_1(x_1, \ldots, x_{n-r}) + y_r f_r(x_1, \ldots, x_{n-r}).$$

The *discriminant* $\Delta(h)$ of h is obtained by eliminating the x_i's and y_i's from the $n+1$ equations

$$h = 0, \partial h/\partial x_i = 0, \partial h/\partial y_j = f_j = 0.$$

Hence $\Delta(h) \sim \mathrm{Res}(f_0, \ldots, f_r)$.

Hyperplane sections and discriminants

Let $P_j \subset \mathbb{R}^{n-r}$, $j = 0, \ldots, r$, $P = \text{Cayley}(P_0, \ldots, P_r)$,
$\varphi : (\mathbb{C}^*)^{n-r} \times (\mathbb{C}^*)^r \to X_P \subseteq \mathbb{P}^N$, $\varphi_j : (\mathbb{C}^*)^{n-r} \to X_{P_j} \subset X_P$.

A hyperplane section of X_P:

$$h(x_1, \ldots, x_{n-r}, y_1, \ldots, y_r) := f_0 + y_1 f_1 + \cdots + y_r f_r = 0,$$

where $f_i = 0$ is a hyperplane section of X_{P_i}.

The hyperplane section is singular (i.e., the hyperplane is tangent to X_P) when $\Delta(h) = 0$.

Geometric interpretation of the Cayley trick: The hyperplane H is tangent to X_P iff $H \supseteq \langle \varphi_0(t), \ldots, \varphi_r(t) \rangle$, for $t \in (\mathbb{C}^*)^{n-r}$.

Degree of dual varieties

Theorem (Gelfand–Kapranov–Zelevinsky)

If $X_P \subset \mathbb{P}^N$ is nonsingular and the dual variety $X_P^\vee := \{H \in (\mathbb{P}^N)^\vee$ is tangent to $X_P\}$ is a hypersurface, then

$$\deg X_P^\vee = \textstyle\sum_{F \prec P} (-1)^{\operatorname{cod} F} (\dim F + 1) \operatorname{Vol}_{\mathbb{Z}}(F),$$

where the F's are the faces of P.

Example

Let $X_P \subset \mathbb{P}^6$ be the Del Pezzo surface. Then

$$\deg X_P = \operatorname{Vol}_{\mathbb{Z}}(P) = 6,$$

$$\deg X_P^\vee = 3 \operatorname{Vol}_{\mathbb{Z}}(P) - 2 \cdot 6 \cdot 1 + 6 = 3 \cdot 6 - 2 \cdot 6 + 6 = 12.$$

A surface in $\mathbb{P}^3$

$$\mathcal{A} = \{(0,0),(1,0),(1,1),(0,2)\}$$

gives

$$X_{\mathcal{A}} : \varphi(t_1,t_2) = (1 : t_1 : t_2t_2 : t_2^2) \in \mathbb{P}^3$$

and (compute!)

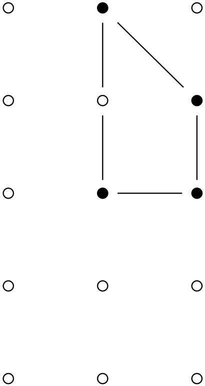
$$X_{\mathcal{A}}^{\vee} : \varphi^{\vee}(t_1,t_2) = (-1 : 2t_1^{-1} : -2t_1^{-1}t_2^{-1} : t_2^{-2}),$$

so $X_{\mathcal{A}}^{\vee} \cong X_{-\mathcal{A}}$, where

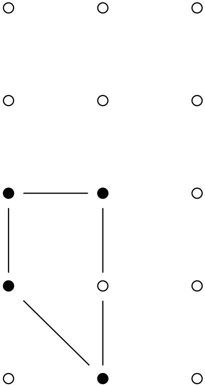
$$-\mathcal{A} = \{(0,0),(-1,0),(-1,-1),(0,-2)\}$$

This surface is *selfdual*: $X_{\mathcal{A}}^{\vee} \cong X_{\mathcal{A}}$.

The corresponding polygons



$\mathcal{A}$



$-\mathcal{A}$

Higher order dual varieties

The *kth dual variety* $X^{(k)}$ is defined as:

$$X^{(k)} = \overline{\{H \in (\mathbb{P}^N)^\vee \mid H \supseteq \operatorname{Osc}_x^k \text{ for some regular } x\}}.$$

In particular, $X^{(1)} = X^\vee$, $X^{(k-1)} \supseteq X^{(k)}$, and $X^{(k)}$ is contained in the singular locus of $X^\vee$ for $k \geq 2$.

The expected dimension of $X^\vee$ is $N-1$ and that of $X^{(k)}$ is $n + N - \dim \operatorname{Osc}_x^k - 1$.

$X_{\mathcal{A}}$ is *k-selfdual* if $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$ for some $\phi \colon \mathbb{P}^N \cong (\mathbb{P}^N)^\vee$.

Characterization of k-selfdual configurations

The lattice configuration $\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ is knap if no basis vector $e_i \in \mathbb{R}^{N+1}$ is in the row span of the matrix $A^k(\mathbf{1})$.

Theorem (Dickenstein–P.)

(1) *$X_\mathcal{A}$ is k-selfdual if and only if $\dim X_\mathcal{A} = \dim X_\mathcal{A}^{(k)}$ and $\mathcal{A}$ is knap.*

(2) *If $\mathcal{A}$ is knap and $\dim \mathrm{Ker} A^k(\mathbf{1}) = 1$, then $X_\mathcal{A}$ is k-selfdual.*

(3) *If $\mathcal{A}$ is knap and k-selfdual, and $\dim \mathrm{Ker} A^k(\mathbf{1}) = r > 1$, then $\mathcal{A} = e_0 \times \mathcal{A}_0 \cup \ldots \cup e_{r-1} \times \mathcal{A}_{r-1}$ is r-Cayley.*

Chasles–Cayley–Bacharach

Observation: Any vector in the rowspan of $A^k(\mathbf{1})$ is of the form $(Q(a_0), \ldots, Q(a_N))$, where Q is a polynomial in n variables of degree $\leq k$. Hence $\mathcal{A}$ is not $k$nap iff there is Q such that $Q(a_i) \neq 0$ and $Q(a_j) = 0$ for all $j \neq i$.

Example

Three quadrics $Q_1, Q_2, Q_3 \in \mathbb{Z}[x_1, x_2, x_3]$ with

$$Q_1 \cap Q_2 \cap Q_3 = \{a_0, \ldots, a_7\} = \mathcal{A} \subset \mathbb{Z}^3 \subset \mathbb{R}^3.$$

Then $X_{\mathcal{A}}$ is a 2-selfdual threefold:

By Cayley–Bacharach, $\mathcal{A}$ is 2nap, and the rank of the (10×8)-matrix $A^2(\mathbf{1})$ is $10 - 3 = 7$, so $\dim \operatorname{Ker} A^2(\mathbf{1}) = 1$.

Polynomials with many integral zeros

Take integers m_1, m_2, m_3 and set $f(x) = \prod_{i=1}^{3}(x - m_i)$.

The quadratic polynomial[1]

$$Q(x,y) = \frac{f(x) - f(y)}{x - y} \in \mathbb{Z}[x,y]$$

vanishes at the 6 lattice points $(m_i, m_j), j \neq i$.

Then $\mathcal{A}$ is 2nap, and $\operatorname{rk} A^2(\mathbf{1}) = 6 - 1 = 5$, so $\dim \operatorname{Ker} A^2(\mathbf{1}) = 1$. Hence $\mathcal{A}$ is 2-selfdual.

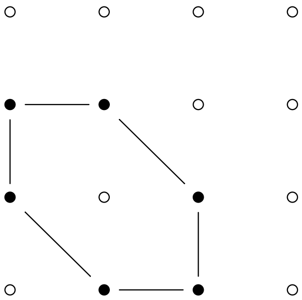
[1] cf. Schaefer, Rodriguez Villegas–Voloch

Togliatti revisited

For $m_1 = 0$, $m_2 = 1$, $m_3 = 2$,

$$Q(x, y) = x^2 + xy + y^2 - 3x - 3y + 2,$$

and $\mathcal{A}$ is the Togliatti configuration:



Thank you for your attention!