

Functors of infinitely near points
on an algebraic surface
and the counting of Gromov–Witten invariants

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Solving equations

Algebraic geometry is about solving polynomial equations; it is the study of the common set of solutions of a given set of polynomials in several variables.

I started solving equations in two variables as a child while skiing in the mountains with my father – he would write the equations in the snow, and ask my brother and me to solve them.

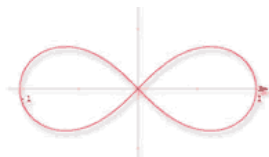


Algebraic geometry

A projective algebraic variety in $\mathbb{CP}^n$ is the common set of zeros of a collection of homogeneous polynomials in $n + 1$ variables.

For example, a *rational curve* is a one-dimensional algebraic variety birationally equivalent to the projective line $\mathbb{CP}^1$.

A rational algebraic plane curve of degree 4, with one real node (and two complex nodes):



Enumerative geometry and Calabi–Yau threefolds

Clemens’ conjecture: There are only finitely many rational curves of degree d on a general quintic hypersurface in $\mathbb{CP}^4$.
Proved for $d \leq 10$ (Clemens, Katz, Kleiman–Johnsen, Cotterill.)

The physicists enter the scene:

In string theory, rational curves on Calabi–Yau threefolds are instantons. Using the principle of *mirror symmetry*, curves on a Calabi–Yau threefold (such as the quintic) can be counted by computing integrals on the mirror manifold. This way Candelas, de la Ossa, Green, and Parkes predicted the generating function of the Clemens problem.

The physicists also got interested in the similar problem for surfaces instead of threefolds. The 2-dimensional analogue of a CY threefold is a K3 surface (e.g. a quartic surface in $\mathbb{CP}^3$), but the problem is equally interesting for *any* surface.

Gromov–Witten invariants

Let S be a nonsingular projective surface, e.g. the complex projective plane $\mathbb{CP}^2$. The *Gromov–Witten invariants* of S can be interpreted as the numbers of curves in a complete linear system $|C|$ of given genus g , or the numbers N_r of curves having r nodes, passing through an appropriate number of points on S .

Vainsencher gave explicit formulas for N_r in terms of the Chern numbers of C and S , for $r \leq 6$. His work inspired Göttsche, Kleiman–P., Ai-Ko Liu, Kazarian, Kerner, among others.

Example

Let $N(d, r)$ = number of plane curves of degree d with r nodes passing through $d(d+3)/2 - r$ points. Then $N(2, 0) = 1$, $N(3, 1) = 12$, $N(4, 1) = 27$, $N(4, 3) = 620$, ...

Generating functions

As in the case of a Calabi–Yau threefold, we hope to find a “nice” generating function for N_r .

Example

Let $V \subset \mathbb{P}^3$ be a quartic surface (a K3 surface).

Let $C \subset V$ be a curve, r an integer such that $C^2/2 = r - 1$, and set

$$N_r := \#\{r\text{-nodal rational curves in } |C|\}.$$

The generating function is (Yau–Zaslow, Bryan–Leung)

$$f(q) = \sum_r N_r q^r = \prod_m (1 - q^m)^{-24}.$$

The conjecture

For more general surfaces, this is hard. Even in the case of $\mathbb{CP}^2$, only a recursive formula is known (Kontsevich, Ran, Caporaso–Harris).

Conjecture. The generating function for nodal curves in a given family of curves on a surface S can be written as

$$\sum N_r t^r = \sum \frac{1}{r!} P_r(a_1, \dots, a_r) t^r,$$

where the a_i are linear combinations (computable from an algorithm) of the Chern numbers of S and the curves C in the family, and the P_i are the Bell polynomials, defined by the formal identity

$$\sum_{i \geq 0} P_i(a_1, \dots, a_i) t^i / i! = \exp\left(\sum_{j \geq 1} a_j t^j / j!\right).$$



The case of ≤ 8 nodes

Theorem. *For $r \leq 8$ the number of nodal curves C on a surface S is equal to*

$$N_r = \frac{1}{r!} P_r(a_1, \dots, a_r)$$

where the a_i are (explicit) linear combinations of the Chern numbers of C and S , and the P_i are the Bell polynomials, defined by the formal identity

$$\sum_{i \geq 0} P_i(a_1, \dots, a_i) t^i / i! = \exp\left(\sum_{j \geq 1} a_j t^j / j!\right).$$

(Cf. also Göttsche's conjecture, proved by Brian–Leung for K3 and Abelian surfaces.)

Plane curves

Consider the family of plane curves of given degree d .

The Chern numbers are

$$C^2 = d^2, C \cdot c_1(T_S) = 3d, c_1(T_S)^2 = 9, c_2(T_S) = 3.$$

The N_r are polynomials in d of degree $2r$:

$$N_1 = 3d^2 - 6d + 3$$

$$N_2 = \frac{9}{2}d^4 - 18d^3 + 6d^2 + \frac{81}{2}d - 33$$

$$N_3 = \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525$$

$$N_4 = \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^5 - 642d^4 - 2529d^3 + \frac{37881}{8}d^2 \\ + \frac{18057}{4}d - 8865$$

$$N_5 = \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5 \\ + \frac{128859}{8}d^4 + \frac{59097}{2}d^3 - \frac{3528381}{40}d^2 - \frac{946929}{20}d + 153513$$

Idea of the proof

A node (an ordinary double point) of a curve C on a nonsingular surface S is resolved by blowing up the singular point. The strict transform $C - 2E$ of the given curve C will thus have one fewer node than C .

Set $Y = |C|$, and consider the family of surfaces

$$\pi': F' \rightarrow F = S \times Y,$$

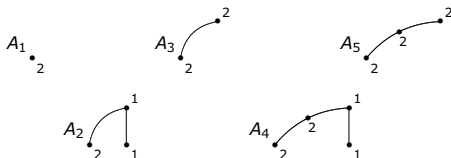
obtained by blowing up the diagonal in $F \times_Y F$. The new family is a family of surfaces S_x , where S_x is the blow up of S in the point x . Let $X \subset F$ be the set of singular points of the curves C ; then the r -nodal curves in $|C|$ correspond to $(r - 1)$ -nodal curves in $|\pi'^*C - 2E|_X$. We get the r -nodal formula by pushing down the $(r - 1)$ -nodal formula, and hence can use induction on the number r of nodes.

Curve singularities and Enriques diagrams

In the proof we also need to consider worse singularities.

Let $x \in C \subset S$ be a singular point on a curve on a nonsingular surface. The singularity can be resolved by a series of blowups, and we get an associated weighted resolution graph (Enriques diagram) $\mathbf{D}$, which determines the topological type of the singularity.

Examples of diagrams



The functor of infinitely near points

Let S be a nonsingular surface and

$$S^{(n+1)} \rightarrow S^{(n)} \rightarrow \dots \rightarrow S^{(0)} = S$$

a sequence of blowups at centers $t_i \in S^{(i)}$. Then $(t_0, \dots, t_n)$ is called a *sequence of infinitely near points* of S .

Theorem. *Let $F \rightarrow Y$ be a family of surfaces. The sequences of infinitely near T -points of F/Y form a functor, which is represented by $F^{(n)}/Y$, where $F^{(n)}$ is defined recursively by letting $F^{(i)} \rightarrow F^{(i-1)}$ be the blow up of the diagonal in $F^{(i-1)} \times_Y F^{(i-1)}$, composed with projection on the second factor.*

For each ordered diagram $(\mathbf{D}, \theta)$, the subfunctor of sequences with ordered diagram $(\mathbf{D}, \theta)$ is represented by a subscheme $F(\mathbf{D}, \theta) \subset F^{(n)}$, which is smooth over Y .

Zariski clusters and the Hilbert scheme

To each point x on the surface S and each diagram $\mathbf{D}$, one can associate an ideal in the local ring $\mathcal{O}_{S,x}$ of finite colength equal to the *degree* d of the diagram $\mathbf{D}$. The corresponding “fat point” (Zariski cluster) has the property that a curve containing this fat point has a singularity of type $\mathbf{D}$.

This gives a map $\Psi_\theta: F(\mathbf{D}, \theta) \rightarrow \mathrm{Hilb}_{F/Y}^d$, which factors through the quotient of the action of $\mathrm{Aut}(\mathbf{D})$.

Theorem. *The map*

$$\Psi: F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D}) \rightarrow \mathrm{Hilb}_{F/Y}^d$$

is universally injective and an embedding in characteristic 0 (but can be purely inseparable in positive characteristic).

The Hilbert scheme of a diagram

Let $H(\mathbf{D}) \subset \mathrm{Hilb}_{F/Y}^d$ denote the set of fat points of colength d with diagram $\mathbf{D}$.

Corollary. *The subset $H(\mathbf{D}) \subset \mathrm{Hilb}_{F/Y}^d$ is a locally closed subscheme, smooth over Y , with geometrically irreducible fibers of dimension $\dim(\mathbf{D})$.*

APPLICATION: Let $C \subset F$ be a family of curves over Y , and consider the natural embedding $\mathrm{Hilb}_{C/Y}^d \subset \mathrm{Hilb}_{F/Y}^d$. The solution to the problem: *enumerate curves in the family C/Y with singularities of type $\mathbf{D}$* is the computation of the pushdown to Y of the class

$$[\overline{H(\mathbf{D})}] \cap [\mathrm{Hilb}_{C/Y}^d].$$

PROBLEM: Compute $[\overline{H(\mathbf{D})}]$.



Configuration spaces

In the case that $\mathbf{D}$ is the diagram corresponding to r nodes, $H(\mathbf{D})$ is equal to the set of distinct r -tuples in F/Y , i.e., to the *configuration space* of r points in F/Y , equal to the open subset

$$F \times_Y \dots \times_Y F \setminus \{\text{diagonals}\} \subset F_Y^r.$$

The space $F^{(r)}$ is a compactification of the configuration space. Other compactifications are $F[r]$ (Fulton–MacPherson) and $F\langle r \rangle$ (Ulyanov).

Another approach: Instead of the recursive procedure, using $F^{(r)}$, one could try directly to use intersection theory on this or another compactification of the configuration space. We want to compute the class

$$m_r := \overline{[X^r \setminus \{\text{diagonals}\}]},$$

where $X \subset F$ is the set of singular points of the curves in the family $C \subset F$.

Excess intersection theory

Consider the natural map $F\langle r \rangle \rightarrow F^r$ and let $p_i: F\langle r \rangle \rightarrow F$ be the composition with the i -th projection. Then

$$p_1^*[X] \cdots p_r^*[X] = m_r + \sum_Z (p_1^*[X] \cdots p_r^*[X])^Z,$$

where the sum is taken over all connected components Z of the intersection

$$p_1^*[X] \cap \cdots \cap p_r^*[X],$$

and where $(p_1^*[X] \cdots p_r^*[X])^Z$ denotes the “equivalence” of Z .

For each multidiagonal Δ_I in F^r , there is a divisor D_I in $F\langle r \rangle$. For each D_I , collect the Z 's contained in it. This gives a contribution b_I , a class supported on D_I .

Diagonals

How many multidagonals of each type exist? By type $\mathbf{k} = (k_1, \dots, k_r)$ we mean that k_2 pairs of points of $(x_1, \dots, x_r)$ are equal, k_3 triples of points of $(x_1, \dots, x_r)$ are equal, and so on. There are precisely

$$\frac{r!}{k_1! \cdots k_r!} \left(\frac{1}{1!}\right)^{k_1} \cdots \left(\frac{1}{r!}\right)^{k_r}$$

multidagonals of type $(k_1, \dots, k_r)$.

By symmetry, the class b_I only depends on the type of the diagonal. The excess intersection formula therefore gives

$$m_r = p_1^*[X] \cdots p_r^*[X] - \sum_{\mathbf{k}} \frac{r!}{k_1! \cdots k_r!} \left(\frac{1}{1!}\right)^{k_1} \cdots \left(\frac{1}{r!}\right)^{k_r} b_{\mathbf{k}}.$$



The conjecture

The conjectured formula was

$$N_r = \frac{1}{r!} P_r(a_1, \dots, a_r) = \sum_{\mathbf{k}} \frac{1}{k_1! \cdots k_r!} \left(\frac{a_1}{1!}\right)^{k_1} \cdots \left(\frac{a_r}{r!}\right)^{k_r},$$

where the sum is taken over $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 0}^r$ with $\sum i k_i = r$.

We know that $\pi_* p_i^*[X] = a_1$, and it remains to show that, more generally, $\pi_* b_{\mathbf{k}} = a_1^{k_1} \cdots a_r^{k_r}$.

My student Nikolay Qviller is trying to do this, using the theory of dominant transforms on the space $F\langle r \rangle$ developed by Li Li. The ambition for the time being is to prove the *shape* of the polynomials N_r , not to compute them.

Note that Kazarian has a topological argument, involving Thom polynomials, for showing that such a formula holds.

Bell polynomials and Faà di Bruno's formula

We have seen two reasons for the Bell polynomials to appear in the conjectured formula. The first (and the reason we found the Bell polynomial form) is the recursive method. It showed that there was a “derivation” process, reminiscent of how one gets Faà di Bruno's formula for the derivatives of a composed function. The other is the “direct approach” via intersections, where the Bell polynomials appear as “generating functions” for the enumeration of multidiagonals.

The beauty of the Bell polynomial shape is that in order to compute N_r from N_{r-1} , one only needs to compute *one* term, namely a_r :

$$P_r(a_1, \dots, a_r) = a_r + \sum_{i=1}^{r-1} \binom{r-1}{i} P_i(a_1, \dots, a_i) a_{r-i}.$$



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