Projective geometry from a toric point of view

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Differential geometry

Let $r : \mathbb{R} \to \mathbb{R}^N$ be a (parameterized) curve, $t \mapsto r(t) = (r_1(t), r_2(t), \dots, r_N(t))$.

The tangent to the curve at the point $r(t)$ is the line $\langle r(t), r'(t) \rangle$, the osculating plane is $\langle r(t), r'(t), r''(t) \rangle$, and so on.

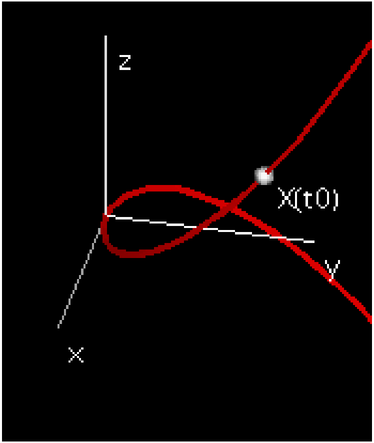
Example

$r(t) = (t, t^2, t^3) \in \mathbb{R}^3$

The tangent line at $(0, 0, 0)$ is $\langle (0, 0, 0), (1, 0, 0) \rangle$ – the x-axis.

The osculating plane at $(0, 0, 0)$ is $\langle (0, 0, 0), (1, 0, 0), (0, 2, 0) \rangle$ – the xy-plane.

Twisted cubic



Projective varieties

Let $X \subset \mathbb{P}^N$ be a (smooth) projective algebraic variety of dimension n over an algebraically closed field $\mathbb{K}$.

Set $\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_X$. The kth *jet bundle* (or principal parts bundle of $\mathcal{L}$) is of rank $\binom{n+k}{n}$ and comes with a jet map

$$j_k \colon \mathcal{O}_X^{N+1} \to \mathcal{P}_X^k(\mathcal{L}),$$

whose fibers are given by Taylor expansions up to kth order of

$$s = (s_0, \ldots, s_N) \colon \mathcal{O}_X^{N+1} \to \mathcal{L}.$$

The exact sequences

$$0 \to S^i\Omega_X^1 \otimes \mathcal{L} \to \mathcal{P}_X^i(\mathcal{L}) \to \mathcal{P}_X^{i-1}(\mathcal{L}) \to 0$$

allow one to compute the Chern classes of the jet bundles in terms of those of X and $\mathcal{L}$.

Tangent and osculating spaces

The embedded tangent space to X at a point x is equal to

$$\mathbb{T}_{X,x} = \mathbb{P}(\operatorname{Im} j_{1,x}) = \mathbb{P}(\mathcal{P}^1_X(\mathcal{L})_x) \cong \mathbb{P}^n.$$

The *kth osculating space* to X at x is the linear space

$$\mathbb{T}^k_{X,x} := \mathbb{P}(\operatorname{Im} j_{k,x}).$$

Note: $\dim \mathbb{T}^k_{X,x} \leq \operatorname{rk} \mathcal{P}^k_X(\mathcal{L}) - 1 = \binom{n+k}{k} - 1.$

Inflections

Let

$$d_k + 1 := \text{ generic rank of } j_k \colon \mathcal{O}_X^{N+1} \to \mathcal{P}_X^k(\mathcal{L}).$$

A point $x \in X$ is an inflection point of order k if $\operatorname{rk} j_{k,x} < d_k + 1$; equivalently, if $\dim \mathbb{T}^k_{X,x} < d_k$.

Question 1: Determine the (class of the) locus of inflection points on X.

Question 2: Classify varieties with special osculating behavior.

Example

A curve $X \subset \mathbb{P}^N$ of degree d and genus g has $(N+1)(d+N(g-1))$ inflection points. So the only uninflected curves in $\mathbb{P}^N$ are the rational normal curves: $d = N$ and $g = 0$.

Three theorems

Theorem (Fulton–Kleiman–P.–Tai)

Let X be a smooth, irreducible variety of dimension n and set $N = \binom{n+k}{k} - 1$. The only embedding $X \to \mathbb{P}^N$ such that $\mathbb{T}^k_{X,x} = \mathbb{P}^N$ for all $x \in X$ is the kth Veronese embedding of $X = \mathbb{P}^n$.

Theorem (Ballico–P.–Tai)

Let $X \subset \mathbb{P}^{2k+1}$ be a smooth surface such that $\dim \mathbb{T}^m_{X,x} = 2m$ for all $x \in X$ and all $m \leq k$. Then X is equal to the balanced rational normal scroll of degree $2k$.

Theorem (Lanteri–Mallavibarrena–P.)

The only uninflected n-dimensional scroll $X \subset \mathbb{P}^{nk+\ell-1}$, $1 \leq \ell \leq n$, is the balanced rational normal scroll of degree nk.

Toric embeddings

$$\mathcal{A} = \{a_0, \dots, a_N\} \subset \mathbb{Z}^n \rightsquigarrow X_{\mathcal{A}} \subseteq \mathbb{P}^N.$$

The associated (equivariantly embedded) projective toric variety $X_{\mathcal{A}}$ is the Zariski closure of the image of all $t = (t_1, \dots, t_n) \in (\mathbb{K}^*)^n$ under the map

$$t \mapsto (t^{a_0} : \cdots : t^{a_N}).$$

E.g., $\mathcal{A} = P \cap \mathbb{Z}^n$, for a lattice polytope P.

The three above examples are toric:

- the kth Veronese of $\mathbb{P}^n$: $P = k\Delta_n$
- a balanced rational normal scroll of dimension n, degree nk: $P = \Delta_{n-1} \times k\Delta_1$.

If we assume X is toric, the theorems are easier to prove.

Togliatti's surface

The lattice point configuration

$$\mathcal{A} = \{(1,0),(0,1),(2,0),(0,2),(2,1),(1,2)\} \subset \mathbb{Z}^2$$

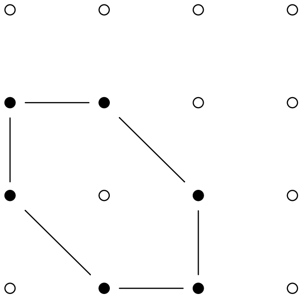
gives the toric embedding

$$(\mathbb{K}^*)^2 \to \mathbb{P}^5$$

given by

$$(x,y) \mapsto (x : y : x^2 : y^2 : x^2y : xy^2).$$

Togliatti lattice point configuration



Polytopes and toric varieties: dictionary

$P \subset \mathbb{R}^n$ lattice polytope, $X_P \subset \mathbb{P}^N$

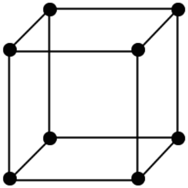
- X_P smooth iff P smooth
- Hilbert polynomial of X_P = Ehrhart polynomial of P
- $\dim H^0(X_P, mL_P) = \#(mP \cap \mathbb{Z})$
- X_P a surface: sectional genus $= \# \operatorname{Int} P \cap \mathbb{Z}$
- $\deg X_P = c_1(L_P)^n = \operatorname{Vol}_{\mathbb{Z}}(P)$
- $c_i(T_{X_P})c_1(L_P)^{n-i} = \sum_{\operatorname{codim} F_i = i} \operatorname{Vol}_{\mathbb{Z}}(F_i)$.
- $c_n(T_{X_P}) = \#$ vertices of P
- Riemann–Roch and Ehrhart series
- Resolution of singularities and continued fractions
- Local Euler obstruction = "corner volume"

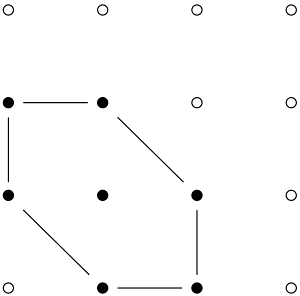
Sections and projections

Let $\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ be a lattice point configuration and let $X_{\mathcal{A}} \subset \mathbb{P}^N$ denote the corresponding toric embedding. Let $\mathcal{A}'$ be a lattice point configuration obtained from $\mathcal{A}$ by removing m points. Then the toric embedding $X_{\mathcal{A}'} \subset \mathbb{P}^{N'}$, where $N' = N - m$, is the (toric) linear projection of $X_{\mathcal{A}}$ with center equal to the linear span of the "removed points".

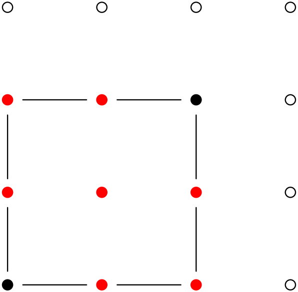
A *toric hyperplane section* of $X_{\mathcal{A}}$ is obtained by taking a hyperplane in $\mathbb{Z}^n$ and "collapsing" the point configuration $\mathcal{A}$ into this lattice hyperplane in such a way that one point is "lost": two points map to the same point.

Del Pezzo lattice configuration

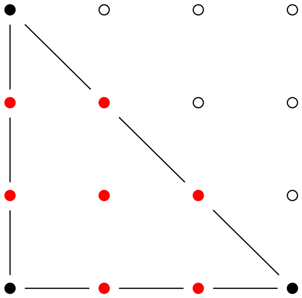




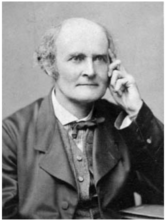
$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ via $\mathcal{O}(2,2)$



Third Veronese: $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$



Cayley polytopes



Let $P_0, \ldots, P_r \subset \mathbb{R}^{n-r}$ be convex lattice polytopes and $e_0, \ldots, e_r$ the vertices of $\Delta_r \subset \mathbb{R}^r$.

The polytope

$$P = \mathrm{Conv}\{e_0 \times P_0, \ldots, e_r \times P_r\} \subset \mathbb{R}^r \times \mathbb{R}^{n-r} = \mathbb{R}^n,$$

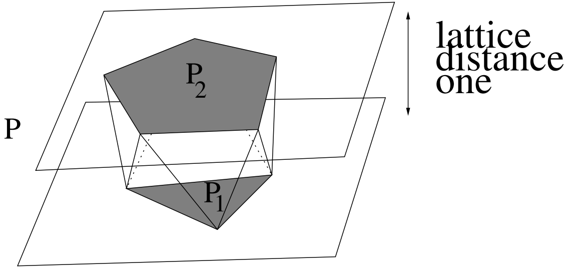
is called a *Cayley polytope.*

We write

$$P = \mathrm{Cayley}(P_0, \ldots, P_r).$$

Hollow polytopes

A Cayley polytope is “hollow”: it has no interior lattice points.



The codegree and degree of a polytope

$$\mathrm{codeg}(P) := \min\{m \,|\, mP \text{ has interior lattice points}\}.$$

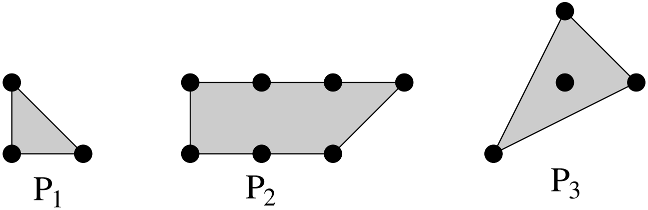
$$\deg(P) := n + 1 - \mathrm{codeg}(P)$$

Example (1)

$$\mathrm{codeg}(\Delta_n) = n + 1 \text{ and } \mathrm{codeg}(2\Delta_n) = \lceil \frac{n+1}{2} \rceil.$$

Example (2)

$$P = \mathrm{Cayley}(P_0, \ldots, P_r) \text{ implies } \mathrm{codeg}(P) \geq r + 1.$$



$\mathrm{codeg}(P_1) = 3$ $\mathrm{codeg}(P_2) = 2$ $\mathrm{codeg}(P_3) = 1$

The Cayley polytope conjecture

Question (Batyrev–Nill): Is there an integer $N(d)$ such that any polytope P of degree d and $\dim P \geq N(d)$ is a Cayley polytope?

Answer (Haase–Nill–Payne): Yes, and $N(d) \leq (d^2 + 19d - 4)/2$

Question: Is $N(d)$ linear in d?

Answer (Dickenstein–Di Rocco–P.): Yes, $N(d) = 2d + 1$ (if P is smooth and $\mathbb{Q}$-normal).

Note that $n \geq 2d + 1$ is equivalent to $\operatorname{codeg}(P) \geq \frac{n+3}{2}$.

Theorem (Dickenstein, Di Rocco, P., Nill)

Let P be a smooth lattice polytope of dimension n. The following are equivalent

(1) $\operatorname{codeg}(P) \geq \frac{n+3}{2}$

(2) $P = \operatorname{Cayley}(P_0, \ldots, P_r)$ *is a smooth Cayley polytope with* $r + 1 = \operatorname{codeg}(P)$ *and* $r > \frac{n}{2}$.

The proof of this combinatorial result is algebro-geometric (adjoints and nef-value maps à la Beltrametti–Sommese, toric fibrations à la Reid).

Higher order dual varieties

The *kth dual variety* $X^{(k)}$ is defined as:

$$X^{(k)} = \overline{\{H \in (\mathbb{P}^N)^\vee \mid H \supseteq \mathbb{T}^k_{X,x} \text{ for some } x \in X_{j_k-\text{cst}}\}}.$$

In particular, $X^{(1)} = X^\vee$, $X^{(k-1)} \supseteq X^{(k)}$, and $X^{(k)}$ is contained in the singular locus of $X^\vee$ for $k \geq 2$.

The expected dimension of $X^\vee$ is $N-1$ and that of $X^{(k)}$ is $n + N - d_k - 1$.

Degree of dual varieties

Gelfand–Kapranov–Zelevinsky:

If X_P is smooth, then

$$\deg X_P^\vee = \sum_{F \prec P} (-1)^{\operatorname{cod} F} (\dim F + 1) \operatorname{Vol}_{\mathbb{Z}}(F)$$

Matsui–Takeuchi:

$$\deg X_P^\vee = \sum_{F \prec P} (-1)^{\operatorname{cod} F} (\dim F + 1) \operatorname{Vol}_{\mathbb{Z}}(F) \operatorname{Eu}(F),$$

where $\operatorname{Eu}(F)$ denotes the generic value of the local Euler obstruction of points on X_P corresponding to the face F.

Weighted projective planes

The weighted projective plane $\mathbb{P}(k, m, n)$ is the toric surface in $\mathbb{P}^N$, with $N = (kmn + k + m + n)/2$, given by the lattice points in the convex hull of the points $\{(mn, 0, 0), (0, kn, 0), (0, 0, km)\}$. This surface has isolated cyclic quotient singularities at the points corresponding to the vertices of the triangle.

Theorem (Nødland)

$\deg \mathbb{P}(k, m, n)^\vee =$
$3kmn - 2(k+n+m) + \sum_{i=1}^{r}(2-a_i) + \sum_{i=1}^{s}(2-b_i) + \sum_{i=1}^{t}(2-c_i)$,
where the (a_i), (b_i), (c_i) *are the integers appearing in the Hirzebruch–Jung continued fractions coming from the three singular points.*

Degree of higher dual varieties

Theorem (Dickenstein–Di Rocco–P.)

Let (X_P, L_P) be a smooth, 2-regular toric threefold embedding $\neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(i))$, $i = 2, 3$, $(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi)$. Then

$$\deg X^{(2)} = 62V - 57F + 28E - 8v + 58V_1 + 51F_1 + 20E_1,$$

where V, F, E (resp. V_1, F_1, E_1) denote the (lattice) volume, area of facets, length of edges of P (resp. the adjoint polytope $\mathrm{Conv}(\mathrm{Int}\, P)$), and $v = \#\{vertices\ of\ P\}$.

k-selfdual toric varieties (joint with A. Dickenstein)

$\mathcal{A} = \{a_0, \ldots, a_N\} \subset \mathbb{Z}^n$ a lattice point configuration, and $X_{\mathcal{A}} \subset \mathbb{P}^N$ the corresponding toric embedding.

Form the matrix A by adding a row of 1's to the matrix $(a_0|\cdots|a_N)$. Denote by $\mathbf{v}_0 = (1, \ldots, 1)$, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Z}^{N+1}$ the row vectors of A.

For any $\alpha \in \mathbb{N}^{n+1}$, denote by $\mathbf{v}_\alpha \in \mathbb{Z}^{N+1}$ the vector obtained as the coordinatewise product of α_0 times the row vector $\mathbf{v}_0$ times $\ldots$ times α_n times the row vector $\mathbf{v}_n$.

Order the vectors $\{\mathbf{v}_\alpha : |\alpha| \leq k\}$. Let $A^{(k)}$ be the $\binom{n+k}{k} \times (N+1)$ integer matrix with these rows.

Rational normal curve

Take $\mathcal{A} = \{0, \ldots, d\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \end{pmatrix},$$

and

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 1 & 4 & 9 & \cdots & d^2 \\ 0 & 1 & 8 & 27 & \cdots & d^3 \end{pmatrix}.$$

Note that

$$A^{(3)} \cong \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & d \\ 0 & 0 & 1 & 3 & \cdots & \binom{d}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{d}{3} \end{pmatrix}.$$

The case $k = 2$

Denote by $\mathbf{v}_i * \mathbf{v}_j \in \mathbb{Z}^{m+1}$ the vector given by the coordinatewise product of these vectors. Define the $\binom{n+2}{2} \times (m+1)$-matrix

$$A^{(2)} = \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{v}_1 * \mathbf{v}_1 \\ \mathbf{v}_1 * \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} * \mathbf{v}_n \\ \mathbf{v}_n * \mathbf{v}_n \end{pmatrix},$$

$\mathbf{v}_i * \mathbf{v}_j$, $1 \leq i \leq j \leq n$. Then, $\mathbb{P}(\text{Rowspan}(A^{(2)})) = \mathbb{T}^2_{X_{\mathcal{A}},\mathbf{1}}$ describes the second osculating space of $X_{\mathcal{A}}$ at the point $\mathbf{1}$.

Non-pyramidal configurations

The configuration $\mathcal{A}$ is a *non-pyramid* (nap) if the configuration of columns in A is not a pyramid (i.e., no basis vector e_i of $\mathbb{R}^{N+1}$ lies in the rowspan of the matrix).

The configuration $\mathcal{A}$ is $k$nap if the configuration of columns in $A^{(k)}$ is not a pyramid.

Note that any vector in the rowspan of $A^{(k)}$ is equal to

$$(Q(a_0), \ldots, Q(a_N)),$$

for some polynomial Q in n variables, of degree $\leq k$.

$A^{(k)}$ is a pyramid iff there exist Q, i such that $Q(a_j) = 0$ for all $j \neq i$ and $Q(a_i) \neq 0$.

Characterization of k-self dual configurations

$X_{\mathcal{A}}$ is *k-selfdual* if $\phi(X_{\mathcal{A}}) = X_{\mathcal{A}}^{(k)}$ for some $\phi\colon \mathbb{P}^N \cong (\mathbb{P}^N)^\vee$.

Theorem (Dickenstein–P.)

(1) *$X_{\mathcal{A}}$ is k-selfdual if and only if $\dim X_{\mathcal{A}} = \dim X_{\mathcal{A}}^{(k)}$ and $\mathcal{A}$ is knap.*

(2) *If $\mathcal{A}$ is knap and $\dim \operatorname{Ker} A^{(k)} = 1$, then $X_{\mathcal{A}}$ is k-selfdual.*

The proof generalizes [Bourel–Dickenstein–Rittatore] ($k = 1$).

A surface in $\mathbb{P}^3$

$$\mathcal{A} = \{(0,0),(1,0),(1,1),(0,2)\}$$

gives

$$X_{\mathcal{A}} : (x,y) \mapsto (1 : x : xy : y^2)$$

and

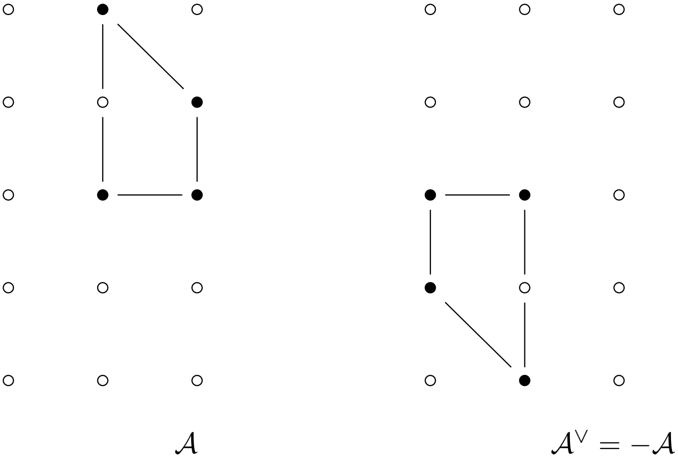
$$X_{\mathcal{A}}^{\vee} \cong X_{\mathcal{A}^{\vee}} : (x,y) \mapsto (-1 : 2x^{-1} : -2x^{-1}y^{-1} : y^{-2}),$$

with

$$\mathcal{A}^{\vee} = \{(0,0),(-1,0),(-1,-1),(0,-2)\} = -\mathcal{A}.$$

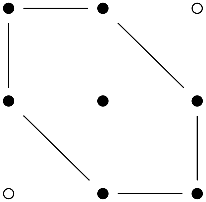
This surface is self dual.

The corresponding polytopes

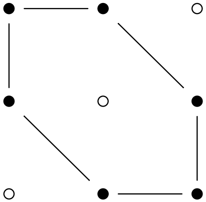


Del Pezzo and Togliatti

Del Pezzo is not 2nap:

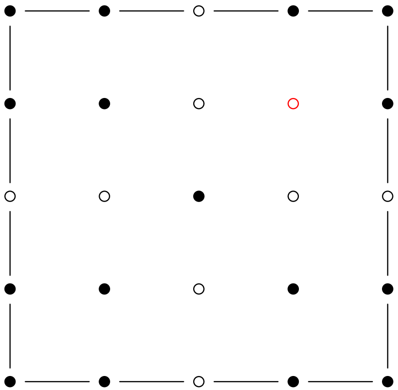


Togliatti is 2nap:



Example

This square is an example of a 4-selfdual smooth surface which is not projectively normal and not centrally symmetric.



The *complete* polytope is 7-selfdual, projectively normal and centrally symmetric.

Chasles–Cayley–Bacharach

Non-trivial linear relations between the rows of $A^{(k)}$ correspond to polynomials of degree $\leq k$ vanishing on $\mathcal{A}$ (D. Perkinson).

Example

Three quadrics $Q_1, Q_2, Q_3 \in \mathbb{Z}[x_1, x_2, x_3]$ with

$$Q_1 \cap Q_2 \cap Q_3 = \{a_0, \ldots, a_7\} = \mathcal{A} \subset \mathbb{Z}^3 \subset \mathbb{R}^3.$$

Then $X_{\mathcal{A}}$ is a 2-selfdual threefold:

The rank of the (10×8)-matrix $A^{(2)}$ is $10 - 3 = 7$, so $\dim \operatorname{Ker} A^{(k)} = 1$.

Connections with number theory

In general it is difficult to find integer polynomials with many integer roots (cf. Rodriguez Villegas, Voloch, Zagier).

Example

Consider 3 integers m_1, m_2, m_3 and $f(x) = \prod_{i=1}^{3}(x - m_i)$.
Consider the quadratic polynomial

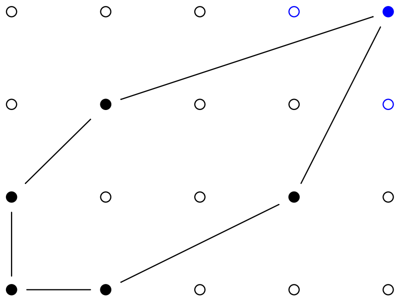
$$Q(x,y) = \frac{f(x) - f(y)}{x - y} \in \mathbb{Z}[x,y]$$

Q vanishes at the 6 lattice points $(m_i, m_j), j \neq i$, while $\binom{2+2}{2}$ is also equal to 6.
The configuration $\mathcal{A}$ given by these 6 points is 2-self dual because it is 2nap and $\dim \operatorname{Ker} A^{(k)} = 1$.

Curves with many lattice points

$$\mathcal{A}' := \{(0,0),(1,0),(0,1),(3,1),(1,2)\}$$

The unique conic through these five points, given by the vanishing of $Q = x^2 - 2xy + 2y^2 - x - 2y$, also goes through the lattice points $a_5 = (3,3)$, $a_6 = (4,3)$ and $a_7 = (4,2)$.
So it is a conic through 8 lattice points.



Adding any one of these three points to $\mathcal{A}'$ gives 3 examples of 2-selfdual surfaces in $\mathbb{P}^5$ that are non-smooth.
If we add all 3 points, we get a 3-selfdual surface.

Joins

Let $V_1, \dots, V_s$ be finite dimensional $\mathbb{K}$-vector spaces and let $X_1 \subseteq \mathbb{P}(V_1)$, …, $X_s \subseteq \mathbb{P}(V_s)$ be projective varieties. The *join* of $X_1, \dots, X_s$ is the projective subvariety of $\mathbb{P}(V_1 \oplus \cdots \oplus V_s)$ defined by

$$\mathrm{J}(X_1, \dots, X_s) = \overline{\left\{ [x_1 : \cdots : x_s] \,\middle|\, [x_i] \in X_i \right\}}.$$

Proposition (Dickenstein–P.)

Assume $\mathcal{A}_1, \dots, \mathcal{A}_s$ are $k$nap and k-selfdual. Then the join $X_{\mathcal{A}} = \mathrm{J}(X_{\mathcal{A}_1}, \dots, X_{\mathcal{A}_s})$ is s-Cayley, $k$nap, and k-selfdual, with

$$\dim \operatorname{Ker} A^{(k)} = \dim \operatorname{Ker} A_1^{(k)} + \cdots + \dim \operatorname{Ker} A_s^{(k)} \geq s.$$

Joins of varieties of degree at least 2 are not smooth.

k-selfdual Cayley polytopes

Proposition (Dickenstein–P.)

Let $\mathcal{B}$ be a lattice configuration of cardinality $m+1$ such that the general kth osculating space of $X_{\mathcal{B}}$ is the whole $\mathbb{P}^m$ and $\dim \operatorname{Ker} B^{(k-1)} = 1$. Let $r \geq 1$ and take $\mathcal{A} = \text{Cayley}(\mathcal{B}, \ldots, \mathcal{B})$ ($r+1$ times), so that

$$X_{\mathcal{A}} = \mathbb{P}^r \times X_{\mathcal{B}} \subset \mathbb{P}^{(r+1)(m+1)-1}.$$

Then, $X_{\mathcal{A}}$ is k-selfdual if and only if $X_{\mathcal{B}}$ is $(k-1)$-selfdual.

Proof.

One checks that $\mathcal{A}$ is $k$nap if and only if $\mathcal{B}$ is $(k-1)$nap, and that $\dim \operatorname{Ker} A^{(k)} = r$. Then use a combinatorial/toric variety argument. □

Segre-Veronese examples

The Segre embedding $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2^n-1}$ is $(n-1)$-selfdual [Vallès, 2006]. More generally:

Proposition (Dickenstein–P.)

Let $\mathcal{A}$ be a lattice point configuration such that $X_{\mathcal{A}}$ is equal to a Segre embedding of the following form:

(i) $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$,

(ii) $\mathbb{P}^r \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$,

with $m \geq 1$ copies of $\mathbb{P}^1$'s, and the embedding is of type $(\ell_1, \ldots, \ell_m)$ with $k := \sum_{i=1}^{m} \ell_i - 1 > 0$ in case (i), or $(1, \ell_1, \ldots, \ell_m)$ with $k := \sum_{i=1}^{m} \ell_i$ in case (ii), $\ell_i \geq 1$.
Then, in both cases $X_{\mathcal{A}}$ is k-selfdual. Moreover, $\dim \operatorname{Ker} A^{(k)} = 1$ in case (i) and $\dim \operatorname{Ker} A^{(k)} = r$ in case (ii).

Towards a classification in the smooth case

Conjecture

The only smooth, projectively normal k-selfdual toric varieties $X_{\mathcal{A}}$ with $\dim \operatorname{Ker} A^{(k)} > 1$ are the Segre-Veronese examples described in the previous Proposition (ii).

For $k = 1$, this holds: the only smooth, projectively normal selfdual toric varieties are: the plane conic $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, the quadric surface in $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ and the Segre embeddings $\mathbb{P}^r \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2r+1}$ for any $r \geq 2$.

For $k > 1$, when $\dim \operatorname{Ker} A^{(k)} = 1$, there is no hope to get a classification, nor is there hope when $\mathcal{A} \neq \operatorname{Conv}(\mathcal{A}) \cap \mathbb{Z}^n$.

Thank you for your attention!